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The Oxford Handbook *of* THE HISTORY OF MATHEMATICS

THE OXFORD HANDBOOK OF

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Edited by

Eleanor Robson and Jacqueline Stedall

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TABLE OF CONTENTS

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INTRODUCTION

Eleanor Robson and Jacqueline Stedall

We hope that this book will not be what you expect. It is not a textbook, an encyclopedia, or a manual. If you are looking for a comprehensive account of the history of mathematics, divided in the usual way into periods and cultures, you will not find it here. Even a book of this size is too small for that, and in any case it is not what we want to offer. Instead, this book explores the history of mathematics under a series of themes which raise new questions about what mathematics has been and what it has meant to practise it. The book is not descriptive or didactic but investigative, comprising a variety of innovative and imaginative approaches to history.

The image on the front cover captures, we hope, the ethos of the *Handbook* (Chapter 1.2, Fig. 1.2.5). At first glance it has nothing to do with the history of mathematics. We see a large man in a headdress and cloak, wielding a ceremonial staff over a group of downcast kneeling women. Who are they, and what is going on? Who made this image, and why? Without giving away too much—Gary Urton's chapter has the answers—we can say here that the clue is in the phrase written in Spanish above the women's heads: *Repartición de las mugeres donzellas q[ue] haze el ynga* 'categorization (into census-groups) of the maiden women that the Inka made'. As this and many other contributions to the book demonstrate, mathematics is not confined to classrooms and universities. It is used all over the world, in all languages and cultures, by all sorts of people. Further, it is not solely a literate activity but leaves physical traces in the material world: not just writings but also objects, images, and even buildings and landscapes. More often, mathematical practices are ephemeral and transient, spoken words or bodily gestures recorded and preserved only exceptionally and haphazardly.

A book of this kind depends on detailed research in disparate disciplines by a large number of people. We gave authors a broad remit to select topics and approaches from their own area of expertise, as long as they went beyond straight 'what-happened-when' historical accounts. We asked for their writing to be exemplary rather than exhaustive, focusing on key issues, questions, and methodologies rather than on blanket coverage, and on placing mathematical content into context. We hoped for an engaging and accessible style, with striking images and examples, that would open up the subject to new readers and challenge those already familiar with it. It was never going to be possible to cover every conceivable approach to the material, or every aspect we would have liked to include. Nevertheless, authors responded to the broad brief with a stimulating variety of styles and topics.

We have grouped the thirty-six chapters into three main sections under the following headings: geographies and cultures, people and practices, interactions and interpretations. Each is further divided into three subsections of four chapters arranged chronologically. The chapters do not need to be read in numerical order: as each of the chapters is multifaceted, many other structures would be possible and interesting. However, within each subsection, as in the book as a whole, we have tried to represent a range of periods and cultures. There are many points of cross-reference between individual sections and chapters, some of which are indicated as they arise, but we hope that readers will make many more connections for themselves.

In working on the book, we have tried to break down boundaries in several important ways. The most obvious, perhaps, is the use of themed sections rather than the more usual chronological divisions, in such a way as to encourage comparisons between one period and another. Between them, the chapters deal with the mathematics of five thousand years, but without privileging the past three centuries. While some chapters range over several hundred years, others focus tightly on a short span of time. We have in the main used the conventional western BC/AD dating system, while remaining alert to other world chronologies.

The *Handbook* is as wide-ranging geographically as it is chronologically, to the extent that we have made geographies and cultures the subject of the first section. Every historian of mathematics acknowledges the global nature of the subject, yet it is hard to do it justice within standard narrative accounts. The key mathematical cultures of North America, Europe, the Middle East, India, and China are all represented here, as one might expect. But we also made a point of commissioning chapters on areas which are not often treated in the mainstream history of mathematics: Russia, the Balkans, Vietnam, and South America, for instance. The dissemination and cross-fertilization of mathematical ideas and practices between world cultures is a recurring theme throughout the book.

The second section is about people and practices. Who creates mathematics? Who uses it and how? The mathematician is an invention of modern Europe. To limit the history of mathematics to the history of mathematicians is to lose much of the subject's richness. Creators and users of mathematics have included cloth weavers, accountants, instrument makers, princes, astrologers, musicians, missionaries, schoolchildren, teachers, theologians, surveyors, builders, and artists. Even when we can discover very little about these people as individuals, group biographies and studies of mathematical subcultures can yield important new insights into their lives. This broader understanding of mathematical practitioners naturally leads to a new appreciation of what counts as a historical source. We have already mentioned material and oral evidence; even within written media, diaries and school exercise books, novels and account books have much to offer the historian of mathematics. Further, the ways in which people have chosen to express themselves—whether with words, numerals, or symbols, whether in learned languages or vernaculars—are as historically meaningful as the mathematical content itself.

