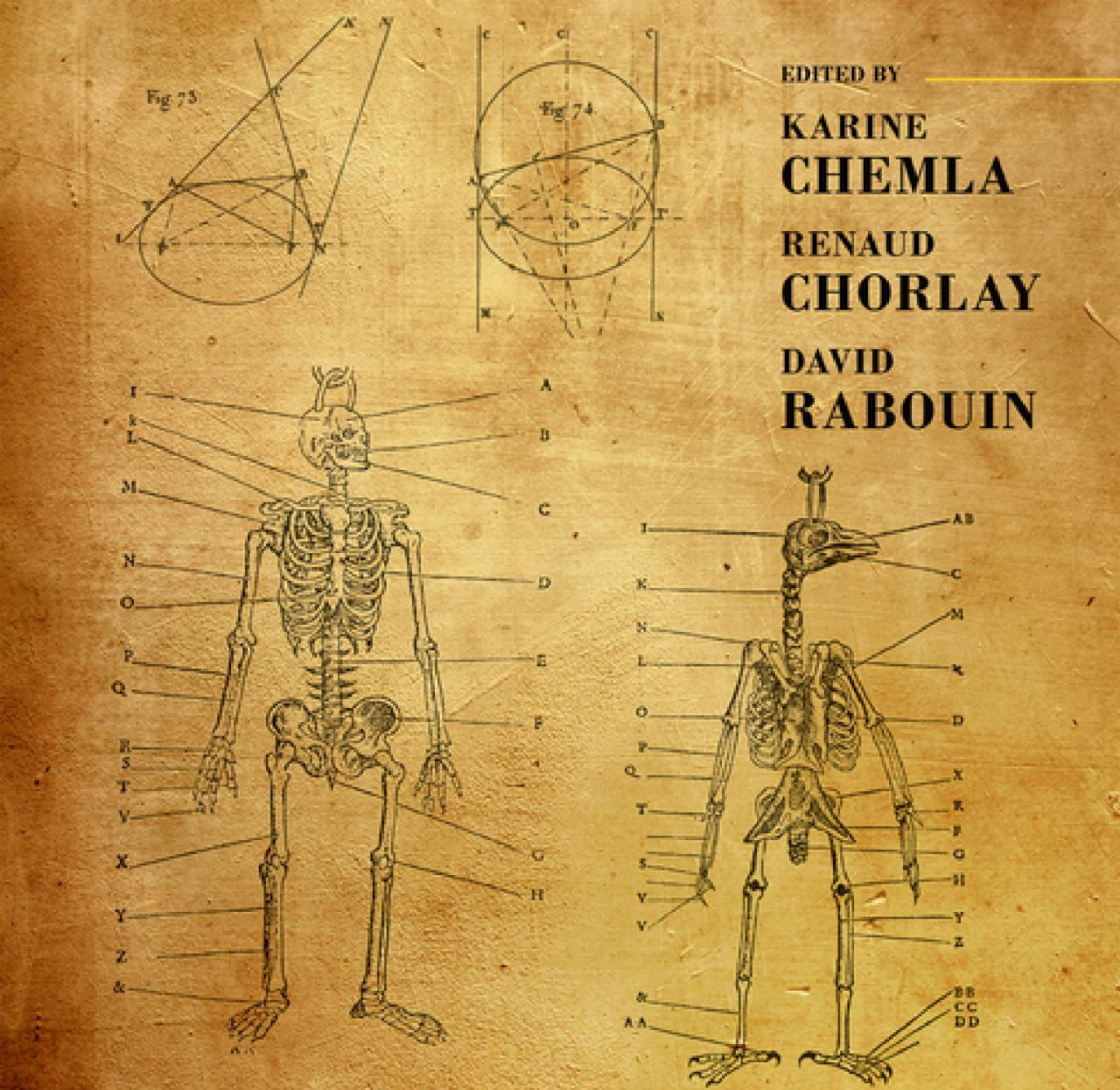

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The Oxford Handbook *of* GENERALITY IN MATHEMATICS AND THE SCIENCES

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The Oxford Handbook of Generality in Mathematics and the Sciences

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THE OXFORD HANDBOOK OF GENERALITY IN MATHEMATICS AND THE SCIENCES

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1

Prologue: generality as a component of an epistemological culture

KARINE CHEMLA, RENAUD CHORLAY, AND DAVID RABOUIN

Generic, general, universal.

Uniform, unified.

"For almost all," "except for a set of measure zero," particular, special, exceptional, pathologic.

Principle, law, general method, ad hoc solution.

Model, example, case, paradigm, prototype.

All these adjectives, terms, and expressions have been used, and sometimes shaped, by actors in the context of scientific activity. However, they do not occur uniformly, independently of the setting. This statement holds true diachronically. It also holds true synchronically: at the same time period, different mathematical milieus, for instance, show collective use of different terms related to the general.[1]

This simple remark takes us to the core issue of this book. It aims to show *how*, in given contexts, actors have valued generality and *how* they worked with specific types of "general" entities, procedures, and arguments. Actors, we claim, have *shaped* these various types of generality. Depending on factors in the context in which they are or were operating to be elucidated, actors have introduced specific terminologies to distinguish

[1] This book is the outcome of a collective work that took place between 2004 and 2009 in the context of the research group of CNRS and University Paris Diderot at the time called REHSEIS. In the meantime, REHSEIS has merged with another research group to constitute a larger entity, newly named SPHERE. The collective work developed in a seminar that was organized by Karine Chemla, Renaud Chorlay, David Rabouin, and Anne Robadey. It allowed us to explore multiple facets of generality. We are happy to thank all the participants and contributors for the insights they gave us, as well as Rebekah Arana, who helped us with the polishing of some of the articles. Karine Chemla was able to benefit from the generous hospitality of Professor Lorraine Daston and the Max Planck Institut für Wissenschaftsgeschichte as well as the unflinching support of the librarians in Berlin during summer 2014 to work on the completion of the book, and in particular its prologue. Our thanks extend to Richard Kennedy for his contribution to the preparation of the final version of this prologue.

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between different levels or forms of generality, and have designed means to work with them, or to work in relation to them. Actors have in some cases discussed which virtues they attached to the general, why it was essential to their project, and in relation to which other values they prized it.[2]

This book aims to inquire into this diversity.[3] It intends to highlight how "the general" does not assume any *a priori* meaning, which would be valid across contexts, at a given time, or even in a single discipline. Nor, in our view, would a history of scientific progress as reflected by the achievement of an ever higher level of generality be faithful to our sources. On the contrary, the goal of this book is to reveal how actors worked out what the meaningful types of generality were for them, *in relation to* their project, and the issues they chose to deal with. If such a view holds true, as we claim, it implies that evidence exists of different ways of *understanding* the general in different contexts. Accordingly, it suggests a nonlinear pattern for a history of generality. The book does not claim to offer such a history. However, it intends to open a space for such a historical approach to generality to become possible.

The claim we make about the various ways of working out and practicing generality is a facet of a more general thesis, which draws on the assumption that the scientific cultures in the context of which actors have operated do not fall ready-made from the skies. The thesis holds that actors have shaped these cultures, and reshaped them constantly, *in relation to* the scientific work they have carried out.[4] These cultures bring into play types

[2] Philosophers, especially those in a pragmatist tradition like Hilary Putnam, have insisted on the importance of taking values into account when dealing with scientific practice. See, for instance, Putnam (2002: 30*ff*). In philosophy of science, in the last decades, "epistemic values" have been at the center of a certain attention. See, for instance, Kuhn (1977), Laudan (1984), and Carrier (2012). In conclusion, we will situate our project with respect to this developing field of study. Let us simply note for now that simplicity, beauty and other values have often been mentioned as examples in this context. However, it seems to us that generality has not often been discussed as a value, except, as we suggest in conclusion, indirectly. This is one of the main claims of the book: that it should be addressed as an epistemic value.

[3] Hagner and Laubichler (2006) is the outcome of a similar project. Outlining some differences between the two projects enables us to be more precise on the specificity of our own project, and also on how the two endeavors complement each other. The book edited by Hagner and Laubichler addresses the issue of generality essentially as a concern that gained momentum, for instance in the second half of the nineteenth century in Germany, in view of the increasing specialization of scientific activity, and the related emphasis placed on detail and accuracy. Here, the main antonyms of the "general" are the "special" or the "partial." Was it possible, some actors worried, to maintain a global approach within a discipline? At an even higher level, they asked: Could one maintain a global and reflexive outlook on science? Was there a discipline able to represent the general level for all other disciplines? How, more generally, could one act in favor of the unity of science despite ever finer disciplinary differentiations? Typical in this respect are the congresses of the Gesellschaft Deutscher Naturforscher und Ärtzte around 1900, described by Ziche (2006), whose actual organization institutionalized the concern by opposing "general" lectures—in a sense of general that Ziche discussed—and specialized sections. Correlatively, the approach to generality most represented in Hagner and Laubichler (2006), which focuses mainly on the nineteenth century and the first half of the twentieth century, is at the level of disciplines or beyond. By contrast, in our book, we place ourselves at the micro-level of scientific activity, in settings much smaller than those of disciplines. As a rule we also examine generality closer to scientific *practice*, aiming to uncover variety behind what has often been assumed to take a rather obvious and uniform meaning. Accordingly, we pay close attention to differences between the constellations of terms linked to generality that vary across settings and reflect the categories of the specific actors. In addition, Hagner and Laubichler (2006) consider disciplines ranging from physics to history and philosophy, whereas we focus on mathematics, physics, and the life sciences.

[4] This is one of the main theses put forward and discussed in Chemla and Fox Keller (2016).

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of texts and inscriptions, instruments, and other material entities. They also bear on ways of engaging with these entities. Further, they include shaping of the social organization of scientific activity as well as epistemological factors.[5] This book focuses in the first place on one such epistemological factor: generality. We do not want only to understand *how* it was conceived and implemented in different contexts, but we are also interested in how actors related this facet of their activity to other facets. It is our hope that our project can inspire similar efforts that will allow us to better understand how epistemological factors are key components of scientific cultures.

The book could have included case studies ranging from antiquity to the modern times, and occurring in any part of the planet.[6] To understand diversity in a more restricted environment, and thus be able to perceive relationships between contexts with respect to generality, we have instead chosen to concentrate on mainly, even if not exclusively, early modern and modern Europe. However, we have found it useful, for the sake of our reflection, to explore our set of issues in the context of various disciplines. The reader will thus find chapters in this volume devoted to mathematics, physics, and the life sciences.

1.1 Actors' historiography of generality and their meditations upon its value

Some actors have shown interest in the history of generality, and some have developed a reflection on it. They sometimes manifested their awareness of differences in how generality had been handled and practiced in the past. As Frédéric Brechenmacher highlights, in a chapter devoted to a controversy on generality to which we return below, for some actors like Leopold Kronecker (1823–1891), in line with a certain way of thinking about history in the nineteenth century, progress in mathematics meant, in particular, progress in the achievement of ever higher levels of generality. This points to one way of perceiving differences in the past. Actors also regularly manifested their awareness that other practitioners at their time did not approach generality in the same way as they did—they sometimes added "in the right way." This is one facet of Kronecker's dispute with Camille Jordan (1838–1922). Kronecker interprets his disagreement with Jordan in this respect as the result of a history of progress in the forms of reasoning. In his outline

[5] Different concepts of "scientific cultures" that could be useful for historians, philosophers, or anthropologists of science were put forward in the last decades. Knorr-Cetina (1999) introduces the concept of "epistemic cultures" based on ethnographic case studies carried out in laboratories working on high-energy physics and molecular biology in the twentieth century. Fox Keller (2002) develops her ideas about "epistemological cultures" through case studies related to the history of biological development. In that publication, she highlights in particular the epistemological factors that differ across scientific cultures and account for the problems that sometimes occur in interdisciplinary exchanges. In particular, she analyzes how in work on the same biological problems, physicists and biologists differ on what counts as an "explanation." See also Chapter 17 by Fox Keller in this book. Chemla (2009) outlines a concept of "mathematical culture," inspired by these previous publications. The introduction to Chemla et al. (2016) discusses the relationship between these concepts and others.

[6] Indeed, one of the editors has intensively published on the valuing of generality attested in mathematical writings from ancient China (see, for instance, Chemla, 2003, 2005).

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of the history of general reasoning, Kronecker attributes to mathematicians of the past (and to Jordan) a practice of reasoning that was flawed—Hawkins (1977: 122) referred to this practice as "generic reasoning." In Kronecker's eyes, the Berlin mathematician Karl Weierstrass (1815–1897) had shaped a new practice, from which essential epistemic benefits derived. Kronecker's views on generality and its history, Brechenmacher argues, have left their imprint on the historiography of mathematics, which embraced them. An interpretation of the history of generality as a linear unfolding thus has roots in the past that can be uncovered. As a result of observers adopting actors' historical accounts in this respect, views like Kronecker's have overshadowed the different conceptions of other actors at the time. Historiography, Brechenmacher adds, has perhaps failed to recognize diversity in this respect.

The same conclusion derives from Olivier Darrigol's description of James Clerk Maxwell's (1831–1879) practice of generality. Historiography, Darrigol emphasizes, has adopted Pierre Duhem's (1861–1916) assessment of this practice, which was formulated from the viewpoint of another conception of how generality should be achieved. As a result, Darrigol suggests, we lack an account of how Maxwell fulfilled the ideal of generality he set himself.

