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Yvette Kosmann-Schwarzbach

The Noether Theorems

Invariance and Conservation Laws in the Twentieth Century

Translated by
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Translation of "Invariante Variationsprobleme" (1918)

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

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1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

First page of "Invariante Variationsprobleme" (reproduced with permission)
Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, 1918, pp. 235–257.

INVARIANT VARIATIONAL PROBLEMS

(For F. Klein, on the occasion of the fiftieth anniversary of his doctorate)

by **Emmy Noether** in Göttingen

Presented by F. Klein at the session of 26 July 1918*

We consider variational problems which are invariant[A] under a continuous group (in the sense of Lie); the consequences that are implied for the associated differential equations find their most general expression in the theorems formulated in §1, which are proven in the subsequent sections. For those differential equations that arise from variational problems, the statements that can be formulated are much more precise than for the arbitrary differential equations that are invariant under a group, which are the subject of Lie's researches. What follows thus depends upon a combination of the methods of the formal calculus of variations and of Lie's theory of groups. For certain groups and variational problems this combination is not new; I shall mention Hamel and Herglotz for certain finite groups, Lorentz and his students (for example, Fokker), Weyl and Klein for certain infinite groups.[1] In particular, Klein's second note and the following developments were mutually influential, and for this reason I take the liberty of referring to the final remarks in Klein's note.

1 Preliminary Remarks and the Formulation of the Theorems

All the functions that will be considered here will be assumed to be analytic or at least continuous and continuously differentiable a finite number of times, and single-valued within the domain that is being considered.

By the term "transformation group" one usually refers to a system of transformations such that for each transformation there exists an inverse which is an element of the system, and such that the composition of any two transformations of the system is again an element of the system. The group is called a *finite continuous* [group] $\mathfrak{G}_\rho$ when its transformations can be expressed in a general form which depends analytically on ρ *essential* parameters ε (i.e., the ρ parameters cannot be represented by ρ functions of a smaller number of parameters). In the same way, one speaks of an *infinite continuous* group $\mathfrak{G}_{\infty\rho}$ for a group whose most general transformations depend on ρ essential arbitrary functions $p(x)$ and their derivatives in a way that is

* The definitive version of the manuscript was prepared only at the end of September.

[A] *gestatten*, to permit, in the sense of admitting [an invariance group] has been translated as "being invariant under [the action of] a group" (Translator's note).

[1] Hamel, Math. Ann., vol. 59, and Zeitschrift f. Math. u. Phys., vol. 50. Herglotz, Ann. d. Phys. (4) vol. 36, in particular §9, p. 511. Fokker, Verslag d. Amsterdamer Akad., 27/1 1917. For a more complete bibliography, see Klein's second note, Göttinger Nachrichten, 19 July 1918.

In a paper by Kneser that has just appeared (Math. Zeitschrift, vol. 2), the determination of invariants is dealt with by a similar method.

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analytical or at least continuous and continuously differentiable a finite number of times. An intermediate case is the one in which the groups depend on an infinite number of parameters but not on arbitrary functions. Finally, one calls a group that depends not only on parameters but also on arbitrary functions a *mixed group*.[2]

Let $x_1,\ldots,x_n$ be independent variables, and let $u_1(x),\ldots,u_\mu(x)$ be functions of these variables. If one subjects the x and the u to the transformations of a group, then one should recover, among all the transformed quantities, precisely n independent variables, $y_1,\ldots,y_n$, by the assumption of invertibility of the transformations; let us call the remaining transformed variables that depend on them $v_1(y),\ldots,v_\mu(y)$. In the transformations, the derivatives of u with respect to x, that is to say $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots$, may also occur.[3] A function is said to be an *invariant* of the group if there is a relation

$$P\left(x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\cdots\right)=P\left(y,v,\frac{\partial v}{\partial y},\frac{\partial^2 v}{\partial y^2},\cdots\right).$$

In particular, an integral I is an invariant of the group if it satisfies the relation

$$(1)\qquad \begin{aligned} I &= \int\cdots\int f\left(x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\cdots\right)dx \\ &= \int\cdots\int f\left(y,v,\frac{\partial v}{\partial y},\frac{\partial^2 v}{\partial y^2},\cdots\right)dy \end{aligned}$$ [4]

integrated over an *arbitrary* real domain in x, and over the corresponding domain in y.[5]