From this perspective the idea of mathematics itself comes under scrutiny. What has it been, and what has it meant to individuals and communities? How is it demarcated from other intellectual endeavours and practical activities? The third section, on interactions and interpretations, highlights the radically different answers that have been given to these questions, not just by those actively involved but also by historians of the subject. Mathematics is not a fixed and unchanging entity. New questions, contexts, and applications all influence what count as productive ways of thinking or important areas of investigation. Change can be rapid. But the backwaters of mathematics can be as interesting to historians as the fast-flowing currents of innovation. The history of mathematics does not stand still either. New methodologies and sources bring new interpretations and perspectives, so that even the oldest mathematics can be freshly understood.

At its best, the history of mathematics interacts constructively with many other ways of studying the past. The authors of this book come from a diverse range of backgrounds, in anthropology, archaeology, art history, philosophy, and literature, as well as the history of mathematics more traditionally understood. They include old hands alongside others just beginning their careers, and a few who work outside academia. Some perhaps found themselves a little surprised to be in such mixed company, but we hope that all of them enjoyed the experience, as we most certainly did. They have each risen wonderfully and good-naturedly to the challenges we set, and we are immensely grateful to all of them.

It is not solely authors and editors who make a book. We would also like to thank our consultants Tom Archibald and June Barrow-Green, as well as the team at OUP: Alison Jones, John Carroll, Dewi Jackson, Tanya Dean, Louise Sprake, and Jenny Clarke.

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GEOGRAPHIES AND CULTURES

1. Global

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`CHAPTER 1.1

What was mathematics in the ancient world? Greek and Chinese perspectives

G E R Lloyd

Two types of approach can be suggested to the question posed by the title of this chapter. On the one hand we might attempt to settle a priori on the criteria for mathematics and then review how far what we find in different ancient cultures measures up to those criteria. Or we could proceed more empirically or inductively by studying those diverse traditions and then deriving an answer to our question on the basis of our findings.

Both approaches are faced with difficulties. On what basis can we decide on the essential characteristics of mathematics? If we thought, commonsensically, to appeal to a dictionary definition, which dictionary are we to follow? There is far from perfect unanimity in what is on offer, nor can it be said that there are obvious, crystal clear, considerations that would enable us to adjudicate uncontroversially between divergent philosophies of mathematics. What mathematics is will be answered quite differently by the Platonist, the constructivist, the intuitionist, the logicist, or the formalist (to name but some of the views on the twin fundamental questions of *what* mathematics studies, and *what* knowledge it produces).

The converse difficulty that faces the second approach is that we have to have some prior idea of what is to count as 'mathematics' to be able to start our cross-cultural study. Other cultures have other terms and concepts and their interpretation poses delicate problems. Faced with evident divergence and heterogeneity, at what point do we have to say that we are not dealing with a different concept of mathematics, but rather with a concept that has nothing to do with mathematics at all? The past provides ample examples of the dangers involved in legislating that certain practices and ideas fall beyond the boundaries of acceptable disciplines.

My own discussion here, which will concentrate largely on just two ancient mathematical traditions, namely Greek and Chinese, will owe more to the second than to the first approach. Of course to study the ancient Greek or Chinese contributions in this area—their theories and their actual practices—we have to adopt a provisional idea of what can be construed as mathematical, principally how numbers and shapes or figures were conceived and manipulated. But as we explore further their ancient ideas of what the studies of such comprised, we can expect that our own understanding will be subject to modification as we proceed. We join up, as we shall see, with those problems in the philosophy of mathematics I mentioned: so in a sense a combination of both approaches is inevitable.

Both the Greeks and the Chinese had terms for studies that deal, at least in part, with what we can easily recognize as mathematical matters, and this can provide an entry into the problems, though the lack of any exact equivalent to our notion in both cases is obvious from the outset. I shall first discuss the issues as they relate to Greece before turning to the less familiar data from ancient China.