These remarks highlight two benefits that derive from our approach. Gaining a better understanding of the diversity of practices of generality promises to yield tools for revisiting, in a critical way, parts of the historiography of science. Historiography has in some way made the past appear uniform, in relation to some actors' representations of generality and its history. The book will help restore some diversity and yield source material for thinking about generality in a broader perspective. Moreover, the episode on which Brechenmacher concentrates shows that the different types of general reasoning practitioners brought into play are correlated with different types of general mathematical features on which they concentrated: invariants, for Weierstrass and Kronecker, and the reduction into simpler elements, for Jordan. In addition to a history of the modes of general reasoning, we thus see the emergence of a history of the various ways of shaping generality in the subject matter itself.

1.1.1 What are the issues at stake in generality? Epistemic and epistemological values

The case studies evoked above already suggest our sources testify to motley practices of generality. Accordingly, actors have offered dissonant historical accounts of generality. These different historiographies naturally constitute an important resource for us and this is in fact where our book begins its inquiry.

Its second chapter is devoted to Michel Chasles's (1793–1880) *Aperçu historique sur l'origine et le développement des méthodes en géométrie*,[7] completed in 1837. The book

[7] The full title is: *Aperçu historique sur l'origine et le développement des méthodes en géométrie, particulièrement de celles qui se rapportent à la géométrie moderne, suivi d'un mémoire de géométrie sur deux principes généraux de la science : la dualité et l'homographie*.

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appeared in a specific context. Chasles was researching on geometry in the framework of a mathematical culture that took shape in France at the end of the eighteenth century and the first half of the nineteenth century in engineering schools, and more specifically in relation to the Ecole Polytechnique. These geometers, who collectively shaped projective geometry, were all obsessed by the question of the relationship between analytic and "purely geometrical" approaches to geometry. In particular, they acknowledged the immense power of analytical approaches in solving geometrical problems, but could not understand why pure geometry was comparatively so weak when dealing with the same questions. For them, generality appeared to be one of the main assets of analytical approaches. Our geometers thus collectively began a reflection on the *sources* of generality in analysis, and the *means* to be developed to equip pure geometry with forms of generality able to compete with those of the rival approach. Their reflection did not develop only from within their mathematical practice, but they also approached the question from a historical perspective. Following predecessors and colleagues like Carnot and Poncelet, Chasles conducted a mathematical and historical reflection on the history of geometry from the viewpoint of generality.

In Chapter 2, Chemla relies on Chasles's reflection to capture an actor's perspective on the various facets of generality in geometry. Indeed, his book provides an amazing source of material for the examination of how a geometer perceives generality in his field, how he understands the shaping of means to achieve generality throughout history, and how he contributes to shaping further means of introducing new forms of generality. In his reflections, generality appears to be multi-faceted. What is essential for us is that it takes different forms from those we see in the context of the Kronecker–Jordan controversy.

In line with his project, Chasles surveys historical transformations of generality in geometry in two historical traditions (the analytic and the geometric approaches), his intention being to deploy another type of generality. For Chasles, ancient Greek geometry seriously lacked generality in many senses of the term. Chasles identifies a turn in this respect in about the sixteenth century.

His historical account, Chemla shows, first identifies a history of geometrical *objects,* which shapes objects with increasing generality. Chasles attributes to Girard Desargues (1591–1661) a key role in this regard. Further, in Chasles's view, the change in geometrical objects Desargues brought about was correlated with the development of means to *transfer* properties between objects formerly perceived as unrelated, or to put these properties in relation with each other. For Chasles, these operations represent another facet of generality. More generally, he underlines the introduction, in the seventeenth century, of *methods*, which established *connections* between objects and also between properties.

Chasles likewise emphasizes the emergence of *uniformity* in the treatment of problems dealing with different objects, as one important aspect of the shaping of generality in geometry, associating it in particular with René Descartes (1596–1650). From the perspective of generality, Chasles thus also sketches a history of the changing means of proving in geometry. He offers in particular a history of reflections on the actual scope of the conclusions that could be derived from a proof.

Like Kronecker's, Chasles's history is a history of progress, but it is not linear. He appreciates how in the context of different approaches, different groups of actors

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operating at the same time shaped different means of achieving different types of generality. Generality was explored in different directions, and each of these lines of inquiry, he shows, contributed to the history of the shaping of generality in geometry.

Chemla further highlights how specific features of Chasles's treatment of the history of geometry can be correlated in various ways with his own contributions to geometry. Let us outline only one of these facets, which will prove important for the argument of the book. Chasles's meditations on general objects in geometry, the uniformity of their treatment and the possible scope of the conclusion of a proof, Chemla suggests, inspired his reformulation of a principle, made explicit by Poncelet in 1822, under the name of "the principle of continuity."

Poncelet's principle had the aim of enabling practitioners to claim a conclusion for a general geometrical object, after having completed a proof of the proposition on a narrower set of objects. First, the principle derived from a reflection on what general objects were in geometry. It was based on a new conception, for which objects were no longer figures, but configurations of geometrical elements (lines, curves, planes, etc.) that could present different ranges of general states. Secondly, Poncelet considered that a conclusion, obtained through reasoning based on one general configuration, could be asserted about any other general configuration, as long as the latter configuration derived from the original through continuous deformation (hence the name given to the principle). In particular, when some relations between elements of the configuration had become "ideal," the conclusion, Poncelet suggested, still applied. The principle clearly contributed to the shaping of a form of generality in geometry. It stated how conclusions about general objects could be derived from a proof, despite the fact that it was not valid for all their states. For Poncelet, the principle simply accommodated in geometry an assumption that was implicitly used in analytical reasoning about geometrical figures and from which this type of reasoning derived its generality. We return to this formulation of the principle below.

The principle embodies but one facet of generality in this context. On a higher level, the reflection on generality that developed within projective geometry is characterized by the emergence of similar "principles," and also by the amount of work carried out in discussing and reformulating them. Chasles ponders this fact more widely. In this specific case, his reflections will precisely lead him to offer a new analysis for the "principle of continuity," and a new formulation, which he will then call "the principle of contingent relationships."

His analysis suggests another conception of a general object in geometry. His key concept in this respect is that of "figures," for which one can distinguish between different "general circumstances of construction." For such objects, Chasles distinguishes between permanent and contingent features, and his principle suggests another way of determining, or interpreting, the generality of the conclusion deriving from a given proof. In his view, the conclusion of a proof that has been carried out on one set of "general circumstances of construction" of a figure can be stated for the figure in any other "general circumstances of construction." Chasles further suggests, on this basis, a new practice of proof: he invites geometers to develop proofs using only permanent properties, which will apply uniformly to the figure in any set of "general circumstances of construction."

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Accounting for this reflection on, and reformulation of, principles belongs to a history of projective geometry that does not only deal with results and theories, but also includes how actors produced, and discussed, the means of achieving types of generality in their practice. In this case, it is particularly important to highlight the history of the principle, since, as we show below, the related reflection on generality, and the concepts the reflection produced, actually inspired similar developments in other mathematical domains.[8] Focusing on generality and actors' reflections on it thus opens a new page in the history of science, by showing that not only concepts, methods, and results circulate, but also actors' reflections on generality.

Chasles's historical considerations are an opportunity for him to discuss the virtues he attaches to the most general statements. In his view, they are also the most simple to formulate, the easiest to prove, the most widely true, as well as being the most fruitful. By the latter term, he means these statements yield, through almost no proof at all, the other true statements in the theory to which they belong. It comes as no surprise that the identification of these statements—the "source" of all the others, whose existence he assumes for all theories—is the goal he sets for his practice. Chemla suggests this goal also echoes the aim pursued in the analytical organization of knowledge at the time.[9]

To approach the issues at stake here in more abstract terms, we suggest the introduction of a distinction, whose usefulness will be further illustrated in what follows. For Chasles, like for Kronecker, we claim generality is an "epistemic value," in that its pursuit, like that, say, of "coherence" in scientific practice, would be conducive to truth. Indeed for actors like Kronecker, as we show in greater detail below, the *types* of general statement and uniform approach he advocated guide us to true knowledge.[10] But for Chasles, generality is not merely an "epistemic value." It is *also*, we emphasize, an "epistemological value,"

[8] See Section 1.3.4.

[9] Lagrange (1799 (Thermidor, An VII): 280) gives a general description of this goal when he deals with the analytical treatment of spherical trigonometry.

[10] The reader should be warned that the expression "epistemic value" is used elsewhere with a markedly different meaning, for instance in the title of the book by Haddock, Millar, and Pritchard (2009). In that book, the expression refers to what makes the value of knowledge, as opposed to true belief, for instance. Knowledge is thus considered given a priori to the analyst, and what is at stake is its *evaluation* by contrast to other forms of beliefs. In our book, in general the term "value" refers to values like simplicity or elegance, which actors prize and put into play in the production of knowledge. What is at stake is how this valuing is correlated with the knowledge produced, and how these values can take different meanings. In our case, the expression "epistemic value" refers to a distinction we establish between values, the second pole of the distinction being "epistemological value" (see below). For a discussion about epistemic values in scientific practice, see Putnam (2002: 30–3). Like Putnam in his book, we distinguish between a distinction and a dichotomy. We are *not* claiming that we establish a dichotomy between values: generality is a perfect example of values that are found in both sides of the distinction. Carrier (2012) also discusses the parts played by "epistemic values" in the production of scientific knowledge. He identifies two main roles. For him, "epistemic values" contribute to the selection of the goals chosen and the evaluation of their significance. Moreover, "epistemic values" are values conducive to truth. Carrier does mention generality as one factor that enables us to capture variety in the goals actors choose as significant (p. 240). This book is predicated on the assumption that one can go further in the analysis of generality as a value. The *ways* in which the values on which Carrier focuses in his analysis are conducive to truth differ from how in our view actors perceive generality sometimes performs the same task. This calls for an analysis of the modalities according to which various types of epistemic values fulfill this function. This issue goes beyond the framework of our book. However, it appears as a promising avenue for a future general inquiry into epistemic values.

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in the sense that its pursuit according to some criteria yields a kind of knowledge suited to *how* we want to know. We can obtain different forms of knowledge in geometry, that is, different types of formulations for theorems and different kinds of proof. Chasles has expectations regarding the satisfactory form of knowledge we should aim for. The link that, as we just described, Chasles establishes between generality, on the one hand, simplicity and fruitfulness as he envisages them, on the other, relates precisely to the latter concern.[11]

In the same way that actors form different historical accounts of generality, they also perceive the forms taken by generality and its virtues in different ways. Chapter 3, by Eberhard Knobloch, illustrates the fact by focusing on Leibniz (1646–1716), a particularly rich case for a discussion of generality in this respect. What is interesting for us is that the virtues Leibniz lends to generality will *at the same time* relate to Chasles's account and yet present important differences.

As we have seen, for Chasles a key manifestation of generality in geometry derived from a credo regarding the structure of knowledge in any theory. By contrast, the assumption from which for Leibniz generality derives is a theologico-philosophical principle, which holds there is a "universal harmony" in the world. This harmony is brought to light through the reduction of the variety of things to the highest order possible—this is another form generality can take. Mathematics can be used to highlight order. Order can also be disclosed within mathematics. This goal assigned to mathematics and mathematical endeavor thus grants generality its value. To use the terminology introduced above, it is in this regard an epistemic value. Knobloch's chapter focuses on the latter dimension: the disclosure of order within mathematics. The possibility and the meaning of the general are thus postulated, and identifying the general in Leibniz's view will allow us to perceive beauty in mathematics. It will take the form of "divine theorems," "laws," "methods," and so on.

A first type of manifestation of the general is the production of theorems that "link together the most dispersed things," or are common to all formulas in a given context. This appears as a first way of reducing a given variety and showing order in it. Although the theme of connecting truths was also present in Chasles's account, Leibniz attaches different virtues to this reduction, which brings to light in which regards generality is also an epistemological value for Leibniz. For him, such theorems are "excellent summaries of human understanding," constituting "abridgements," which ease memorization and the work of thought. With reductions of this type, the practitioner is saved the labor of repeating similar treatments in situations that are shown to be related.

For Leibniz, the general also has the property of being simple, however, in his case, prominently in relation to the fact that it is *concise*: all irrelevant details have been

[11] Values of this type are also those that are at play when actors choose between competing theories that equally account for facts. Simplicity, beauty, and other values have been evoked in this respect. We suggest that these values can be epistemic or epistemological, depending on how actors justify their use. They have been discussed mainly in the philosophy of physics and in relation to theory choice. The case in mathematics that we discuss here indicates that these values are used at different levels (or scales) and play different parts. This also awaits further description.

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eliminated. In this manner, Leibniz suggests, fruitfulness is implemented, and this value takes meanings close to those we outlined above. However, in line with Leibnizian reflections on the characteristic, it comes as no surprise that the meditation takes a specific course. For him—a fact that played no role in Chasles's reflections—adequate notations yield conciseness. They thus play a key role in the exhibition of the general. Once notations paint the "intimate nature of things," they disclose the universal in them, and reasoning becomes a computation. The reduction that adequate notation carries out also allows again the expression to "be easily retained" and to "the labor of thinking" to be "diminished." As Leibniz shows, a reduction of this type also provides help in carrying out further reductions of the first type. They are both tools in the service of the art of invention.

Finally, for Leibniz as for Chasles, the general is the subject of an unending quest, the assumption being that there can always be higher levels of generality, yielding even more powerful resources. In this quest, a field appears as fundamental for Leibniz, namely, combinatorics. The key position of this field in relation to generality can be correlated to a specific mode of expression of the general: "laws" of formation. Laws of this kind highlight patterns in general expressions, formulas, or tables, allowing practitioners to produce these entities, without memorizing them, and thus dispense with them. The fact that these laws represent higher forms of generality echoes the fundamental position Leibniz grants to combinatorics in mathematics.

In conclusion, we see clearly that the meaning of the general, the values attached to it, as well as the forms it can take vary significantly from one context to another, in close relation to actors' scientific activity.