On the other hand, I calculate for an arbitrary integral I, which is not necessarily invariant, the first variation δI, and I transform it, according to the rules of the

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[2] Lie defines, in the "Grundlagen für die Theorie der unendlichen kontinuierlichen Transformationsgruppen" ["Basic Principles of the Theory of Infinite Continuous Transformation Groups"], Ber. d. K. Sächs. Ges. der Wissensch. 1891 (to be cited henceforth as "Grundlagen"), the infinite continuous groups as transformation groups whose elements are given by the most general solutions of a system of partial differential equations provided that these solutions do not depend exclusively on a finite number of parameters. Thus one obtains one of the above-mentioned cases distinct from that of a finite group, while, on the other hand, the limiting case of an infinite number of parameters does not necessarily satisfy a system of differential equations.

[3] I omit the indices here, and in the summations as well whenever it is possible, and I write $\frac{\partial^2 u}{\partial x^2}$ for $\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\gamma}$, etc.

[4] I write dx, dy for $dx_1\ldots dx_n$, $dy_1\ldots dy_n$ for short.

[5] All the arguments x, u, ε, $p(x)$ that occur in the transformations must be assumed to be real, while the coefficients may be complex. Since the final results consist of *identities* among the x, the u, the parameters and the arbitrary functions, these identities are valid as well for the complex domain, once one assumes that all the functions that occur are analytic. In any event, a major part of the results can be proven without integration, so a restriction to the real domain is not necessary for the proof. However, the considerations at the end of §2 and at the beginning of §5 do not seem to be valid without integration.

caculus of variations, by integration by parts. Once one assumes that δu and all the derivatives that occur vanish on the boundary, but remain arbitrary elsewhere, one obtains the well-known result,

$$(2) \qquad \delta I = \int \cdots \int \delta f \, dx = \int \cdots \int \left(\sum \psi_i \left(x, u, \frac{\partial u}{\partial x}, \cdots \right) \delta u_i \right) dx,$$

where ψ represents the *Lagrangian expressions*, that is to say, the left-hand side of the Lagrangian equations of the associated variational problem $\delta I = 0$. To that integral relation there corresponds an *identity* without an integral in δu and its derivatives that one obtains by adding the boundary terms. As an integration by parts shows, these boundary terms are integrals of *divergences*, that is to say, expressions

$$\operatorname{Div} A = \frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n},$$

where A is linear in δu and its derivatives. From that it follows that

$$(3) \qquad \sum \psi_i \delta u_i = \delta f + \operatorname{Div} A.$$

In particular, if f contains only the *first* derivatives of u, then, in the case of a simple integral, identity (3) is identical to Heun's "central Lagrangian equation,"

$$(4) \qquad \sum \psi_i \delta u_i = \delta f - \frac{d}{dx} \left(\sum \frac{\partial f}{\partial u_i'} \delta u_i \right), \qquad \left(u_i' = \frac{du_i}{dx} \right),$$

while for an n-fold integral, (3) becomes

$$(5) \qquad \sum \psi_i \delta u_i = \delta f - \frac{\partial}{\partial x_1} \left(\sum \frac{\partial f}{\partial \frac{\partial u_i}{\partial x_1}} \delta u_i \right) - \cdots - \frac{\partial}{\partial x_n} \left(\sum \frac{\partial f}{\partial \frac{\partial u_i}{\partial x_n}} \delta u_i \right).$$

For the simple integral and κ derivatives of the u, (3) yields

$$(6) \qquad \sum \psi_i \delta u_i = \delta f -$$

$$- \tfrac{d}{dx} \left\{ \Sigma \left(\binom{1}{1} \tfrac{\partial f}{\partial u_i^{(1)}} \delta u_i + \binom{2}{1} \tfrac{\partial f}{\partial u_i^{(2)}} \delta u_i^{(1)} + \cdots + \binom{\kappa}{1} \tfrac{\partial f}{\partial u_i^{(\kappa)}} \delta u_i^{(\kappa-1)} \right) \right\} +$$

$$+ \tfrac{d^2}{dx^2} \left\{ \Sigma \left(\binom{2}{2} \tfrac{\partial f}{\partial u_i^{(2)}} \delta u_i + \binom{3}{2} \tfrac{\partial f}{\partial u_i^{(3)}} \delta u_i^{(1)} + \cdots + \binom{\kappa}{2} \tfrac{\partial f}{\partial u_i^{(\kappa)}} \delta u_i^{(\kappa-2)} \right) \right\} +$$