Greek perspectives

Our term 'mathematics' is, of course, derived from the Greek *mathēmatikē*, but that word is derived from the verb *manthanein* which has the quite general meaning of 'to learn'. A *mathēma* can be any branch of learning, anything we have learnt, as when in Herodotus, *Histories* 1.207, Croesus refers to what he has learnt, his *mathēmata*, from the bitter experiences in his life. So the *mathēmatikos* is, strictly speaking, the person who is fond of learning in general, and it is so used by Plato, for instance, in his dialogue *Timaeus* 88c, where the point at issue is the need to strike a balance between the cultivation of the intellect (in general) and that of the body—the principle that later became encapsulated in the dictum *mens sana in corpore sano* 'a healthy mind in a healthy body'. But from the fifth century BC certain branches of study came to occupy a privileged position as the *mathēmata* par excellence. The terms mostly look familiar enough, *arithmētikē*, *geomētrikē*, *harmonikē*, *astronomia*, and so on, but that is deceptive. Let me spend a little time explaining first the differences between the ancient Greeks' ideas and our own, and second some of the disagreements among Greek authors themselves about the proper subject-matter and methods of certain disciplines.

Arithmētikē is the study of *arithmos*, but that is usually defined in terms of positive integers greater than one. Although Diophantus, who lived at some time in late antiquity, possibly in the third century AD, is a partial exception, the Greeks did not normally think of the number series as an infinitely divisible continuum, but rather as a set of discrete entities. They dealt with what we call fractions as ratios between integers. Negative numbers are not *arithmoi*. Nor is the number one, thought of as neither odd nor even. Plato draws a distinction, in the *Gorgias* 451bc, between *arithmētikē* and *logistikē*, calculation, derived from the verb *logizesthai*, which is often used of reasoning in general. Both studies focus on the odd and the even, but *logistikē* deals with the pluralities they form while *arithmētikē* considers them—so Socrates is made to claim—in themselves. That, at least, is the view Socrates expresses in the course of probing what the sophist Gorgias was prepared to include in what he called 'the art of rhetoric', though in other contexts the two terms that Socrates thus distinguished were used more or less interchangeably. Meanwhile a different way of contrasting the more abstract and practical aspects of the study of *arithmoi* is to be found in Plato's *Philebus* 56d, where Socrates distinguishes the way the many, *hoi polloi*, use them from the way philosophers do. Ordinary people use units that are unequal, speaking of two armies, for instance, or two oxen, while the philosophers deal with units that do not differ from one another in any respect; abstract ones in other words.[1]

At the same time, the study of *arithmoi* encompassed much more than we would include under the rubric of arithmetic. The Greeks represented numbers by letters, where α represents the number 1, β the number 2, γ 3, ι 10, and so on. This means that any proper name could be associated with a number. While some held that such connections were purely fortuitous, others saw them as deeply significant. When in the third century AD the neo-Pythagorean Iamblichus claimed that 'mathematics' is the key to understanding the whole of nature and all its parts, he illustrated this with the symbolic associations of numbers, the patterns they form in magic squares and the like, as well as with more widely accepted examples such as the identification of the main musical concords, the octave, fifth, and fourth, with the ratios 2:1, 3:2, and 4:3. The beginnings of such associations, both symbolic and otherwise, go back to the pre-Platonic Pythagoreans of the fifth and early fourth centuries BC, who are said by Aristotle to have held that in some sense 'all things' 'are' or 'imitate' numbers. Yet this is quite unclear, first because we cannot be sure what 'all things' covers, and secondly because of the evident discrepancy between the claim that they *are* numbers and the much weaker one that they merely *imitate* them.

1. Cf. Asper, Chapter 2.1 in this volume, who highlights divergences between practical Greek mathematics and the mathematics of the cultured elite. On the proof techniques in the latter, Netz (1999) is fundamental.

What about 'geometry'? The literal meaning of the components of the Greek word *geōmetria* is the measurement of land. According to a well-known passage in Herodotus, 2 109, the study was supposed to have originated in Egypt in relation, precisely, to land measurement after the flooding of the Nile. Measurement, *metrētikē*, still figures in the account Plato gives in the *Laws* 817e when his spokesman, the Athenian Stranger, specifies the branches of the *mathēmata* that are appropriate for free citizens, though now this is measurement of 'lengths, breadths and depths', not of land. Similarly, in the *Philebus* 56e we again find a contrast between the exact *geometria* that is useful for philosophy and the branch of the art of measurement that is appropriate for carpentry or architecture.