How do actors understand the way(s) in which the general can be established and the way(s) in which the extension of its validity may be captured? How do they consider it can be worked with appropriately? These are the issues to which the next chapters turn, while displaying yet other meanings attached to, and other forms taken by, the general in other contexts.

1.1.2 Actors' reflections about generality

Actors do not always formulate their reflections on generality explicitly, nor do they always make explicit which options they take in this respect. Yet, aspects of their reflections and choices on this issue can very often be gathered from clues we find in their writings. In some cases, background knowledge about actors and their immediate context, especially the scholarly culture in the context of which they operated, can complement the sources that come down from them and help us describe how they understood generality and how they worked with it. However, this is not always the case.

As usual, ancient history provides the most critical examples in this respect, where we have isolated documents, with no meta-level statements that might reveal actors' reflections on generality and their practice of it. How can we, in such cases, interpret the clues and describe our actors' take on generality? The book illustrates the problem, and examines this issue, with an example taken precisely from ancient history. This constitutes its main incursion into earlier time periods. In fortunate cases, we can find other documents that, although they may have been written centuries apart from the sources

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under consideration, present connections with them regarding the issue of generality and, further, make reflections on this topic more explicit. Even though this information must clearly be used with discernment, it yields precious evidence to interpret the clues our sources contain on actors' understanding of, and practice with, generality.

Chinese mathematical texts from antiquity offer an example of a situation of this kind. For example, the classic *The Nine Chapters on Mathematical Procedures* (completed ca. first century CE) gives many clues that generality was a major epistemological value for its authors. And yet, at first sight, the book includes no comments on this fact. Nor can we find, strictly speaking, any contemporary document that would fill this gap in our documentation. However, we are fortunate enough to have another classic, probably completed a century earlier, *The Gnomon of the Zhou [Dynasty]*, as well as commentaries from the third and the seventh centuries on *The Nine Chapters*, handed down with that classic. These documents all yield essential information enabling us to interpret features of the approach to generality in the context of *The Nine Chapters*, as well as in its commentaries (Chemla, 2003, 2005). Greek mathematical texts of antiquity confront us with a similar situation.

Clearly, Euclid's *Elements* also reflects careful consideration, as well as practice, of generality.[12] However, no immediately related evidence can allow us to describe Euclid's cogitations other than clues in his text. In this case, Aristotle's detailed discussions on generality, which additionally often evokes mathematics as a key example, provide crucial information for the interpretation of these clues. Chapter 4, by David Rabouin, is devoted to this case study. Rabouin considers features of Euclid's approach to generality more specifically in the example of the theory of ratios.

As is well known, the issue of generality is essential to Aristotle's discussion of science, since for him the principal characteristic feature of scientific knowledge is that it is knowledge of the general. In this context, Aristotle frames the issue in a very specific way. He considers that the general is attached to "genres," and that it formulates essential attributes of entities falling under a "genre," which derive from a "demonstration" holding for all of them. This specific approach to generality nicely correlates with the structure of Euclid's *Elements*. Indeed, the *Elements* can be decomposed into two parts, one dealing with geometrical objects and the other with numbers. Moreover, proportions are defined for these two domains of entities in two different ways, and similar properties of proportions are proved using wholly different characteristics of objects in each case.

At first sight, the correlation between the clues given by the mathematical text about the treatment of the general and the theory of "genres" expounded by the philosophical text thus appears to be obvious. However, Euclid's *Elements* also contains a puzzling passage, in which proportions on magnitudes and proportions on numbers are brought into relation. This point seems to challenge the interpretation of the general in Euclid's *Elements* as conforming to what Aristotle describes. It further raises problems for the interpretation of the *Elements stricto sensu*.

[12] *Pace* Chasles, from whose perspective Greek texts of antiquity were deficient in this regard.

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It is interesting to note how the difficulty in interpretation is in line with a problem in understanding how Euclid deals with the general. In fact, Rabouin highlights that this difficulty is *itself* correlated with vexing issues in the interpretation of Aristotle's problematic statements on the general. Accordingly, the solution Rabouin offers questions received views on the general in Aristotle's theory. Widening the documentary basis on which this question has been approached, Rabouin offers a modified account of how Aristotle understands the general in science and how he prescribes it should be pursued. It is noteworthy that this modified account enables us to make sense of the features of the Euclidean text that were perplexing. Perplexing, Rabouin stresses, only for the modern reader, since we have no evidence that ancient commentators took issue with these features of Euclid's *Elements*. Perhaps, Rabouin suggests, false expectations with respect to generality in the ancient texts have created challenges for modern readers.

This conclusion is worthy of greater consideration by historians dealing with values such as generality. Different understanding and practices of a value in different contexts demand methodological prudence in order to avoid reshaping the past on the basis of our expectations. We have already met with similar concerns above.

This ancient discussion touches the question of the *entity* to which general properties can be legitimately attached. It also brings into focus the issue of which *procedures* can legitimately be used to establish a property of that kind. It sets the frame for a practice of the general. Closer to us, we still find actors, like the mathematician Henri Poincaré (1854–1912), who wrote many texts discussing the meaning of general statements and ruminating on their legitimacy. Igor Ly's analysis of these reflections, in Chapter 5, shows that the issues Poincaré addresses in this respect in his philosophical writings are wholly different from the issues outlined above. For Poincaré, as for Aristotle, there is no science without the general. However, Poincaré is interested in understanding the different meanings of generality in different disciplines, and comparing the practices of generalization in these different contexts. His aim is, in particular, to characterize the meaning of generality in mathematics and to understand the part played by mathematics in practices of generalization in other disciplines.

To begin with, what does it mean, Poincaré asks, to speak of "all the integers", for instance? And, how should we interpret a mathematical statement asserting that a property is shared by the elements of this infinite collection? Here, as elsewhere, Poincaré's discussions about generality involve the infinite. A key point in Poincaré's answer to these questions consists, Ly stresses, in his interpretation of the infinite: for him, it is never "given," but is endlessly in construction, referring in fact to the potential infinity of a sequence of operations. Accordingly, generality is thus not the *result* of, but rather the *operation* of generalization itself. With respect to integers, for instance, Poincaré suggests their collection is conceived through the "power of the mind" to repeat the addition of 1 indefinitely. This power of the mind, of which we have the intuition, is, for Poincaré, what gives meaning to such expressions as "all the integers." It also allows us to shape and grasp the mathematical continuum and other mathematical concepts. The same "power of the mind" is brought into play in mathematical induction, which for Poincaré is *not* a logical, but a purely mathematical type of reasoning. The reason for this is that induction requires an indefinite combination of the same or of similar acts, which in Poincaré's

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view characterizes mathematical generalization, by comparison to generalization in other scientific fields. In fact induction provides mathematics not only with an essential tool for generalization, but also for shaping mathematical concepts. These arguments help explain why in Poincaré's reflections, generality, generalization, and the infinite are always associated with each other.

Poincaré is also interested in generalization in physics and especially in the part played by mathematics in these generalizations. As Ly shows, Poincaré develops careful and specific analyses with respect to both mathematical physics and experimental physics. In the latter case, Poincaré offers an extremely subtle theory of the part played by mathematics in inductions carried out on the basis of experimental measurements. Here again, his conclusion leads Poincaré to oppose this type of induction and a generalization that operates by means of the extension of the domain of a predicate.

In all these cases, the scientist Poincaré discusses philosophical questions as an observer of scientific activity. He takes as his starting point the practices of generalization in mathematical, or experimental, physics. And he develops quite specific interpretations of what generality means and how generalization functions in different contexts, thereby justifying the legitimacy of these operations in mathematics and in physics.

Quite interestingly, the mathematician Poincaré is also essential in another respect. We have abundant evidence that in his work he proceeds in a specific way with respect to generality, constantly keeping an awareness of the generality of the situations he is dealing with. Even though, to the best of our knowledge, he remains silent about this, there is ample evidence of the fact in the statements he uses and the structure of his writings. This takes us to another range of issues that the book addresses.

1.2 Statements and concepts: the formulation of the general

We have seen so far the variety of reflections actors developed with respect to the general. These reflections partly overlap and partly diverge. They clearly constitute a precious asset for our project of a historical study of generality. However, how did actors express and state the general? How did they write it down? Part II of the book examines this question, emphasizing actors' roles in shaping concepts and statements to grasp, and express, the general. On this issue, Poincaré will provide us with a magnificent example, which highlights a key phenomenon for our history: the historicity of the statements actors used to formulate the general.

1.2.1 Developing new kinds of statement

In Chapter 6, Anne Robadey provides evidence documenting the circumstances in which a practitioner introduced a new type of general statement. The statement in question asserts that a proposition holds true for "almost all" the objects considered, where the meaning of "almost all" is *quantified precisely* using mathematical tools. Robadey notices

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statements of this kind are also characterized by the fact that no attempt is made to individually identify the "exceptions," the existence of which is referred to.

The practitioner to whom Robadey ascribes the invention of this type of general statements is none other than Poincaré himself. She is even able to situate the moment of this invention between early December 1889 and 5 January 1890. The historical background for it is clear. In 1888, Poincaré competed for a prize offered by King Oscar II of Sweden. Having been awarded the prize, he submitted the text of the *memoir* for publication. At the end of 1889, he became aware of a mistake in the proof of the main theorem and withdrew his publication. Within a month, he struggled to fix the problem and, on 5 January 1890, he was able to send the new version of the *memoir*, which was published. The introduction of the new type of statement was part of the resources on which Poincaré drew to formulate and establish a new result in place of the erroneous one. Robadey can document subsequent episodes more finely in the shaping of the statement under consideration.

In her chapter she emphasizes several points that are essential for our purpose. First, in his formulation of the new type of statement, Poincaré uses the word "exceptional" to refer to the cases for which the general property does not hold. He recurrently emphasizes that in these statements, this word, which is taken from ordinary language, is a technical term, making clear the mathematical meaning he has ascribed to it. Interestingly enough, Robadey notes, this fact contrasts with Poincaré's way of using another technical expression: "the most general polynomials of their degree." Clearly, the latter expression relates to yet another specific type of statement of the general. For us, its sense requires some explanation. However, Poincaré seems to take the technical meaning of that other expression for granted, never stressing it, nor even defining it. This suggests the conclusion that the latter technical expression was in common use in the mathematical culture for the members of which Poincaré writes. This thus gives us hints that in given contexts, actors use shared sets of specific technical expressions of the general. By contrast, when Poincaré introduces a new type of formulation of the general, he feels compelled to warn his readers of the technical dimension of its terms.

Second, Poincaré relies on knowledge in probability theory to ascribe a meaning to the term "exceptional." In brief, he defines "exceptional" as that which arrives with a probability equal to zero. Moreover, in the second version of the *memoir* where he uses these concepts, he makes clear how he suggests defining such a probability. This, Robadey emphasizes, is in fact the first piece of evidence we have of Poincaré's reflections on probabilities. In fact, when he brings probabilities into play to fix the flaw discovered in the first version of his *memoir*, Poincaré does not use knowledge on the topic that would be readily available. Robadey is able to show how the new version evidences Poincaré's research work on probabilities that remained otherwise unpublished at the time. What is more, Robadey suggests the critical situation of having to correct his mistake prompted Poincaré's personal work in this field, which he would later revisit. In fact, he puts his research on the topic into play to give meaning to the new type of statement he introduces. This fact highlights a key conclusion: shaping a new statement of this kind is not only a matter of formulation but also requires mathematical knowledge, which in this case was developed for this purpose. Actually, it is precisely this facet of the new statement—the

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definition from probability it brings into play—that underwent transformations in Poincaré's successive statements of his main new theorem between 1890 and 1891.

Third, through a remarkably fine analysis of a corpus of texts, Robadey establishes that the introduction of the new type of statement is correlated with other facets of Poincaré's mathematical work to correct his result. Poincaré modifies the meaning of the key concepts at stake in his *memoir*—like that of stability. He restructures the organization of his text, redefining the goal to be achieved. The way in which he chooses what will become the new essential result of stability is correlated with the possibility of stating something strong, which is general enough to be meaningful, even though it does not hold universally.

Finally, the nature of the statement, which points to exceptional cases without identifying precisely what they are, is correlated with the type of proof Poincaré develops, which, as Robadey stresses, is non-constructive. The new statement is altogether produced in the context of this gigantic reshaping and is what makes it possible. Again, the cultural artifacts actors produce in the context of their activity cannot be dissociated from the questions they address and the goals they set themselves. The modes of stating generality illustrate this more general assertion.

To recapitulate, the type of statement we are interested in was introduced as a resource to shape a new approach in line with the result that needed to be established. Although Robadey accurately establishes the circumstances for its introduction, we still lack a precise historical account of how other practitioners picked it up, and began using it. However, the fact is, as we shall see below (see Section 1.2.3), that statements of this kind circulated widely, being reworked and extended in various types of context, in which they served as inspiration to actors structuring collective research programs.

Such case studies are essential for our purpose, since they highlight a phenomenon that, to the best of our knowledge, has been overlooked. Statements of forms of generality have their history, which is worth addressing, and their production is as important a part of scientific work as is the production of new concepts and results. Once they have become part of the tools adopted within a given scientific culture, they yield key resources for the practice. But there is more. Robadey shows the approach to mathematical situations that statements of this kind disclose is not exceptional in Poincaré's mathematics. To the contrary, it fits with a systematic attitude toward his research topics, to which many of his writings testify. One could refer to it as a *style* of dealing with generality. Indeed, Poincaré's writings are full of explicit remarks he systematically adds to his reasoning, assessing degrees of generality of a situation under consideration. Robadey highlights the rich terminology omnipresent under his pen: "general," "particular," "exceptional," "most general," and so on. Moreover, she identifies three types of resource Poincaré puts into play to quantify a degree of generality. In addition to probability, she notes he regularly counts the number of arbitrary constants that the expression of a solution to a problem involves, as an assessment of the size of the set of solutions thereby found. He also uses insights from Georg Cantor's (1845–1918) research, using concepts that would soon become essential in topology (dense set, perfect set, and so on).[13]

[13] We return to these concepts in Sections 1.2.3 and 1.3.2.