$$+ \cdots + (-1)^\kappa \tfrac{d^\kappa}{dx^\kappa} \left\{ \Sigma \binom{\kappa}{\kappa} \tfrac{\partial f}{\partial u_i^{(\kappa)}} \delta u_i \right\},$$

and there is a corresponding identity for an n-fold integral; in particular, A contains δu and its derivatives up to order $\kappa - 1$. That the Lagrangian expressions ψ_i are actually defined by (4), (5) and (6) is a result of the fact that, by the combinations of the right-hand sides, all the higher derivatives of the δu are eliminated, while, on the other hand, relation (2), which one clearly obtains by an integration by parts, is satisfied.

In what follows we shall examine the following two theorems:

I. *If the integral I is invariant under a [group] $\mathfrak{G}_\rho$, then there are ρ linearly independent combinations among the Lagrangian expressions which become divergences—and conversely, that implies the invariance of I under a [group] $\mathfrak{G}_\rho$. The theorem remains valid in the limiting case of an infinite number of parameters.*

II. *If the integral I is invariant under a [group] $\mathfrak{G}_{\infty\rho}$ depending on arbitrary functions and their derivatives up to order σ, then there are ρ identities among the Lagrangian expressions and their derivatives up to order σ. Here as well the converse is valid.*[6]

For mixed groups, the statements of these theorems remain valid; thus one obtains identities[B] as well as divergence relations independent of them.

If we pass from these identity relations to the associated variational *problem*, that is to say, if we set $\psi = 0$,[7] then Theorem I states in the one-dimensional case—where the divergence coincides with a total differential—the existence of ρ first integrals among which, however, there may still be nonlinear identities;[8] in higher dimensions one obtains the divergence equations that, recently, have often been referred to as "conservation laws." Theorem II states that ρ Lagrangian equations are a consequence of the others.[C]

The simplest example for Theorem II—without its converse—is Weierstrass's parametric representation; here, as is well known, the integral is invariant in the case of homogeneity of the first order when one replaces the independent variable x by an arbitrary function of x which leaves u unchanged $(y = p(x);\ v_i(y) = u_i(x))$. Thus *one* arbitrary function occurs though none of its derivatives occurs, and to this corresponds the well-known linear relation among the Lagrangian expressions themselves, $\sum \psi_i \frac{du_i}{dx} = 0$. Another example is offered by the physicists' "general theory of relativity"; in this case the group is the group of *all* the transformations of the $x : y_i = p_i(x)$, while the u (called $g_{\mu\nu}$ and q) are thus subjected to the transformations induced on the coefficients of a quadratic and of a linear differential form, respectively, transformations which contain the first derivatives of the arbitrary functions $p(x)$. To that there correspond the n known identities among the Lagrangian expressions and their first derivatives.[9]

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[6] For some trivial exceptions, see §2, note 13.

[B] *Abhängigkeit*, dependence, has been translated by "identity." *Identität* has been translated by "identity" or "identity relation." Both *Relation* and *Beziehung* have been translated by "relation" and *Verbindung* by "combination" (Translator's note).

[7] More generally, one can also set $\psi_i = T_i$; see §3, note 15.

[8] See the end of §3.

[C] I.e., among the Lagrangian equations, ρ equations are consequences of the remaining ones (Translator's note).

[9] For this, see Klein's presentation.

If, in particular, one considers a group such that there is no derivative of the $u(x)$ in the transformations, and that furthermore the transformed independent quantities depend only on the x and not on the u, then (as is proven in §5) from the invariance of I, the relative invariance of $\sum \psi_i \delta u_i$[10] follows, and also that of the divergences that appear in Theorem I, once the parameters are subjected to appropriate transformations. From that it follows as well that the first integrals mentioned above are also invariant under the group. For Theorem II, the relative invariance of the left-hand sides of the identities, expressed in terms of the arbitrary functions, follows, and consequently another function whose divergence vanishes identically and which is invariant under the group—which, in the physicists' theory of relativity, establishes the link between identities and law[D] of energy.[11] Theorem II ultimately yields, in terms of group theory, the proof of a related assertion of Hilbert concerning the lack of a proper law of energy in "general relativity." As a result of these additional remarks, Theorem I includes all the known theorems in mechanics, etc., concerning first integrals, while Theorem II can be described as the maximal generalization in group theory of "general relativity."