Those remarks of Plato already open up a gap between practical utility—mathematics as securing the needs of everyday life—and a very different mode of usefulness, namely in training the intellect. One classical text that articulates that contrast is a speech that Xenophon puts in the mouth of Socrates in the *Memorabilia*, 4 7 2–5. While Plato's Socrates is adamant that mathematics is useful primarily because it turns the mind away from perceptible things to the study of intelligible entities, in Xenophon Socrates is made to lay stress on the usefulness of geometry for land measurement and on the study of the heavens for the calendar and for navigation, and to dismiss as irrelevant the more theoretical aspects of those studies. Similarly, Isocrates too (11 22–3, 12 26–8, 15 261–5) distinguishes the practical and the theoretical sides of mathematical studies and in certain circumstances has critical remarks to make about the latter.

The clearest extant statements of the opposing view come not from the mathematicians but from philosophers commenting on mathematics from their own distinctive perspective. What mathematics can achieve that sets it apart from most other modes of reasoning is that it is exact and that it can demonstrate its conclusions. Plato repeatedly contrasts this with the merely persuasive arguments used in the law-courts and assemblies, where what the audience can be brought to believe may or may not be true, and may or may not be in their best interests. Philosophy, the claim is, is not interested in persuasion but in the truth. Mathematics is repeatedly used as the prime example of a mode of reasoning that can produce certainty: and yet mathematics, in the view Plato develops in the *Republic*, is subordinate to dialectic, the pure study of the intelligible world that represents the highest form of philosophy. Mathematical studies are valued as a propaedeutic, or training, in abstract thought: but they rely on perceptible diagrams and they give no account of their hypotheses, rather taking them to be clear. Philosophy, by contrast, moves from its hypotheses up to a supreme principle that is said to be 'unhypothetical'.

The exact status of that principle, which is identified with the Form of the Good, is highly obscure and much disputed. Likening it to a mathematical axiom immediately runs into difficulties, for what sense does it make to call an axiom 'unaxiomatic'? But Plato was clear that both dialectic and the mathematical sciences deal with independent intelligible entities.

Aristotle contradicted Plato on the philosophical point: mathematics does not study independently existing realities. Rather it studies the mathematical properties of physical objects. But he was more explicit than Plato in offering a clear definition of demonstration itself and in classifying the various indemonstrable primary premises on which it depends. Demonstration, in the strict sense, proceeds by valid deductive argument (Aristotle thought of this in terms of his theory of the syllogism) from premises that must be true, primary, necessary, prior to, and explanatory of the conclusions. They must, too, be indemonstrable, to avoid the twin flaws of circular reasoning or an infinite regress. Any premise that can be demonstrated should be. But there have to be *ultimate* primary premises that are evident in themselves. One of Aristotle's examples is the equality axiom, namely if you take equals from equals, equals remain. That cannot be shown other than by circular argument, which yields no proof at all, but it is clear in itself.

It is obvious what this model of axiomatic-deductive demonstration owes to mathematics. I have just mentioned Aristotle's citation of the equality axiom, which figures also among Euclid's 'common opinions',[2] and most of the examples of demonstrations that Aristotle gives, in the *Posterior analytics*, are mathematical. Yet in the absence of substantial extant texts before Euclid's *Elements* itself (conventionally dated to around 300 BC) it is difficult, or rather impossible, to say how far mathematicians before Aristotle had progressed towards an explicit notion of an indemonstrable axiom. Proclus, in the fifth century AD, claims to be drawing on the fourth century BC historian of mathematics, Eudemus, in reporting that Hippocrates of Chios was the first to compose a book of 'Elements', and he further names a number of other figures, Eudoxus, Theodorus, Theaetetus, and Archytas among those who 'increased the number of theorems and progressed towards a more epistemic or systematic arrangement of them' (*Commentary on Euclid's Elements I* 66.7–18).

That is obviously teleological history, as if they had a clear vision of the goal they should set themselves, namely the Euclidean *Elements* as we have it. The two most substantial stretches of mathematical reasoning from the pre-Aristotelian period that we have are Hippocrates' quadratures of lunes and Archytas' determining two mean proportionals (for the sake of solving the problem of the duplication of the cube) by way of a complex kinematic diagram involving the intersection of three surfaces of revolution, namely a right cone, a cylinder, and a torus. Hippocrates' quadratures are reported by Simplicius (*Commentary on Aristotle's Physics* 53.28–69.34), Archytas' work by Eutocius (*Commentary on*

2. Often translated as 'common notions'.

Archimedes' On the sphere and cylinder II, vol. 3, 84.13–88.2), and both early mathematicians show impeccable mastery of the subject-matter in question. Yet neither text confirms, nor even suggests, that these mathematicians had defined the starting-points they required in terms of different types of indemonstrable primary premises.