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The same kind of view on mathematical situations can also be perceived in how Poincaré writes down his exploration of a problem, when the treatment requires cases to be distinguished. The article by Robadey (2015) is devoted to how Poincaré proceeds when he presents his reasoning in the form of an enumeration of cases, as he often does. She shows that Poincaré *systematically* lists cases in an order of decreasing generality, opposing a general case to particular cases, whose generality *relative* to each other must also have been assessed, since they are listed accordingly. Incidental remarks throughout texts of this type reveal Poincaré's awareness of relative degrees of generality among different types of particular cases.

This type of assessment of generality can be documented from Poincaré's early writings on differential equations, in 1878 and 1879. Robadey establishes that, for this case, this feature distinguishes Poincaré from his predecessors. It constitutes the basis for the development of a new type of reasoning using the relative degrees of generality of particular cases. In fact, the related hierarchy of cases, presented in the form of an enumeration, is a key resource on which Poincaré draws to develop his new global approach to differential equations. This new approach depends on his ability to focus on what is both tractable and essential in the mathematical situation, as determined by the assessment of degrees of generality.

This remark allows Robadey (2015) to characterize the nature of the general reasoning that Poincaré carries out. It is by no means a kind of "generic reasoning" of the type Hawkins (1977) showed that Weierstrass had criticized. In fact, Robadey suggests that under the label "generic reasoning" Hawkins might have put together types of reasoning that differ substantially, and she outlines the beginning of a typology which is most interesting for the project of our book. In this setting, Poincaré's reasoning appears to be a general reasoning carried out within a framework defined by a quantification of the generality of the cases left aside, by comparison to the size of the collection of cases dealt with. In conclusion, we see a strong connection between some early works by Poincaré and his *memoir* evoked above: in the latter case, the specific concern about generality takes the shape of a statement, whereas, in the former, it takes the shape of the structure of a text. Moreover, Robadey emphasizes, the project to shape an enumeration in this way and the criteria put into play to do so are *not* stated discursively: *only* the text of the enumeration and incidental remarks on it reveal this part of Poincaré's work. The writing of the general is not only located in concepts and statements. It can also take textual forms, whose interpretation becomes more difficult for historians. In fact, elsewhere Robadey sheds light on another phenomenon of exactly the same kind, when she shows how a *memoir* by Poincaré is not in fact devoted to the topic it apparently deals with. Indeed, she establishes the topic under consideration is a *paradigm* in the context of which Poincaré chooses to present a general method.[14]

Frédéric Jaëck addresses a similar issue in the next chapter (Chapter 7) of the book, which he devotes to the introduction of what is for us today an "abstract mathematical

[14] Robadey (2004) further endeavors to account for why Poincaré chose to write his memoir in this way. Again, the interpretation of such texts is challenging, in relation to the fact that they express the general using a textual form.

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structure." Incidentally, by contrast to the previous cases, here one would be tempted to recognize forms of generality related to abstraction. We have seen so far that these were clearly not the only types of generality. We will now see that the matter is more subtle. In the publication of his PhD thesis, in 1922, Stefan Banach (1892–1945) introduces a system of "axioms," in which we could be tempted to identify what today is called "Banach spaces." Part of these axioms relate to the fact that Banach spaces are vector spaces. The other part introduces a norm and the property of completeness. However, in contrast to what previous historians of mathematics have claimed, Jaëck suggests that in this early paper, Banach does *not* introduce the object now given his name. In his view, the axioms play a different role in 1922 from the one they play in the *Théorie des opérations linéaires* which Banach published ten years later. They do not have the same *generality*, and this remark leads him to distinguish two stages in the process of emergence of Banach spaces. Jaëck suggests introducing a distinction between *forms* of generality, which yields a different periodization in the history of science. Again, the distinction between these forms depends crucially on suggestions regarding how historians should interpret their sources.

More precisely, in 1922, Banach's aim in introducing these axioms is to identify key properties, shared by different collections of functions (a list of which is provided at the beginning of his article, and to each of which a specific norm can be attached). The introduction of these properties is tied to a specific organization that Banach intends to give to mathematical knowledge in the 1922 article. Banach's aim was to deal with integral equations. For this, in a first part, he wants to establish certain theorems that hold true for various collections of functions. His ambition is to establish these theorems once and for all, solely by using in his proof the axioms brought to light. By proving that the various collections of functions, each with a specific norm attached, satisfy the axioms, he can apply the theorems to them. The key part played by the axioms in the 1922 publication is thus to allow Banach to make *general proofs* for theorems. As a result, the generality of the axioms is *bounded* by the list of collections of functions.

All the theorems proved in relation to the axioms are used, in the second part of the article, to deal with the integral equations. Hence the consequences of the axioms are only considered in relation to a preassigned goal. In this sense, there is no study of the "Banach space" object as such, in contrast to the 1932 book. Jaëck captures this latter feature of the 1922 article by stating that it does not manifest "reflexivity" with respect to these axioms. These remarks define a first type of generality that Jaëck identifies in Banach's writings.

In the 1932 book, the same axioms have an entirely different meaning, which can be captured in the structure of the text. To begin with, in 1932, the axioms allow Banach to introduce a general object, which will later be called "Banach spaces." The book studies some properties of this object as such, without attaching the axioms to a closed list of collections of functions. In Jaëck's terms, the generality of the axioms is now "open." Moreover, the results obtained about the objects manifest "reflexivity," in that they betray an interest in their properties, rather than an intention of deriving specific applications. This is the point where generality is achieved by means of abstraction. "Reflexivity" and "openness" are the two criteria that lead Jaëck to identify, in the 1932

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book, a second kind of generality. Indeed, between 1922 and 1932, the meaning of the term "general" has changed. From a textual viewpoint, the 1922 and 1932 publications present the same axioms, which in the present day we associate with "Banach spaces." However, the meaning and status of these axioms, and as a result, the kind of generality they embody differ. The same conclusion holds true for the various parts into which the 1922 axioms can be grouped. We can find the axioms of vector spaces in Peano's *Calcolo geometrico* and Riesz introduces the key properties of norms in 1916. However, the meaning and status of these statements, their generality differ from what we find in Banach's successive publications.

Jaëck raises an essential problem for history of science. How can historians determine the nature and degree of the generality of statements, like axioms, they read in their sources? Jaëck illustrates the key fact that the generality of a statement derives not only from the words composing it, but also from the *context* in which it is used and the *way* in which it is used in this context. In line with this concern, Jaëck pays specific attention to the organization of the various texts he takes into account, bringing to light that the organization with which knowledge is presented, and the deductive structure are, in this case, essential ingredients in determining the kind of generality that its statements assert. Such a method proves here to be a useful tool for conceptual history. In a sense, the process Jaëck describes is of the type the philosopher Jean Cavaillès (1903–1944) referred to as "thematization" in mathematics.[15] From this standpoint, the micro-historical analysis Jaëck develops here suggests more historical work can be carried out to observe in greater detail how precisely thematizations occurred.

But this is far from the end of the story. First, with respect to the 1932 book: its organization in fact bears witness to a more global change of perspective. Its successive chapters are devoted to various structures that are defined by a subset of the axioms introduced in 1922 (groups, general vector spaces, normed spaces, etc.). The intention, in 1922, of crafting general proofs paved the way to the introduction, in 1932, not only of a general object, but of *different* general objects of the same kind. These objects all embody the second kind of generality. Moreover, these general objects are connected with each other in a scale of decreasing generality. Secondly, now, with respect to the 1922 article: Jaëck remarks that, when dealing with this document, historians have previously focused their attention on the axioms in relation to the question of the origin of "Banach spaces." However, the article gives a prominent role to linear operations. These are also general objects Banach considers in his PhD thesis. Further, even though most theorems about them are clearly motivated by the intention of solving functional equations, some theorems seem to indicate that Banach also considers them for their own sake. Perhaps, Jaëck suggests, in the 1922 article, one can detect some reflexivity vis-à-vis operations and thus the constitution of a general object of the second type. Perhaps can we perceive here the beginning of what later would be called the theory of operators. Thematization processes might be sometimes intertwined.

[15] See Cavaillès (1938: 177–8).

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1.2.2 A diachronic approach: continuity and reinterpretation

In Chapters 6 and 7, we have considered how specific actors introduce ways of *stating* something general. We have examined the case of expressions using a specific type of concept or statements of a technical kind. We have also advocated why we *have* to consider other types of general formulations that used more macroscopic textual features. Once these types of general entities and statements are introduced, how do other actors appropriate and rework them? In other terms, what is the historicity of these entities and statements? This is the specific issue examined in Part II.2, in which we adopt a diachronic perspective.

The first case study in this regard deals with a topic that is somewhat paradigmatic in the context of our project. It is an investigation of the concept of "genre" in natural history. Indeed, grouping living objects into species and genera can be taken as one of the paradigmatic activities, if not *the* very activity, embodying a search for generality in the sciences. In Chapter 8, Yves Cambefort analyzes the constitution of these "genres" in the long term. What the examination of this type of activity shows, which is of wider interest for our theme, is that the practices leading to the constitution of these groups in different contexts and at different time periods—practices of generality, we might say, or practices of genre making—present essential differences, despite significant continuity in the groups shaped and in the general terms used to refer to them. To highlight this point, Cambefort's chapter sketches, mainly for zoology, how a similar ambition of, and a quest for, generality have been pursued with entirely different methods, and diverging interpretations of the groups constituted.

For Plato and Aristotle, Cambefort argues, the introduction of genres and species for animals did not aim to *classify*, but rather to *differentiate* between living objects. This is coherent with what we have seen above for Aristotle, in relation to mathematics. This approach to "genres" and "species" echoes their "downward" practice of differentiation: Plato, like Aristotle, started from genera (in their sense), and introduced criteria of differentiation within these genera, to distinguish between species. Despite this similarity in the procedures of grouping, Cambefort stresses, their practices of dealing with *differentia* were not the same. However, neither the constitution of a classification nor that of a terminology were related concerns for them. Only later, did these artifacts become explicit goals for naturalists.

For example, Aristotle's concept of genre had no absolute value, each genus being understood in relation to the species into which it was divided. Any genus could, in fact, be considered a species with respect to another, higher cluster, which then was considered a genus. A key change occurs in this respect in the context of seventeenth-century botany, before it was adopted for animals.[16] Cambefort suggests *classification* then becomes a central concern. Correlatively, the *identification* of natural objects becomes a key task.

[16] Note the circulation of a practice of generality from one context to another. We return to this issue subsequently.

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In this new environment, genres become more decisively attached to the classificatory activity, and they are interpreted as a level in a more general arborescent scheme. The latter feature illustrates how, depending on the context, the shaping of general entities can be associated with different spectra of scholarly operations. Further, at the time genres were considered to be absolute, occurring all at the *same* level (or rank) in the classification of living beings. In other words, the same "general entities" became less logical and, by contrast, more loaded with meanings related to the natural world.

In the subsequent century, we note another shift in the shaping of genres, in relation to changes in the ways of carrying out these activities. This is what can be gathered from the observation of Carl Linnaeus' (1707–1778) specific practice of identification and classification. While Linnaeus adopts some features and practices for genres and species deriving from seventeenth-century botany, he nevertheless suggests a new interpretation for genres, taking them to be entities created by God and hence natural facts the naturalist must discover. With this particular interpretation, the sort of naturalistic operations put into play to establish these groups—our general entities—undergo a transformation. The *discovery* of these genres and species becomes the purpose of the naturalist's classificatory activities, now practiced in an upward way. The effort bears on identifying key features that enable the naturalist to recognize genres. Moreover, the system of names Linnaeus suggests is tightly related to this practice of grouping, since it aims to help practitioners situate genres and species as groups that present natural divisions with one another, as reflected in the terminology. Seen from a higher perspective, these names can be considered as symbolic tools shaped to facilitate the circulation within the system of general entities. Again, we see how in each context specific practices relate to ways of making genres, which are interpreted in different ways.

The contrast between Linnaeus and Georges-Louis Leclerc de Buffon (1707–1788) illustrates clearly how other assumptions about the natural world can lead to an entirely different practice with "general entities." As Cambefort emphasizes, Buffon's assumptions are in sharp opposition with Linnaeus'. For Buffon the natural world is a continuum—and in his eyes, Linnaeus' "systematic" approach is creating arbitrary and artificial classifications. Accordingly, for Buffon, classification is not a primary activity, and interestingly for us genres—the shaping of these "general entities"—are not prominent in the naturalists' work. We return to his approach to generality subsequently.

Nineteenth-century zoology inherited the organization of activities around classification and the creation of genres from Linnaeus. However, most practitioners gave a different interpretation to genres, highlighting their *conventional* meaning. Accordingly, they also placed emphasis on *practical* considerations attached to the definition of genres, an issue that remains meaningful to the present day. Notably, the size of the groups created can make significant differences in the practice, which sheds light on the key part played by the genres and species in the operations of situating given entities in the classification. We thus see how an activity with the aim of creating genres and species can interpret, and accordingly shape, them in entirely different ways.