2 Divergence Relations and Identities

Let $\mathfrak{G}$ be a continuous group—finite or infinite; one can always assume that the identity transformation corresponds to the vanishing of the parameters ε, or to the vanishing of the arbitrary functions $p(x)$,[12] respectively. The most general transformation is then of the form

$$y_i = A_i\left(x, u, \frac{\partial u}{\partial x}, \cdots\right) = x_i + \Delta x_i + \cdots$$
$$v_i(y) = B_i\left(x, u, \frac{\partial u}{\partial x}, \cdots\right) = u_i + \Delta u_i + \cdots,$$

where Δx_i, Δu_i are the terms of lowest degree in ε, or in $p(x)$ and its derivatives, respectively, and we shall assume that in fact they are *linear*. As we shall show further on, this does not restrict the generality.

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[10] This is to say that $\sum \psi_i \delta u_i$ is invariant under the transformation up to a multiplicative factor.

[D] *Energiesatz* has been translated literally as "law of energy," in the sense of "law of conservation of energy," just as, *infra*, in §6, *eigentlich Energiesatz*, has been translated as "proper law of energy," in the sense of "proper law of conservation of energy" (Translator's note).

[11] See Klein's second note.

[12] Cf. Lie, "Grundlagen," p. 331. When dealing with arbitrary functions, it is necessary to replace the special values a^σ of the parameters by fixed functions $p^\sigma, \frac{\partial p^\sigma}{\partial x}, \cdots$; and correspondingly the values $a^\sigma + \varepsilon$ by $p^\sigma + p(x), \frac{\partial p^\sigma}{\partial x} + \frac{\partial p}{\partial x}$, etc.

Now let the integral I be invariant under $\mathfrak{G}$; then relation (1) is satisfied. In particular, I is also invariant under the infinitesimal transformations contained in $\mathfrak{G}$,

$$y_i = x_i + \Delta x_i; \qquad v_i(y) = u_i + \Delta u_i,$$

and therefore relation (1) becomes

$$(7) \qquad 0 = \Delta I = \int \cdots \int f\left(y, v(y), \frac{\partial v}{\partial y}, \cdots\right) dy - \int \cdots \int f\left(x, u(x), \frac{\partial u}{\partial x}, \cdots\right) dx,$$

where the first integral is defined on a domain in $x + \Delta x$ corresponding to the domain in x. But this integration can be replaced by an integration on the domain in x by means of the transformation

$$(8) \qquad \int \cdots \int f\left(y, v(y), \frac{\partial v}{\partial y}, \cdots\right) dy = \int \cdots \int f\left(x, v(x), \frac{\partial v}{\partial x}, \cdots\right) dx + \int \cdots \int \mathrm{Div}(f.\ \Delta x)\, dx,$$

which is valid for infinitesimal Δx, If, instead of the infinitesimal transformation Δu, one introduces the variation

$$(9) \qquad \bar{\delta} u_i = v_i(x) - u_i(x) = \Delta u_i - \sum \frac{\partial u_i}{\partial x_\lambda} \Delta x_\lambda,$$

(7) and (8) thus become

$$(10) \qquad 0 = \int \cdots \int \{\bar{\delta} f + \mathrm{Div}(f.\ \Delta x)\} dx.$$

The right-hand side is the classical formula for the simultaneous variation of the dependent and independent variables. Since relation (10) is satisfied by integration on an *arbitrary* domain, the integrand must vanish identically; Lie's differential equations for the invariance of I thus become the relation

$$(11) \qquad \bar{\delta} f + \mathrm{Div}(f.\ \Delta x) = 0.$$

If, using (3), one expresses $\bar{\delta} f$ here in terms of the Lagrangian expressions, one obtains

$$(12) \qquad \sum \psi_i \bar{\delta} u_i = \mathrm{Div}\, B \qquad (B = A - f.\ \Delta x),$$

one obtains a relation, $\bar{\delta} f + \mathrm{Div}(A - B) = 0$. Let us then set $\Delta x = \frac{1}{f} \cdot (A - B)$; one obtains (11) immediately; finally, by integration, we obtain (7), $\Delta I = 0$, which is to say, the invariance of I under the infinitesimal transformations determined by Δx and Δu, where the Δu may be calculated from Δx and $\bar{\delta} u$ by means of (9), and Δx and Δu are *linear* in the parameters. But it is well known that $\Delta I = 0$ implies the invariance of I under the finite transformations which may be obtained by integrating the system of simultaneous equations[G]

$$(17) \qquad \frac{dx_i}{dt} = \Delta x_i; \quad \frac{du_i}{dt} = \Delta u_i; \quad \begin{pmatrix} x_i = y_i & \\ & \text{for } t = 0 \\ u_i = v_i & \end{pmatrix}.$$

These finite transformations contain ρ parameters $a_1, \ldots, a_\rho$, i.e., the combinations $t\varepsilon_1, \ldots, t\varepsilon_\rho$. By the assumption of the existence of ρ and only ρ linearly independent divergence relations (13), it follows that the finite transformations always form a group if they do not contain the derivatives $\frac{\partial u}{\partial x}$. In the contrary case, in fact, there would be at least one infinitesimal transformation, obtained as a Lie bracket, which would not be linearly dependent on the remaining ρ; and since I remains invariant under this transformation as well, there would be more than ρ linearly independent divergence relations; otherwise this infinitesimal transformation would have the particular form $\bar{\delta} u = 0$, $\mathrm{Div}(f. \Delta x) = 0$, but, in this case, Δx or Δu would depend on derivatives, which is contrary to the assumption. The question whether this case can occur when derivatives occur in Δx or Δu is still open; it is then necessary to add all the functions Δx such that $\mathrm{Div}(f. \Delta x) = 0$ to the preceding Δx to obtain the group property, but, by convention, the supplementary parameters must not be taken into account. *Therefore the converse is proven.*

From this converse it further follows that Δx and Δu may actually be assumed to be *linear* in the parameters. In fact, if Δx and Δu were expressions of a higher degree in ε, one would simply have, because of the linear independence of the powers of ε, a greater number of corresponding relations of the type (13), from which one would deduce, by the converse, the invariance of I with respect to a group whose infinitesimal transformations depend *linearly* on the parameters. If this group must have exactly ρ parameters, then there must exist linear identities among the divergence relations originally obtained for the terms of higher degree in ε.

It still must be observed that in the case where Δx and Δu contain also derivatives of u, the finite transformations may depend on an infinity of derivatives of u; in fact, when one determines the[H] $\frac{d^2 x_i}{dt^2}, \frac{d^2 u_i}{dt^2}$, the integration of (17) leads in this case to $\Delta\left(\frac{\partial u}{\partial x_\kappa}\right) = \frac{\partial \Delta u}{\partial x_\kappa} - \sum_\lambda \frac{\partial u}{\partial x_\lambda} \frac{\partial \Delta x_\lambda}{\partial x_\kappa}$, so that the number of derivatives of u increases

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[G] Below, the original text reads $\frac{dx}{dt} = \Delta x_i$, then $x_i = y$ (Translator's note).

[H] The original text reads $\frac{{}^2 x_i}{dt^2}$ (Translator's note).

in general with every step. Here is an example:

$$f = \frac{1}{2}u'^2\,; \quad \psi = -u''\,; \quad \psi.\,x = \frac{d}{dx}(u - u'x)\,; \quad \bar{\delta}u = x.\,\varepsilon\,;$$

$$\Delta x = \frac{-2u}{u'^2}\varepsilon\,; \quad \Delta u = \left(x - \frac{2u}{u'}\right)\cdot\,\varepsilon.$$

Finally, since the Lagrangian expressions of a divergence vanish identically, the converse shows the following: if I is invariant under a $\mathfrak{G}_\rho$, then every integral which differs from I only by an integral on the boundary, which is to say, the integral of a divergence, is itself invariant under a $\mathfrak{G}_\rho$ with the same $\bar{\delta}u$, whose infinitesimal transformations will in general contain derivatives of u. Thus, in the above example, $f^* = \frac{1}{2}\left\{u'^2 - \frac{d}{dx}\left(\frac{u^2}{x}\right)\right\}$ is invariant under the infinitesimal transformation $\Delta u = x\varepsilon$, $\Delta x = 0$, while in the corresponding infinitesimal transformations for f, there occur derivatives of u.