With the publication of Charles Darwin's (1809–1882) *Origin of Species* (1859), new ideas were introduced in the life sciences, which brought about yet another mutation in the interpretation of genres. Genres were thus again maintained as meaningful "general

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entities." However, by means of a genealogical reading, the trees of earlier classifications became interpreted as temporal patterns, in which the upper-level nodes—for instance, the genres—represented common ancestors for lower-level nodes—for instance, species. Consequently this new approach led to changes in the scholarly practices, through which these "general entities" were understood and thus shaped. Noteworthy is the fact, discussed by Cambefort, that in the life sciences in the present day, different groups of practitioners of systematics have developed different practices for the making of "general entities," owing to their diverging way of striking a balance between several criteria. Notably, phylogenetic considerations do not represent the only meaningful criterion for all practitioners. This evidences *contention* among diverging collective ways of shaping genres. The contenders are characterized by different ways of inheriting from past scholarship and different scientific practices of cluster making. However, if "genres" are contested, the making of such types of "general entities" remains a shared goal. Through his diachronic outline on how this task was fulfilled in different contexts, Cambefort thus highlights the wealth of factors that are likely to enter into the shaping of general entities.

Similar conclusions derive from the case study Stéphane Schmitt presents in Chapter 9 on the concept of "homology"—which embodies another way of looking for the general in the life sciences. Broadly speaking, this search attempts to identify parts in the *structures* of different species of living organisms that present a *similarity*, independent of their function. The interest in bringing to light such "homologies" has been correlated with a persisting working assumption that has taken different forms in history. This assumption basically asserts that there are a limited number of organizational plans on which all living beings are built. Aristotle had already held an assumption of this kind. For him, there even existed a single such plan. Naturalists like Etienne Geoffroy Saint-Hilaire (1772–1844) also adopted the idea of a single plan of organization for all living organisms. Other naturalists opted for the slightly different assumption that a small number of such plans existed. For instance, Georges Cuvier (1769–1832) held he could establish the existence of four basic schemes. These assumptions clearly relate to a search for generality in the life sciences. They have guided work on living organisms, leading practitioners to focus on how parts of different organisms correspond to one another.

The key thesis for which Schmitt's chapter argues is that this type of search for generality has changed meaning, and even content, throughout history, in relation to the changing contexts within which it was carried out in the life sciences. However, the basic idea persisted, and even the fundamental practices of naturalists survived key changes in the theoretical framework.

In pre-transformist comparative anatomy, Schmitt notes, scholars looked for homologies in a formal way, without attempting to interpret the results. A search of this kind can be identified in the work of Renaissance naturalist Pierre Belon (1517–1564). In order to express homologies, Belon designed specific kinds of diagrams, with which he displayed, for instance, the similarity between a human being's skeleton and that of a bird (see Figure 9.1, in Chapter 9). This practice illustrates the invention of a way of writing down the general. The argument the diagrams aimed to make led to changes in some features in the description and the drawing of skeletons in unusual ways, so that the similarity appeared more clearly. In other words, here, the search for generality is connected with

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a work practice, and it relates to modes of both organizing data and observing. However, in Belon's works, the remarks about similarity are not developed systematically, the comparison remaining local and bearing on a sample of only a few skeletons.

In the eighteenth century, the search for resemblances between parts of different organisms (the arm of the man and the wing of the bird, for instance) became a significant issue for practitioners. This increased interest in generality seems to have been promoted by a faith inspired by how the successes of Newtonian physics had brought order and unity into the conception of the physical world. Practitioners of the life sciences also had the aim of finding order in the living world. Like systematics, which we examined above, comparative anatomy developed in new and more methodical ways within this context. These developments illustrate how ambitions of generality circulate between fields. However, they took various forms in different life sciences: Systematics and comparative anatomy developed, so to speak, in opposition to each other.[17] This is where we return to Buffon.

Comparisons seeking to establish homologies were carried out on a large scale, and they showed what Buffon emphasized as the anatomical similarity of all living organisms. This led him to put forward for the first time a global *interpretation* of the *meaning* of this manifestation of generality. For Buffon, the global similarity derived from the fact that there had been a single "primitive and general design," on the basis of which all living beings had been created by variation. Comparisons could show at the same time the *variations* in the application of the same design and also the "hidden *resemblance*," which highlighted how the "Divine being" created. Clearly Buffon's general entities differ strikingly from Linnaeus'. Likewise, Buffon's program was an attempt to bring generality to the description of nature, but in a different way. In fact, Buffon pursued generality on different levels, which are in turn addressed in the general chapters distributed among the various volumes of the *Histoire Naturelle*.

Descriptions of animals were a key tool to achieving generality, and it had to be carried out in a specific way. This remark leads to a general conclusion, which we have already emphasized above: bringing to light a type of generality requires specific practices. Let us dwell here for a few lines on description in this context, as another practice of generality. Arbitrariness, Buffon emphasized, should be avoided as much as the excessive accumulation of unorganized information. Buffon's collaborator, Louis Jean-Marie Daubenton (1716–1799), was commissioned to write the morphological and anatomical description of animals, from book III of the *Histoire Naturelle* onward, that is, from 1753. Generality appears to have been a key epistemological value inspiring his work. In a methodological chapter devoted to how description should be carried out, Daubenton makes clear how the method of complete description he advertises has "universal value." He also emphasizes how comparison is, in his view, key to the description.

Instead of piling up facts without any hierarchy between them, he prescribes that description should rather put forward constant properties. *Terms* should be chosen to designate parts in such a way that the same term could designate parts of distinct animals

[17] Here and in what follows, in addition to Schmitt's chapter in this volume, we rely on Schmitt (2010: 16–17, 44–54).

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that correspond to one other. In Daubenton's words, one should proscribe "particular terms," designating the "same thing" with different names, and promote instead "simple" and "universal" denominations. Clearly, the choice of terms is correlated to a systematic practice of *comparison* between animals.

The *order* of the description should also follow constant principles, in order to allow comparison: first describe the whole, and then the parts; first the external characteristics, then the internal organs, and only lastly the skeleton. In the first part of such a description, the position chosen for the animal to be described is systematically the same, both for the *text* and for the *illustration*. This practice went against previous standards, but was later adopted in zoology.

The inclusion of the anatomical part in the description was also meant to counter the practices of both those who merely observed and those who worked on systematics, on the grounds that they only remained at the surface of things. In Daubenton's hands, anatomy was also conducted in a comparative way, inaugurating what soon after came to be known as "comparative anatomy." To allow this comparison, the *text of the description* had to strictly follow the same structure. It also had to focus on elements essential to comparison, leaving aside the elements that brought nothing to the discussion of similarities and dissimilarities. In this sense, the structure of the description was thus dictated by the program of comparison and the search for generality.

Specific animals were chosen as points of reference for the description of others.[18] This feature of the practice relates to the *order* adopted in the *Histoire Naturelle* to present animals. It also relates to specificities in the text of the descriptions: in his work, Daubenton identifies stronger similarities between some groups of animals as opposed to others. These divisions are made manifest by the fact that only the first animal in the division is described, the description of the others being abridged and turned into tables of numbers, thereby instituting an animal as "model." These characteristics of the practice of description illustrate another way of taking the general into account. This form of generality is correlatively materialized in the overall structure of the text. This textual expression of a form of generality evokes what Jaëck describes with respect to Banach.

More broadly, Schmitt shows, the idea of generality captured by the concept of homology between organisms was widespread in the second half of the eighteenth century and beyond, each author subscribing to a specific idea regarding its *nature*. Evolution did *not* derive from such developments, but as soon as the idea of evolution was adopted, it inspired an interpretation of the *meaning* of this form of generality, in terms of descent. Similarities were thus enrolled as arguments in favor of evolution, which conversely offered a totally new perspective on these similarities (their distribution and their meaning). Archetypes became "common ancestors,"—a change in the underlying meaning. However, Schmitt emphasizes the work of bringing to light similarities—the *practices* of looking for the general—did not fundamentally change.

[18] For this and other features of the practice of description and their relation to the purpose of comparing, see Schmitt (2010: 53–57, 59–60, 66–68).

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We recognize a pattern similar to that sketched out above, with respect to systematics. Similarly, in the first place, geneticists attempted to get rid of homology, with the idea it was a superficial link between organisms. However, homology resurfaced at a genetic level. Unexpectedly, the same type of search for kinds of generality continued, even though in this case the reference of the term "homology" underwent a radical mutation.

1.2.3 Circulation between epistemological cultures

The two case studies examined in Section 1.2.2 both deal with long-term continuities in ways of approaching the general in the life sciences. In both cases, a single term (genre or homology) was used in the long term and embodies a type of generality being pursued. How a concept of this type, introduced in a given cultural context, is appropriated into another requires a finer-grain analysis. This is the issue addressed in Part II.3, in Chapter 10, in which Tatiana Roque concentrates on concepts of "genericity" that were introduced into the study of dynamical systems and were constantly reshaped throughout the second half of the twentieth century.

Concepts of genericity are closely related to a type of general statement discussed above, the emergence of which, in mathematics, Robadey's chapter documents. We recall that a statement of this kind typically asserts that a property P holds true for all objects in a domain D, except for a group of objects that can be "neglected." It is thus not universally true, but true with a generality, the assessment of which requires mathematical work. Likewise, Roque studies a case in which the description and classification of *all* objects seems to be out of reach. Consequently, mathematicians settle for a classification of *almost all* objects. In fact, with this case, we return to Henri Poincaré, who is a key figure in our history.

Concepts of "genericity," Roque argues, were brought into the theory of dynamical systems in *relation* to a research strategy adopted by a collective of mathematicians. In this respect, her chapter echoes Schmitt's chapter, which describes the unfolding of a research program drawing on a shared hypothesis regarding general features living beings have in common.

Roque argues the strategy adopted in the study of dynamical systems relied on an essential *a priori* decision, that of *not* considering *single* systems in and of themselves, but *sets* of systems collectively. On this basis, the strategy consisted mainly of two key ideas. First, actors aimed to identify a collection of dynamical systems "large enough" to allow them to approximate, as closely as one might want, *any* dynamical system by one belonging to the collection. This defines a form of generality actors refer to as "genericity." It requires introducing a notion of "closeness" between systems. This notion was shaped using techniques similar to those discussed in Jaëck's chapter. Secondly, actors aimed to choose this collection of dynamical systems in such a way that it proved amenable to description. In this case, it meant that they were driven by the hope of possibly giving a classification to the systems in this collection in such a way that equivalent systems—in a sense of equivalence to be defined—would belong to a same class, and salient features would allow characterization of systems in any class.

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As Roque shows, the *overall strategy*, which relied in a crucial fashion on a type of generality and a method of bringing this generality into play, was in fact initially inspired by another mathematical domain, singularity theory. In that other domain, René Thom (1923–2002) had followed the same strategy and introduced the term "generic" to refer to a collection of tractable geometric objects. The term "generic," and the type of meaning it referred to, were not appropriated alone. We see in fact the importation of a notion of generality, in relation to a strategy, from one context into the other. From this remark derives an important conclusion: Seen from the perspective of the shaping of general entities and research strategies to use them, scientific cultures do not appear as closed cells. They are in conversation with each other and regularly appropriate not only ideas and results, but also concepts and ways of working.

In the case of "genericity," Roque emphasizes the importance of the places (mainly Bures-sur-Yvette, Princeton, and Rio) in which, and the personal connections through which, the transfer was able to take place. Interestingly enough, this was not the first migration of the term "generic," since Thom had borrowed the term, and the idea, from algebraic geometry, thanks to discussions with Claude Chevalley.[19]

While terms referring to forms of generality migrated from one domain to another, in fact their actual meanings were reshaped in each case to suit the new context of their use. Subsequent research on dynamical systems testifies to exactly the same phenomenon. Roque documents the stability of the epistemic tactics outlined above in the history of the theory of dynamical systems, from Poincaré to Smale. However, she shows how research along these lines regularly highlighted problems of two kinds.

First, actors hoped to establish the "genericity" of a collection of dynamical systems they were concentrating on, but failed repeatedly in their attempts. Situations of this kind led researchers to attempt to define the collection of reference in another way that would still be appropriate for the two tasks for which the collection was meant to serve. In other terms, they strove to redefine the reference of the term "generic." These situations also led actors to attempt to identify which general phenomena had been overlooked in the shaping of the former collection. This inquiry focused frequently on "prototypes," illustrating phenomena that had mistakenly been neglected: these objects illustrate other kinds of entities meaningful for a search for the general that actors introduce in some contexts. In relation to the work done to redefine the collection of dynamical systems, the salient features on which to concentrate also underwent transformations.

Second, actors frequently felt the need to rethink the concept of "genericity" they were using, in their attempt to define the representativeness of the collection of objects—or, alternatively, to define the negligible aspects of those outside the collection—in ways that could be better suited to the difficulty they were meeting. In other terms, they strove to reshape the concept with which to capture the general. Particularly important in this respect is the fact that in two different collectives, different sets of mathematical tools were explored. In the network that took shape around Bures-sur-Yvette, Princeton, and Rio, "genericity" was approached as it had been in singularity theory, namely, with topological

[19] On the history of concepts of genericity in algebraic geometry, see Schappacher (2010).

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tools. However, in the Soviet Union, around Andreï Kolmogorov (1903–1987), probability theory, and accordingly measure theory, was favored to capture a similar idea in the study of dynamical systems. These tools were in line with those Robadey shows Poincaré used in his study of problems in celestial mechanics. This different approach soon circulated westward, and it proved to offer new opportunities, at a moment when the topological approach to representativeness seemed to meet with intractable problems.

Thus with evolution in research, and the changing range of phenomena being concentrated on, we see actors constantly reshaping the concept of generality with which they work, and its reference, in relation to the problems dealt with, what the research has shown about these problems, and the types of mathematical knowledge used in the definition of the concept. Further, the case studied by Roque illustrates a key fact: different collectives of actors sometimes shape generality in the same domain in different ways and with different tools. This is a phenomenon to which we return below.