If one passes to the variational *problem*, which is to say if one lets $\psi_i = 0$,[15] then (13) yields the equations Div $B^{(1)} = 0, \ldots,$ Div $B^{(\rho)} = 0$, which are often called "conservation laws." In the one-dimensional case, it follows that $B^{(1)} = \text{const.}, \ldots, B^{(\rho)} = \text{const.}$, and from this fact, the B contain the derivatives of order at most $(2\kappa - 1)$ of the u (by (6)) whenever Δu and Δx do not contain derivatives of an order higher than κ, the order of those derivatives that occur in f. Since, in general, the derivatives of order 2κ occur in ψ,[16] the *existence of ρ first integrals* follows. That there may be nonlinear identities among them is proven once again by the aforementioned f. To linearly independent $\Delta u = \varepsilon_1$, $\Delta x = \varepsilon_2$ there correspond linearly independent relations $u'' = \frac{d}{dx}u'$; $u''.u' = \frac{1}{2}\frac{d}{dx}(u')^2$, while there exists a nonlinear identity among the first integrals $u' = \text{const.}$; $u'^2 = \text{const.}$ Furthermore, we are dealing here only with the elementary case in which Δu, Δx do not contain derivatives of the u.[17]

4 Converse in the Case of an Infinite Group

Let us first show that the assumption of the linearity of Δx and Δu does not constitute a restriction because, even without recourse to the converse, it is an immediate result of the fact that $\mathfrak{G}_{\infty\rho}$ depends formally on ρ and *only* ρ arbitrary functions.

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[15] $\psi_i = 0$ or, in a slightly more general fashion, $\psi_i = T_i$, where T_i are functions recently introduced, are called in physics "field equations." In the case where $\psi_i = T_i$, the identities (13) become the equations Div $B^{(\lambda)} = \sum T_i\,\delta u_i^{(\lambda)}$, which are also called "conservation laws" in physics.

[16] Once f is nonlinear in the derivatives of order κ.

[17] Otherwise, one still obtains that $(u')^\lambda = \text{const.}$ [The original text reads u'^λ (Translator's note).] for every λ from $u''.\ (u')^{\lambda-1} = \frac{1}{\lambda}\frac{d}{dx}(u')^\lambda$.

One shows in fact that, in the nonlinear case, in the course of the composition of transformations whereby the terms of lower order are added, the number of arbitrary functions would increase. In fact, let

$$y = A\left(x, u, \frac{\partial u}{\partial x}, \cdots; p\right) = x + \sum a(x,u,\cdots)p^{\nu} + b(x,u,\cdots)p^{\nu-1}\frac{\partial p}{\partial x}$$

$$+ cp^{\nu-2}\left(\frac{\partial p}{\partial x}\right)^2 + \cdots + d\left(\frac{\partial p}{\partial x}\right)^{\nu} + \cdots \quad (p^{\nu} = (p^{(1)})^{\nu_1}\cdots(p^{(\rho)})^{\nu_\rho});$$

and corresponding to that, $v = B\left(x, u, \frac{\partial u}{\partial x}, \cdots; p\right)$; by composition with $z = A\left(y, v, \frac{\partial v}{\partial y}, \cdots; q\right)$ one obtains, for the terms of lower order,

$$z = x + \sum a(p^{\nu} + q^{\nu}) + b\left\{p^{\nu-1}\frac{\partial p}{\partial x} + q^{\nu-1}\frac{\partial q}{\partial x}\right\}$$

$$+ c\left\{p^{\nu-2}\left(\frac{\partial p}{\partial x}\right)^2 + q^{\nu-2}\left(\frac{\partial q}{\partial x}\right)^2\right\} + \cdots.$$

If any of the coefficients different from a and b is nonvanishing, one obtains in fact a term $p^{\nu-\sigma}\left(\frac{\partial p}{\partial x}\right)^{\sigma} + q^{\nu-\sigma}\left(\frac{\partial q}{\partial x}\right)^{\sigma}$ for $\sigma > 1$, which cannot be written as the differential of a *unique* function or of a power of such a function; the number of arbitrary functions would thus have increased, contrary to the hypothesis. If all the coefficients different from a and b vanish, then, according to the value of the exponents $\nu_1, \ldots, \nu_\rho$, either the second term is the differential of the first (which, for example, always occurs for a $\mathfrak{G}_{\infty 1}$) so that in fact there is linearity, or the number of arbitrary functions increases here as well. The infinitesimal transformations thus satisfy a system of linear partial differential equations because of the *linearity* of the $p(x)$; and since the group properties are satisfied, they form an "infinite group of infinitesimal transformations" according to Lie's definition (Grundlagen, §10).

The *converse* is proven by considerations similar to those of the case of finite groups. The existence of the identities (16) leads, after multiplication by $p^{(\lambda)}(x)$ and summation, and by identity (14), to $\sum \psi_i \bar{\delta} u_i = \text{Div}\, \Gamma$; and from there follow, as in §3, the determination of Δx and Δu and the invariance of I under infinitesimal transformations which effectively depend linearly on ρ arbitrary functions and their derivatives up to order σ. That these infinitesimal transformations, when they do not contain any derivatives $\frac{\partial u}{\partial x}, \cdots$, certainly form a group follows, as it did in §3, from the fact that otherwise, by composition, *more* than ρ arbitrary functions would occur, whereas, by assumption, there are only ρ identities (16); they form in fact an "infinite group of infinitesimal transformations." Now such a group consists (Grundlagen, Theorem VII, p. 391) of the most general infinitesimal transformations of some "infinite group $\mathfrak{G}$ of finite transformations," in the sense of Lie. Each finite transformation is generated by infinitesial transformations (Grundlagen, §7)[18] and can then be obtained by integration of the simultaneous system[I]

$$\frac{dx_i}{dt} = \Delta x_i\ ;\ \frac{du_i}{dt} = \Delta u_i \quad \begin{pmatrix} x_i = y_i & \\ & \text{for } t = 0 \\ u_i = v_i & \end{pmatrix},$$

in which, however, it may occur that it is necessary to assume that the arbitrary $p(x)$ also depend on t. *Thus* $\mathfrak{G}$ *actually depends on* ρ *arbitrary functions*; it suffices in particular to assume that $p(x)$ is independent of t for that dependence to be analytic in the arbitrary functions $q(x) = t.p(x)$.[19] If the derivatives $\frac{\partial u}{\partial x}, \cdots$ are present, it may be necessary to add the infinitesimal transformation $\bar{\delta} u = 0$, $\mathrm{Div}(f.\,\Delta x) = 0$ in order to be able to formulate the same conclusions.

Let us add, following an example of Lie (Grundlagen, §7), a fairly general case where one can obtain an explicit formula which shows as well that the derivatives up to order σ of the arbitrary functions occur, and where the converse is thus complete. These are groups of infinitesimal transformations to which there corresponds the group of *all* the transformations of the x and those of the u "induced" by them, i.e., the transformations of the u for which Δu and therefore u only depend on those arbitrary functions that occur in Δx; there, once more, let us assume that the derivatives $\frac{\partial u}{\partial x}, \cdots$ do not occur in Δu. Then we have

$$\Delta x_i = p^{(i)}(x); \quad \Delta u_i = \sum_{\lambda=1}^{n} \left\{ a^{(\lambda)}(x,u) p^{(\lambda)} + b^{(\lambda)} \frac{\partial p^{(\lambda)}}{\partial x} + \cdots + c^{(\lambda)} \frac{\partial^\sigma p^{(\lambda)}}{\partial x^\sigma} \right\}.$$

Since the infinitesimal transformation $\Delta x = p(x)$ generates every transformation $x = y + g(y)$ with arbitrary $g(y)$, one can, in particular, determine $p(x)$ that depends on t in such a way that the one-parameter group will be generated by

$$x_i = y_i + t.g_i(y), \tag{18}$$

which becomes the identity for $t = 0$, and the required form $x = y + g(y)$ for $t = 1$. In fact, from the differentiation of (18), it follows that:

$$\frac{dx_i}{dt} = g_i(y) = p^{(i)}(x,t), \tag{19}$$

[18] From that it follows in particular that the group $\mathfrak{G}$ generated by the infinitesimal transformations Δx, Δu of a $\mathfrak{G}_{\infty\rho}$ recovers $\mathfrak{G}_{\infty\rho}$. In fact, this $\mathfrak{G}_{\infty\rho}$ does not contain any infinitesimal transformations other than $\Delta x, \Delta u$ depending on arbitrary functions, nor can it contain any which are independent of these functions and which would depend on parameters, because it would be a case of a mixed group. Now, according to the above, the finite transformations are determined from the infinitesimal transformations.

[I] Below, the original text reads $u_i = v$ (Translator's note).

[19] The question whether this last case always occurs was raised by Lie in another formulation (Grundlagen, §7 and §13, conclusion).