1.3 Practices of generality

The previous sections have given insight into actors' own reflections on generality as well as their shaping of concepts, statements, and forms of texts to express the general, refer to it, or work with it. Some of these tools, they inherit, rethink, or even reshape. Others, they simply invent in relation to the challenges they meet, or the project they set themselves. Poincaré's enumerations and Daubenton's descriptions are excellent illustrations of actors shaping of types of writing in this respect. Our analysis showed that these concepts, statements and forms of text are in most cases *related* to specific practices. Poincaré's enumerations derive from how he deals with cases and how he intends to use them. Daubenton's descriptions relate to how he organizes his naturalistic practice. Concepts of genericity are also meaningful in relation to a specific type of research program that relies on them. These examples show how the making of practices is a dimension of actors' work, inseparable from other dimensions of their work. It illustrates more widely, we claim, how they shape ways of carrying out scientific activity.

Part III of the book brings practices linked to generality into focus, to examine them in the context of the scholarly cultures in which they can be observed. In addition to analyzing how actors dealt with the general using specific practices, we are interested in how they shaped these practices in relation to the issues they selected as meaningful, and especially how their results present correlations with the practices used.

1.3.1 Scientists at work

We have previously seen Leibniz, as a practitioner of philosophy and mathematics among other things, developing a reflection on the virtues attached to generality in mathematics and the forms the general could take. We have also seen how his specific understanding of generality was related to key facets of his philosophy. The first chapter (Chapter 11) of Part III, by Emily Grosholz, now focuses on how Leibniz's reflection meshes with his own practice in mathematics. For this, Grosholz concentrates on the specific type of analysis

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that Leibniz shaped, and she does so by considering one kind of mathematical object to which Leibniz devoted a great deal of effort: curves.

The contrast Leibniz draws between his own approach and Descartes's analysis is telling. For Leibniz, Grosholz emphasizes, Descartes artificially restricted himself to a range of curves (the so-called "geometric curves," to which we return below) to achieve universality for his method. This is not the first time we see an actor criticizing another actor's practice of generality as artificially imposing boundaries on facts. We also return to these criticisms below. It is only in this context, Leibniz stresses, that Descartes's analysis could proceed by a systematic and uniform reduction of problems. By contrast, Leibniz's practice of generality develops in the context of specific situations, treated as *paradigms*, and aims to find patterns in them, or relating to them, that make them intelligible. Connecting a curve to sequences of numbers, to other kinds of geometrical figures, including other curves, as well as to mechanical problems, all give ways of *understanding* the curve and highlighting its hybrid character. Taken separately or in conjunction, these patterns offer resources for solving problems about the curve. Further, they frequently appear to be shared with other problems and other mathematical objects, thereby yielding means of establishing bridges between the curve and other curves, or between problems. Conversely, establishing connections between curves or between problems, and thereby studying them in analogy with each other, allows that such patterns circulate and have an extended fruitfulness.

The general is thus approached, and dealt with, in the context of a particular—a particular that is treated in a general way. In this way of proceeding, the generality of the patterns highlighted is established by progressive extension. Patterns shared across contexts shed light on links that relate the different objects at the level of their *intimate nature*. This is *how* a search for generality of this type meshes with gaining *understanding* in all related contexts simultaneously. Clearly, a procedure of this type ascribes no a priori limit to the connections that can be built. Accordingly, Grosholz can assert that analysis, as practiced by Leibniz, lends itself to generalization. Generality is in this way explored not by abstraction, but by the analysis of the conditions of intelligibility of the paradigm, conducted in a never ending process. This description of Leibniz's practice of generality powerfully evokes features of Poincaré's practice outlined above, as well as a practice evidenced in ancient Chinese mathematical texts (Chemla, 2003). In all these cases, *understanding* appears to be the crux of the matter. Perhaps however, a finer-grained analysis of these practices would reveal differences in the choice of paradigms on which to focus, and the ways of using them. Leibniz appears to focus on what Grosholz calls "canonical objects." Characterized by their simplicity, these objects become more meaningful with time, their canonicity being thus shaped through history.

Several features of Leibniz's practice of generality present an interesting parallel with a practice identified in a completely different setting, which Darrigol analyzes in Chapter 12: Physicist James Clerk Maxwell's (1831–1879) use of analogy and, one could say, of "canonical" models. In this case too, generality is not an observers' category. In 1856, Maxwell explicitly stated that a way of proceeding which he had opted for in his practice of physics aimed to "attain generality and precision," while "avoid[ing] the dangers arising from a premature theory professing to explain the

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cause of the phenomena." The statement is striking, since it makes explicit how an actor clearly identifies several epistemological factors that are meaningful for him (generality, precision, avoiding the dangers arising from a premature theory). It further explains how Maxwell's choice for a given practice strikes a balance between these three requirements.

Interestingly enough, as we mentioned at the beginning of this prologue, half a century later Pierre Duhem perceived a practice of physics like Maxwell's as having given up "generality and rigor." In opposition to a historiography of physics that has adopted Duhem's view, Darrigol sets himself the goal of interpreting Maxwell's own statement about his way of achieving generality. Noteworthy is Darrigol's remark that whereas, for Duhem, abstraction was essential to the practice of generality, it was not for Maxwell. This clearly captures a key difference between the two practices, which also makes sense in other contexts.

To begin with, Darrigol shows how in 1856 Maxwell describes his shaping of a practice in physics as inspired by a fruitful analogy established by William Thomson (Lord Kelvin) (1824–1907). Noticing a parallel between the search for thermal equilibrium and that of an electrical potential, which derives from structurally identical differential equations, Thompson had drawn conclusions based on a physical interpretation in the context of the former and stated, without further proof, the same conclusion in the latter. While at first sight Maxwell repeats a similar theoretical gesture as Thomson, the replacement of heat by a geometrical approach to fluid flows is not innocuous and goes along with a key feature of his practice of generality.

Indeed, Maxwell substitutes Thomson's "analogy" with the establishment of an ideal mechanical model, which he then relates to three different physical situations. The ideal model is now what will embody the generality sought. In these three contexts, elements of the model are put in correspondence with concepts derived from experiments. Further, the three sets of physical phenomena can thereby mathematically be treated conjointly. This property of the *dispositif* illustrates one feature of Maxwell's practice of generality. Interestingly enough, for Maxwell, staying only to the mathematical level would prevent the establishment of "connections" between the different situations—a benefit in terms of generality which thus in his view derives from his practice.

Darrigol then highlights how Maxwell's practice of generality developed gradually in the subsequent years, gaining additional facets. The historian's approach thus discloses a historicity in an actor's practice of generality. In fact, the same remark applies to Jaëck's discussion of Banach: his practice of generality changes in line with the change of meaning and status of the axioms he introduces. This general issue points out a most promising future research program. To return to the specific case of Maxwell, to a local use of the mechanical model as a tool to inquire further into various physical situations, he adds a global model capturing the mechanical nature of the whole range of phenomena dealt with in different domains. The fruitfulness of the "generality" of the practice is manifest through the unification between theories Maxwell achieves in this way. However, his awareness that many different mechanisms could be responsible for the "mechanical connections" uncovered leads him to add yet another facet to his practice of generality, when he attempts to capture the "general structure" common to all these models.

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The way in which Maxwell conducted this search led him to move from an assumption of an underlying mechanical model to the general requirement that all these fields' fundamental equations had to have a Lagrangian structure and involve mechanical variables of a generalized type. His multi-faceted search for generality thus led Maxwell to the identification of a highly general principle, inspired by one of his earlier methods of mechanical modeling. A key conclusion emerges from the case study: it brings to light a correlation between Maxwell's practice of generality, shaped to fulfill specific epistemological constraints, and his eventual introduction of a general *principle*, which incorporates features of this practice.[20]

1.3.2 A diachronic approach: continuities and contrasts

Practices of generality sometimes present a form of diachronic stability, despite the fact that they migrate from one context into a wholly different one. This is the key issue addressed by Jean-Gaël Barbara in Chapter 13, in which he focuses on a recurring practice of shaping general objects in the life sciences. The practice under consideration, he suggests, is characterized by how it identifies objects through the converging approaches of several disciplines. Xavier Bichat (1771–1802) is the first practitioner Barbara examines in greater detail from this perspective, and more specifically Bichat's practice in what he called "*anatomie générale*."

Bichat inherited from various practices of generality before him. To begin with, he inherited from (early) eighteenth century attempts to combine anatomical and physiological approaches. These attempts manifest a transformation in the relationship between these two domains of inquiry. Prior to this, physiological studies had been based on anatomy. Anatomy highlighted general facts, on which practitioners relied to discuss the causes of the functions of the organs. In the eighteenth century, physiological discussions freed themselves from anatomy, and the actors likewise set out to identify general principles that might characterize living things as such. These reflections introduced the idea of, and quest for, "general physiological facts." As a result, instead of having generality defined only by anatomical considerations, two different ranges of general facts could be brought together, and contrasted with each other, in the investigation. In that way, physiology was *combined* to anatomy, and did not only *derive* from it. Bichat followed such a trend and Albrecht von Haller's (1708–1777) investigations in particular. However, the specificity of Bichat's approach lay in a systematic attempt to correlate general physiological facts with general anatomical facts. As a consequence, where for example, Haller had only seen one kind of tissue in a particular study, his cross-disciplinary approach allowed Bichat to subdivide it into three types, each type referring to a similarity in function *and* in pathological transformation. Tissues as "general objects" were born at the convergence between the two domains of inquiry.

[20] This book does not systematically inquire into the reflections on, and practices with, principles in physics and beyond. This is, however, an important topic for a systematic study of generality. On this question, see, for instance, Seth (2006).

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Bichat also had the aim of achieving generality through his practice of observation. He expected that many repeated dissections would shape "clear and general ideas" through relying on the senses. Additionally, Bichat actively sought analogies between different observations. Barbara thus concludes Bichat combined two types of generality: the general as that which derives from similar observations; and the general as what occurs in different parts of an organism and is identified through the combination of features brought to light by different ranges of issues. Tissues are one such example.

Finally, Bichat's valuing of generality has an immediate context. For the purpose of teaching anatomy, Bichat's master, Pierre-Joseph Desault (1738–1795), dissatisfied with the state of anatomical knowledge, which required large bodies of facts to be memorized, aimed to reorganize anatomical knowledge into chapters starting with general facts. Interestingly, teaching appears here as an activity, in the context of which the value of generality plays a key part. Moreover, we meet again with the correlation, encountered above, between valuing generality and aiming to alleviate the burden of memorization. Accordingly, Desault called for an "analytical surgical anatomy" in order to simplify and rationalize knowledge. Here, the term "analytical" likewise echoes a type of approach tightly linked with generality. We have seen above that Chasles likewise attempted to emulate it in geometry, for all the epistemological virtues he attached to it. We will soon return to this "analytical" type of approach in mathematics.

Bichat published Desault's lectures after the latter's death. Like Desault, after whom he taught anatomy in Paris, Bichat also promoted the quest for generality as an organizing principle in teaching. However, he took this task of importing analytical treatments into anatomy one step further, since he adopted the ideal in his own investigations. The transfer of generality practices from the activity of teaching to that of inquiry is noteworthy here. Bichat modeled his practice of inquiry on two complementary ideals, which he put into play using elementary practices of generality encountered above: looking for the *right language* that could provide an analytical tool, and looking for the *elementary components* into which to decompose reality and then to recompose it.

Tissues were precisely, in his view, the "elements" with which to carry out decomposition and recomposition. Bichat's project to classify them is in line with attempts at classification in natural history we have evoked above. In this context, Bichat follows those who value the use of a "natural method" and assumes the types of tissue identified are "real objects." In fact, it is important for us to note that the specific practice of generality for which Bichat opts, that is, approaching tissues from the perspective of different fields, appears precisely to be what *grounds* his conviction that these objects are real. In this respect, generality also constitutes for him what we have called an epistemic value.

Barbara suggests the practice of generality thereby defined was later appropriated by other scientists. He establishes his claim, by examining another practitioner's approach to general anatomy: Louis-Antoine Ranvier (1835–1922). It is to be emphasized that Ranvier focused on a level of inquiry different from Bichat's, the microscopic scale, and he begins with another general hypothesis: the generality of the cell. At stake for him was to discover general structures, in the form of parts of cells and various types of cell. To achieve this goal, like Bichat, he also combined the different perspectives that anatomy

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and physiology yielded on the same situation. For him, a combination of approaches of that kind likewise ensured that the objects identified were "real."

The name that Joseph-Louis Renaut (1844–1917) attributed to the practice in question, which he described explicitly, captures its essence in an interesting way: "principle of converging methods." In both cases, the key idea of the practice is the same. Ranvier's references to Bichat seem to indicate, Barbara suggests, that Ranvier actually perceives his practice of defining general biological objects as being inspired by that of his predecessor in the same field. By contrast, another case study, presented by Renaud Chorlay in Chapter 14, shows actors shaping practices of generality in opposition to that of their predecessors, which they reject.

We have evoked above how the practitioners who established projective geometry derived inspiration for their work from a careful examination of how "analytical" approaches brought into geometry types of generality that earlier geometrical approaches failed to achieve. The practice of generality in "analysis" that, among other geometers, Poncelet and then Chasles had observed in their reflections is precisely one of the three main practices on which Chorlay focuses. Investigating a classical corpus—that of the *foundations* of mathematical analysis in the nineteenth century—from a non-classical perspective—that of issues of generality—, Chorlay aims to capture key features of the scientific work bearing on generality, and related features, in what he refers to as three "epistemic configurations." His strategy is to contrast how practitioners in these three contexts approached, and worked with, the notion of "function," which had become the central notion in analysis from the mid-eighteenth century onward. The historical question of the foundations of analysis is usually investigated in terms of rigor, arithmetization, or set-theoretic thinking. Chorlay endeavors to show that questions of generality provide fresh and relevant interpretation frames.