5 Invariance of the Various Elements of the Relations

Upon restriction to the simplest case for the group $\mathfrak{G}$, the case that is usually treated, in which one does not admit any derivatives of the u in the transformations, and where the transformed independent variables depend only on x and not on u, one may conclude that the various terms in the formulas are invariant. First one deduces from known laws the invariance of $\int \cdots \int (\sum \psi_i \delta u_i)\, dx$, whence the relative invariance of $\sum \psi_i \delta u_i$,[21] where δ denotes an arbitrary variation. In fact, on the one hand,

$$\delta I = \int \cdots \int \delta f\left(x, u, \frac{\partial u}{\partial x}, \cdots\right)\, dx = \int \cdots \int \delta f\left(y, v, \frac{\partial v}{\partial y}, \cdots\right)\, dy,$$

and on the other, for a $\delta u, \delta \frac{\partial u}{\partial x}, \cdots$ which vanishes on the boundary and which, because of the homogeneous linear transformation of $\delta u, \delta \frac{\partial u}{\partial x}, \cdots$, corresponds to a $\delta v, \delta \frac{\partial v}{\partial y}, \cdots$ that also vanishes on the boundary:

$$\int \cdots \int \delta f\left(x, u, \frac{\partial u}{\partial x}, \cdots\right)\, dx = \int \cdots \int \left(\sum \psi_i(u, \ldots)\delta u_i\right)\, dx;$$

$$\int \cdots \int \delta f\left(y, v, \frac{\partial v}{\partial y}, \ldots\right)\, dy = \int \cdots \int \left(\sum \psi_i(v, \ldots)\delta v_i\right)\, dy,$$

$$\sum \mathfrak{K}_{\mu\nu} g_\tau^{\mu\nu} + 2 \sum \frac{\partial g^{\mu\sigma} \mathfrak{K}_{\mu\tau}}{\partial w^\sigma} = 0,$$

which are equation (30) in Klein. [Above, the original text reads $\partial g^{\mu\nu}$ (Translator's note).] Now let $I^* = \int \cdots \int \mathfrak{K}^* dS$, where $\mathfrak{K}^* = \mathfrak{K} + \mathrm{Div}$ and thus $\mathfrak{K}^*_{\mu\nu} = \mathfrak{K}_{\mu\nu}$, where $\mathfrak{K}^*_{\mu\nu}$, $\mathfrak{K}_{\mu\nu}$ are the respective Lagrangian expressions. The identities derived above are also satisfied by $\mathfrak{K}^*_{\mu\nu}$; and after multiplication by p^τ and summation, one obtains, when one recognizes the differential of a product,

$$\sum \mathfrak{K}_{\mu\nu} p^{\mu\nu} + 2\, \mathrm{Div}\left(\sum g^{\mu\sigma} \mathfrak{K}_{\mu\tau} p^\tau\right) = 0\,;$$

$$\delta \mathfrak{K}^* + \mathrm{Div}\left(\sum \left(2 g^{\mu\sigma} \mathfrak{K}_{\mu\tau} p^\tau - \frac{\partial \mathfrak{K}^*}{\partial g_\sigma^{\mu\nu}} p^{\mu\nu}\right)\right) = 0.$$

[The original text omits the parentheses within the summation symbol (Translator's note).] Comparing the above with Lie's differential equation, $\delta \mathfrak{K}^* + \mathrm{Div}(\mathfrak{K}^* \Delta w) = 0$, one obtains

$$\Delta w^\sigma = \frac{1}{\mathfrak{K}^*} \cdot \left(\sum \left(2 g^{\mu\sigma} \mathfrak{K}_{\mu\tau} p^\tau - \frac{\partial \mathfrak{K}^*}{\partial g_\sigma^{\mu\nu}} p^{\mu\nu}\right)\right); \quad \Delta g^{\mu\nu} = p^{\mu\nu} + \sum g_\sigma^{\mu\nu} \Delta w^\sigma$$

[The original text omits the last parenthesis but one (Translator's note).] as infinitesimal transformations that leave I^* invariant. These infinitesimal transformations thus depend on the first and second derivatives of the $g^{\mu\nu}$, and contain the arbitrary functions p and their first derivatives.

[21] That means that $\sum \psi_i \delta u_i$ is invariant up to a factor, which is what one calls relative invariance in the algebraic theory of invariants.

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