For Joseph Louis de Lagrange (1736–1813), whose *Théorie des fonctions analytiques* (1797) illustrates the earliest practice examined by Chorlay, the introduction to the notion of function assigns no strict boundary to the object. However, essential in Lagrange's approach is a *principle* of representation of a function by a form of development holding true *uniformly* for all the functions dealt with. Faithful to his practice of analytical treatment, in a sense already encountered above, Lagrange derives from this principle the whole "calculus of functions." The way in which the development "holds true" for a function in particular also characterizes Lagrange's work with generality. It holds true *with full generality* at the level of the *form*, granted that when concrete values are given to variables, the representation *sometimes* fails to have any meaning. Actors refer to this way of dealing with the general as deriving from the "generality of algebra." Chorlay adds a description of how Lagrange captures *singular cases* by means of carefully designed *examples*, which are the simplest cases able to exemplify the phenomena.

It is this practice of generality, from which geometers like Poncelet and Chasles drew inspiration. As Chorlay emphasizes, it is also *this* practice that in Analysis Augustin Louis Cauchy (1789–1857) criticized as mere "induction," and against which he established a new practice of generality. Interestingly enough, as Chemla's chapter recalls, in 1820 Cauchy wrote a negative report on a memoir in which Poncelet introduced the "principle

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of continuity" into geometry, to emulate how analysis achieved generality. In this report, Cauchy addressed *exactly* the same criticism to Poncelet's principle. From the perspective of generality, we thus see how practices circulate from one culture to another. Cauchy's criticisms show how an actor perceives this circulation quite clearly. This remark sheds new light on Cauchy's criticism of Poncelet's principle. We also see how new practices are explicitly designed in opposition to earlier ways of handling generality. Whereas in the earlier context, uniformity was highly valued, in the later context, generality was redefined in relation to the valuing of rigor.

Cauchy's criticism paved the way for the shaping of the second "epistemic configuration" Chorlay examines. In this second context, the notion of function does not have strict boundaries either. However, new types of statement appear, explicitly formulating conditions on the functions as well as on their variables for a proposition asserted to be true. They derive from a new form of proof in analysis, which examines the conditions required for each step in the proof to be valid. Chorlay thus highlights a correlation between the conduct of proof and the form of general statements formulated. These statements set limits to the extension of the group of *functions* for which the proposition as established holds true. Moreover, for given functions, these statements also have the aim of defining the class of *values* of the variables, for which a proposition can be asserted. This type of inquiry will be instrumental in the later development of set theory. The practice of generality in this second context thus appears to be closely related to other features of the "epistemic configuration." Chorlay further emphasizes how in this context, a new use of *examples* emerges: singular functions are shaped as *counterexamples*, used to explore the limits of validity of propositions. This new practice would be of tremendous importance for analysis in subsequent decades.

The third "epistemic configuration" Chorlay analyzes presents several features of wider relevance for our inquiry. In this context, the approach to the notion of function has been entirely renewed. However, in line with the systematic exploration of the conditions of validity of statements, a new feature in the practice of generality has appeared: the classification of functions into classes. Accordingly, the statement of a theorem makes clear for which class it can be asserted. Further, late-nineteenth-century actors like Borel developed an interest in comparing the relative generality of a class with respect to a larger class in which it is contained—what Chorlay calls "embedded generality." Chorlay emphasizes how various types of mathematical means are put into play, and even shaped, to carry out this new task in a precise way. Depending on the purpose, the means chosen to assess the generality will differ. Moreover, Chorlay shows how in analysis new practices of proof emerge, which made use of these assessments of generality to conduct a reasoning. We have already seen how Robadey's chapter documented the emergence of a form of statement and proof of this kind in Poincaré's work. Chorlay's case study thus shows how mathematical work was carried out to develop further means to achieve similar ends. Finally, this type of reasoning pinpointed by Chorlay will precisely be, as we have seen above, an essential ingredient in the deployment of the study of "generic" cases, examined by Roque. Seen from the viewpoint of generality, history of science thus displays circulations, between contexts, not only of concepts and statements, but also of reasoning and other practices relying on a type of generality, or aiming to achieve a

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generality of a certain kind. It also evidences explicit disagreements between actors. They are topics essential for our inquiry.

1.3.3 A synchronic approach: controversies

Several case studies already mentioned evoke actors' criticisms of predecessors' practices of generality and their ensuing adherence to another, possibly new, practice (Leibniz criticizing Descartes, Cauchy criticizing Lagrange, and Duhem criticizing Maxwell). These episodes provide extremely useful evidence for an approach like ours, which aims to identify how actors shaped modes of expression and practices of generality in different contexts. Likewise, disagreements and debates on these matters are very interesting for the differences among cultures of scientific practice they reveal in this respect. We have already evoked a disagreement of this kind, between Linnaeus and Buffon. Their conflicting approaches to the general yielded quite different scientific outcomes, both meaningful from the viewpoint of present-day biology. This part of the book examines more closely three other disputes of that kind, the first of which takes us back to seventeenth-century geometry.

In his survey of the history of geometry from the viewpoint of the value of generality, Chasles stresses the seventeenth century as a turning point. One of the key facts he brings forth is the introduction at that time of a specific type of general procedure, namely, a "method," which allowed practitioners to deal with different objects *uniformly*, and accordingly *connect* the objects as well as the propositions established about them. Barbin's chapter (Chapter 15) takes a closer look at an episode that clearly illustrates Chasles's thesis, while shedding interesting light on differences between actors in this respect.

The episode occurred in the 1630s, and it is related to the problem of finding tangents to curves. What matters most for us here is that it gave rise to a controversy, in which the main protagonists were led to make their views explicit on what a general method should be and how it should be shaped. We can thus observe actors' shaping of their practice in the making. The main protagonists, Descartes and Pierre de Fermat (early seventeenth century–1665), both clearly valued general procedures, whose power extended beyond the treatment of single cases. They each designed a general method for the general problem, but they did so in different ways, echoing the fact they used different categories to describe them.

Descartes referred to his method for finding tangents as "universal." Accordingly, he explicitly defined the range of curves to which it applied: the curves he called geometrical, in relation to the fact that an algebraic equation could be attached to them. Descartes' method relies precisely on the equation and follows a uniform procedure to exhibit the tangent. It could thus be used equally for all these curves, but only for them: the framework was fixed in advance—we have evoked Leibniz's criticism of what he perceives as artificial in this procedure. Interestingly enough, although Desargues had a completely different approach to curves, he shared key features of Descartes's practice of generality. As Chasles emphasizes, Desargues also devised *uniform* ways of defining conical sections and accordingly developed uniform reasoning that could establish related properties of different curves in exactly the same way. He too described his approach as "universal."

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Despite differences between Descartes and Desargues in the approach to curves and the way of dealing with them, Barbin shows that Desargues expressed his support to Descartes in the controversy. The conception and the practice of the general is what the two practitioners share in this case.

Fermat's way of proceeding to approach the tangent problem stands in contrast to this practice of generality. His own practice displays a general method in the context of a specific problem. We have already encountered procedures of this type several times above. Yet Fermat's practice is specific: his presentation of the general method in context establishes connections *at a higher level* between the method and another general method, associated to his name: that of *de maximis and minimis*. Further, Fermat's practice of generality consists in unfolding the potentialities of the general method, through extension and adaptation to different cases. Fermat does not set limits to the group of curves to which the method can apply. As a result, its power extends beyond the set of curves to which Descartes had from the beginning limited the scope of his method. In correlation with this "open" feature—to use the term introduced by Jaëck, which proves relevant in this context too—, Fermat does not consider it to be a priority to highlight the foundations of his method. The controversy with Descartes will compel Fermat, and also Roberval who sides with him, to formulate explicitly their views on what grounds the generality of a method. In conclusion, the episode nicely illustrates how actors shape general objects and general modes of proceeding in different ways. What is more, these different practices of generality mesh with, for example, different approaches to curves and different ways of working with them, and different working techniques to deal with equations. The type of generality pursued is also correlated with the emphasis on other epistemological values and goals. Whereas for Descartes, the uniformity of the procedure matters, for Fermat, the achievement of an ever broader or higher generality appears more meaningful. We have chosen to refer to such ways of doing mathematics, characterized by features of this kind, as "epistemological cultures." This example illustrates how, even when actors operate at the same time, on the same objects and the same problems, the epistemological cultures, in the context of which they are active, differ. Accordingly, in each context, the value of generality displays different forms.

A similar conclusion emerges from the account Frédéric Brechenmacher gives in Chapter 16 of the Jordan–Kronecker dispute in the 1870s, which we evoked in the introduction to this prologue. The conflict breaks out as a priority dispute. What is important for us is that in this context, the actors perceive that part of the dissension relates to how they practice generality. This leads them to make explicit how they believe generality should be pursued. As Brechenmacher makes clear, Jordan in Paris and Weierstrass and Kronecker in Berlin have developed different approaches to a subject (what we understand today as the various types of reduction of matrices). They grasp that their results relate to each other, since they address problems deriving from the same tradition. However, they struggle to understand fully the relationship between their results. We will focus only on what the dispute tells us about our main topic, that is, the competing practices of generality at play in mathematics at the time and the distinct epistemological values actors associate to them.

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As we recalled in the introduction to this prologue, Kronecker's formulation of how he understands the difference between the two practices of generality can be interpreted as a rejection of an old practice—that of a "generic reasoning"—in favor of another one, introduced by Weierstrass a few years before, and prized for its rigor. Kronecker criticizes Jordan for a reasoning that only aims to solve a problem in general, and does not care about exceptions where the reasoning fails to apply. This was in a nutshell the criticism Cauchy, before Weierstrass, formulated against an earlier way of proceeding in mathematics. The historiography of mathematics has strongly emphasized these episodes as testifying to the development of rigor throughout the nineteenth century. For Kronecker, Jordan's approach is thus not truly general. However, it would be mistaken to believe that Kronecker's goal in this case is limited to achieving a greater certainty. Incidentally, if such were the case, the demand for generality would only be a superficial requirement and would not touch the substance of the matter.

In fact, Kronecker's demand of a full generality can be interpreted *only* if we associate it with the other value that gives meaning to it, that is, *uniformity* of the reasoning. Indeed, the fully general reasoning Kronecker expects does not only deal with all cases, but it also deals with them *uniformly*. In his view, the singular cases, for which a reasoning fails, are a precious indication that the practitioner's understanding has not yet reached the crux of the matter. They point to "the real difficulties of the study," and their dissolution, which is carried out only when the "true generality" has been achieved, is a criterion indicating that one has obtained a deeper understanding of the subject and discovered "the wealth of new viewpoints and phenomena which lie in its depths." The generality envisioned is an epistemic value: it appears to be a guide toward the essential features of a situation. The butt of Kronecker's criticism is thus not merely rigor. Kronecker's practice, like Weierstrass', also requires that problems be solved with effective means of computation. The urge to develop such means leads them to opt for an approach in terms of arithmetic *invariants*, that is, one figure of generality in mathematical terms. Accordingly, Kronecker criticizes Jordan's approach for its lack of effectiveness.

Jordan, for his part, manifests another perception of generality. The key other value that he correlatively prizes is simplicity. In his eyes, the Berlinese's computations are hard to understand and lack the simplicity of his approach. Accordingly, Jordan develops a mode of reduction of the objects involved into simpler pieces. In addition, these pieces are of the same kind as those analyzed, which embodies another figure of generality in mathematical terms. For Jordan, an approach of this kind allows the practitioner to "see" what is happening. When he defines his own general approach, simplicity and the possibility of understanding in similar terms appear to be guiding values. Further, from his perspective one virtue of this approach by reduction is that it highlights the relationship between problems that were understood as different and yields related solutions to them: generality in this context also takes the form of unifying a wider set of problems. From Kronecker's perspective, although the *existence* of the reduction is established, Jordan's approach to the problem is flawed, since it makes the actual *exhibiting* of this reduction impossible for theoretical reasons.

Brechenmacher's analysis thus highlights that in the two situations generality belongs to different complexes of values, and it is correlatively understood in different terms. These

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features can be further related to the fact that actors favor different types of procedures, choose different foci for mathematical research, and in the end obtain different kinds of results.

The two studies of disputes examined so far took place within the framework of the *same* discipline. Even when actors work on the same object (curve), or the same problems (which we interpret as matrix reduction), practices and ideals, goals, and values differ. In this context, the ways of understanding and practicing generality also differ. The same conclusion derives from Evelyn Fox Keller's analysis in Chapter 17, which is devoted to practices of generality in the context of two *different* disciplines, namely, physics and biology. More precisely, her study begins by examining a case in which actors belonging to these two fields diverge in their appreciation of what a general treatment of a biological problem should be.

The episode takes place in 1934. Physicist Nicolas Rashevsky (1899–1972) presents a piece of research to biologists, in which he attempts to derive features of the phenomenon of cell division from a model in which he assumes some physical forces being applied to an idealized, and simplified, cell. For him, the cell that is the basis of his work is a model. It is a tractable simplification of the object. Rashevsky thereby puts into play a common practice in physics, which aims to capture the mechanism accounting for a phenomenon. In the context of a practice of this kind, the (re-)production of a given phenomenon, using a few factors that might be at play in a situation, is perceived to provide an explanation of the phenomenon. Such a result suggests a distinction between factors that seem to matter and those that are irrelevant with respect to the phenomenon under consideration. Moreover, the simplicity of the model is itself perceived as an argument in favor of the possible generality of the mechanism. With these few elements, Fox Keller sketches features of the epistemological culture in which Rashevsky usually works. They include practices of research and epistemological factors, in this case, ideals of understanding and values. For the biologists who hear Rashevsky, his ideal cell is not interpreted as a model, but as a cell. As a result, in their views, the import of his results is completely different. For some, what Rashevsky talks about simply does not refer to any living organism: this cell does not exist. For others, his results are fine, but have no generality, their validity being restricted to the special case dealt with. Since the cell fails to take into account the fine details of the general cell, there appears to them to be no way in which the result can be generalized. Its relevance is minimal, if not insignificant. Through her study of the episode, Fox Keller captures the diverging expectations entertained in the two contexts with respect to generality, and she suggests this divergence partly accounts for misunderstandings that develop on the two sides of the disciplinary boundary.

These observations lead Fox Keller to concentrate on the *subject matters* dealt with in the context of the two disciplines. By contrast to the phenomena on which physicists concentrate, taken to be the products of logical and physical necessity, the properties of biological organisms (the objects biologists study) are never static, shaped by the inherently contingent nature of evolution. Clearly, this key difference between the subject matters implies that generality cannot present the same features in the two contexts. Fox Keller asks, what then are the forms generality can take in the life sciences, if one takes this key feature into account? She draws on resources provided by the history of science

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to offer some quite innovative suggestions. Her reflection thus illustrates the resources an inquiry like that presented in this book could offer for practicing scientists.

What is more, the conclusion Fox Keller derives from her observations has a validity that extends far beyond the case study on which she focuses in Chapter 17. Indeed, she stresses that, if the epistemological cultures on which she concentrated present notable differences, this is due in particular to the specificities of the subject matters they deal with. This yields support to, and also accounts for, the thesis we have repeatedly emphasized: collectives of actors shape their ways of doing research in *relation* to the questions they select as being meaningful. Fox Keller draws our attention to the fact that, in this process, the subject matter with which they struggle does play a key part, not least for us in contributing to the determination of forms of generality that are meaningful.

1.3.4 Circulation between epistemological cultures

We have set our inquiry into the value of generality in the context of epistemological cultures, emphasizing how depending on the context, different ways of shaping generality, interpreting this value, and working with it have been devised. However, the various case studies have regularly evidenced that these scholarly cultures are not worlds closed to one another. We have seen resources introduced in the context of one appropriated by another. This is a conclusion holding true more generally. We have seen how it is also valid with respect to an epistemic and epistemological value like generality.

In the final chapter of Part II (Chapter 10), we have examined a case where a concept, that of "genericity," was borrowed from one context to be adapted and used in another. Moreover, the concept was not adopted alone. It was used in relation to a collective research program whose broad outline was similar to the strategy followed in the former context. Likewise, the final chapter of the book (Chapter 18) highlights a striking case of appropriation of a practice of generality, shaped in a given epistemological culture, into a new culture. This case again displays the porosity of these cultures with respect to one another, precisely for the epistemological factors that are the focus of our book. The circulation in question had remained so far unnoticed. It was brought to light in the context of our collective research. Its significance illustrates the benefits that can be derived from the systematic study of a value like generality.

The case in question is Ernst Kummer's (1810–1893) introduction of the notion of ideal numbers into higher arithmetic, which represented a turn in the history of the concept of number as well as in the history of number theory. In this last chapter, Jacqueline Boniface describes the context in arithmetic, in which this innovation took place. Kummer's work followed in the path opened by Carl Friedrich Gauss (1777–1855), when the latter introduced into ordinary arithmetic the entities now called "Gaussian integers" (namely, a type of imaginary number). In 1811, Gauss had justified the introduction into analysis of imaginary magnitudes, by considerations of generality: for him, they had the virtue of bringing into the field a general and uniform validity for truths. In 1825, he further advocated the admission into higher arithmetic of "Gaussian integers" for the generality and simplicity they allowed him to introduce to the theory.

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Kummer follows this direction, when, as Boniface explains, he forms the project of shaping a new form of arithmetic for "complex numbers" (with a specific meaning he gives to the expression), in analogy with ordinary arithmetic. His key idea is a hypothesis related to generality, since he assumes that the fundamental theorem of arithmetic, which asserts the unique decomposition of an integer into prime factors, should hold in that other domain.

Kummer introduces ideal numbers, in addition to imaginary numbers, as the entities necessary to ensure the uniform validity of that fundamental theorem for the class of "complex numbers" he studies. The introduction is thus premised on the idea that "complex numbers" ought to present the same properties as integers in ordinary arithmetic. The failure of some "complex numbers" to satisfy this fundamental theorem is felt as an "anomaly" to be eliminated. Ideal factors are thus introduced to give the fundamental theorem a full generality.

To capture the ideal factors, Kummer proceeds through identifying the adequate properties of the usual factors that could hold for the ideal factors. Noteworthy is the fact that Kummer, precisely like Chasles in relation to his "principle of contingent relationships," distinguishes here between permanent and contingent properties of numbers. He suggests that the properties that do not hold *uniformly* should be discarded in favor of those that are *permanently* valid for all numbers, ideal or not. This remark points to a close parallel between Kummer's reflections and what, as we have seen above, occurred in the context of projective geometry, in fact only a few years earlier. This observation leads us to notice that the term "ideal," which Kummer chose to use to refer to the new entities, also evokes projective geometry: Poncelet had introduced the concept of "ideal" into geometry in relation to his "principle of continuity." This principle, as we mentioned above, was introduced precisely to guarantee that the purely geometrical treatment of geometrical configurations has the *same generality* as the analytical treatment. It was this principle that Chasles reformulated using his "principle of contingent relationships" and the related concepts. Is this mere coincidence? As Boniface mentions, in the publication in which Kummer introduces his ideal numbers, he *explicitly* compares them to geometrical ideals, referring to ideas developed in the context of projective geometry.

If we observe the correlation between the two domains more closely, we see that Kummer borrows the term "ideal" from Poncelet. However, Kummer's interpretation of the related elements follows Chasles's approach and analysis, as formulated in his "principle of contingent relationships" discussed in Chapter 2 of this book. Chasles did not want to adopt the terminology of "ideal elements." It thus appears that Kummer somehow makes a synthesis between various means, shaped to introduce generality in the context of projective geometry. What circulated between the two contexts was a *dispositif* for mathematical work, associated with a *hypothesis* on the nature of ideality as well as a conception of *proof*, all deriving from an emphasis placed on generality. The philosophical analysis of some principles, carried out by the practitioners themselves, yielded a diagnosis regarding the means of bringing generality to a field. This diagnosis allowed the importation, into other domains of mathematics, not of *results,* but of *practices* linked to generality. The conclusion indicates how the influence of projective geometry on subsequent mathematics needs to be approached in a broader perspective and especially in relation to the reflection on the value of generality which it promoted.

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We see how valuing generality leads to introducing homogeneity through the introduction of, in the first place, new relations, and, later on, of new elements. And we see that the techniques for doing so circulated from one domain of mathematics into another, before becoming a "general method" beyond the boundaries of any domain, like the "method of ideal elements" devised by David Hilbert (1862–1943).

1.4 Conclusion

The exploration of generality carried out in this book could certainly have been broader. Indeed, many chapters in the history of science that might look essential for an inquiry like ours were left aside.[21] Accordingly, many practices of generality were hardly evoked. In particular, practices of generality that were given pride of place elsewhere were evoked only tangentially here. We think, for instance, of the many studies that have analyzed the use of laws, cases, or models, in scientific activity. As we explained at the outset, we did not aim at exhaustiveness.

We have placed our collective study of generality under the auspices of a more global project that aims to understand the part played by values in scientific practice and knowledge. With this term, we did not mean economic value, or value defined in terms of usefulness, or even ethical values, even though these other values are certainly also important topics of research.[22] Instead, our project aimed at contributing to the effort of making sense of *how* actors opt for *ways of knowing* and shape them, and *which difference* it makes for the knowledge thereby produced.

This facet of scientific activity has appeared as significant in the last decades and prominently so when historians and philosophers like Thomas Kuhn (1922–1996) have attempted to account for the choice among theories in view of the underdetermination of theories by evidence.[23] Values then appeared useful as tools that could help us to understand what it meant that there can be "alternative roads to knowledge,"[24] or why dissensions over knowledge sometimes could not be solved.[25] As a contribution to this emerging field of research, we have chosen to illustrate how a value—in our case, generality, which from early on has seemed to be inseparable from scientific activity—could be explored in relatively great detail in a historical and epistemological fashion.[26] In contrast to Kuhn, however, the scale at which we have worked was not that of a whole theory, but that of a smaller scale of scientific practice.

[21] The reader will find additional studies in the book edited by Hagner and Laubichler (2006), which complements ours in many respects. Mathematics is not dealt with in this other volume, whereas more weight is put on human and social sciences.

[22] As is clearly illustrated by Putnam (2002), a reflection about any type of value is likely to yield insight for the study of other types.

[23] Kuhn (1977). Note that by bringing "the scope" of a theory into focus, Kuhn touches a value that has relationship with generality as discussed in this book.

[24] Carrier (2012: 242).

[25] Laudan (1984).

[26] In this respect, our project is part of another research program, which is thriving again and in many ways, after decades of quasi dormancy, namely, historical epistemology.

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derive from considering, like Chasles, but with a wider focus, the history of science from the perspective of generality.

Another fact of tremendous importance has also appeared throughout our study. We have noticed recurring practices of generality in contexts that at first sight seem to have been far removed from each other. This is the case, for instance, of the choice of working with, and exploring, the general in the context of a paradigm, which we have emphasized can be evidenced in ancient China, in Leibniz's practice as well as in Poincaré's.[29] This remark suggests that there could be basic modes of working with the general, whose identification remains a task for the future. It strikes us that a study like that presented in this book is an indispensable basis for such a research program to be possibly developed.

Similarly, the various case studies have shown that generality was often valued *in combination with* other values. In different contexts, we have observed different constellations of values. Yet some recurring associations have emerged, like the simultaneous valuing of generality and simplicity. Again, this remark calls for a systematic inquiry into the key reasons for the recurrence of similar constellations of values. Here too, historical and epistemological fieldwork about values like simplicity, rigor, or fruitfulness is a prerequisite for such an inquiry to become genuinely possible.[30] This is a research program whose development we call for, and to which we hope this book will contribute.

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[29] Aby Warburg's practice of the detail in his historical work, described in Grafton (2006), appears as another example thereof.

[30] We have begun research on the value of simplicity along the same lines as those followed in our study of generality. This was, for instance the aim of the workshop "Simplicity as an Epistemological Value in Scientific Practice", organized in 2009 by Karine Chemla and Evelyn Fox Keller in the context of the Institute for Advanced Study, Paris (http://www.paris-iea.fr/en/events/simplicity-as-an-epistemological-value-in-scientific-practice-79, accessed May 16, 2016).

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Part I

The meaning and value of generality

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Section I.1

Epistemic and epistemological values

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> … One can also relate them to *one and the same proposition*, which expresses a *general* property of four points arbitrarily taken on a straight line […Chasles inserts here a reference to the theorem meant]
>
> In that way, propositions 123 and 124, which express a relation between four points arbitrarily taken on a line and a fifth point determined through a certain condition, are *easy consequences* of this theorem.
>
> Propositions 125 and 126 express a relation between four points arbitrarily taken on a straight line and one easily recognizes that this relation is nothing but a *very simple transformation* of the *same theorem*.
>
> … It is remarkable enough that these four propositions (i.e., propositions 119–22, note by KC), which *look so different* from the others and *seem* to have *no relationship with them,* are also *consequences* of the *same theorem*…" (Chasles, 1837: 42–3, Section 36, my emphasis).

Several points are worth noticing here.

In this case, Chasles emphasizes that coming to know the relevant general theorem allows one to *easily* derive from it several propositions that Pappus presented as *distinct* and even *unrelated*. The generality of the theorem is here correlated to the number of different propositions deriving from it, that is, to its fruitfulness. We shall see that, in his own contributions, Chasles looks precisely for theorems that are the most general in the same sense, that is, those that by mere transformations—the word meaning here "reformulations"— can be changed into several propositions formerly thought of as being different. In Section 2.3, by providing one such example from Chasles's work, we shall analyze the tools that can be put into play to achieve this end, and we shall come back to the notion of "transformation."

Moreover, in his comment on Pappus, Chasles puts a proposition being a (logical) "consequence" of a general theorem on a par with a proposition being a "transformation" of the general theorem. One can be more specific here. The single proposition to which Chasles relates all of Pappus's lemmas has the shape of expressing "a general property of four points arbitrarily taken on a straight line." This analysis provides Chasles with a tool with which to identify how propositions relate to this general one. With this tool, some propositions appear to carry out the same task and, through further examination, Chasles identifies that they amount to "a very simple transformation of the same theorem." With the same tool, other propositions can be understood as being "easy consequences." Whether propositions are derived from transformation or as a consequence, Chasles insists on the *ease* with which one obtains them: a simple consequence appears to be a mere change of form. This is the first occurrence of the value of "simplicity" in relation to that of generality. In fact, both epistemological values will prove to have, in Chasles's conception, deep connections.

Finally, Chasles links relating distinct propositions to a single theorem and bringing them into relation with each other, highlighting a connection and a degree of similarity between them. This way of reading ancient sources and analyzing them already betrays distinctive features of Chasles's approach to generality in geometry.

In other examples Chasles examines how propositions that look different in ancient Greek books turn out, according to his analysis, merely to be various *particular cases* of

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