Of course the principles set out in Euclid's *Elements* themselves do not tally exactly with the concepts that Aristotle had proposed in his discussion of strict demonstration. Euclid's three types of starting-points include definitions (as in Aristotle) and common opinions (which, as noted, include what Aristotle called the equality axiom) but also postulates (very different from Aristotle's hypotheses). The last included especially the parallel postulate that sets out the fundamental assumption on which Euclidean geometry is based, namely that non-parallel straight lines meet at a point. However, where the philosophers had demanded arguments that could claim to be incontrovertible, Euclid's *Elements* came to be recognized as providing the most impressive sustained exemplification of such a project. It systematically demonstrates most of the known mathematics of the day using especially reductio arguments (arguments by contradiction) and the misnamed method of exhaustion. Used to determine a curvilinear area such as a circle by inscribing successively larger regular polygons, that method precisely did *not* assume that the circle was 'exhausted', only that the difference between the inscribed rectilinear figure and the circumference of the circle could be made as small as you like. Thereafter, the results that the *Elements* set out could be, and were, treated as secure by later mathematicians in their endeavours to expand the subject.

The impact of this development first on mathematics itself, then further afield, was immense. In statics and hydrostatics, in music theory, in astronomy, the hunt was on to produce axiomatic-deductive demonstrations that basically followed the Euclidean model. But we even find the second century AD medical writer Galen attempting to set up mathematics as a model for reasoning in medicine—to yield conclusions in certain areas of pathology and physiology that could claim to be incontrovertible. Similarly, Proclus attempted an *Elements of theology* in the fifth century AD, again with the idea of producing results that could be represented as certain.

The ramifications of this development are considerable. Yet three points must be emphasized to put it into perspective. First, for ordinary purposes, axiomatics was quite unnecessary. Not just in practical contexts, but in many more theoretical ones, mathematicians and others got on with the business of calculation and measurement without wondering whether their reasoning needed to be given ultimate axiomatic foundations.[3]

3. Cuomo (2001) provides an excellent account of the variety of both theoretical and practical concerns among the Greek mathematicians at different periods.

Second, it was far from being the case that all Greek work in arithmetic and geometry, let alone in other fields such as harmonics or astronomy, adopted the Euclidean pattern. The three 'traditional' problems, of squaring the circle, the duplication of the cube, and the trisection of an angle were tackled already in the fifth century BC without any explicit concern for axiomatics (Knorr 1986). Much of the work of a mathematician such as Hero of Alexandria (first century AD) focuses directly on problems of mensuration using methods similar to those in the traditions of Egyptian and Babylonian mathematics by which, indeed, he may have been influenced.[4] While he certainly refers to Archimedes as if he provided a model for demonstration, his own procedures sharply diverge, on occasion, from Archimedes'.[5] In the *Metrica*, for instance, he sometimes gives an arithmetized demonstration of geometrical propositions, that is, he includes concrete numbers in his exposition. Moreover in the *Pneumatica* he allows exhibiting a result to count as a proof. Further afield, I shall shortly discuss the disputes in harmonics and the study of the heavens, on the aims of the study, and the right methods to use.

Third, the recurrent problem for the model of axiomatic-deductive demonstration that the *Elements* supplied was always that of securing axioms that would be both self-evident and non-trivial. Moreover, it was not enough that an axiom set should be internally consistent: it was generally assumed that they should be true in the sense of a correct representation of reality. Clearly, outside mathematics they were indeed hard to come by. Galen, for example, proposed the principle that 'opposites are cures for opposites' as one of his indemonstrable principles, but the problem was to say what counted as an 'opposite'. If not trivial, it was contestable, but if trivial, useless. Even in mathematics itself, as the example of the parallel postulate itself most clearly showed, what principles could be claimed as self-evident was intensely controversial. Several commentators on the *Elements* protested that the assumption concerning non-parallel straight lines meeting at a point should be a theorem to be proved and removed from among the postulates. Proclus outlines the controversy (*Commentary on Euclid's Elements I* 191.21ff.) and offers his own attempted demonstration as well as reporting one proposed by Ptolemy (365.5ff., 371.10ff.): yet all such turned out to be circular, a result that has sometimes been taken to confirm Euclid's astuteness in deciding to treat this as a postulate in the first place. In time, however, it was precisely the attack on the parallel postulate that led to the eventual emergence of non-Euclidean geometries.

These potential difficulties evidently introduce elements of doubt about the ability of mathematics, or of the subjects based on it, to deliver exactly what

4. Cf. Robson (Chapter 3.1), Rossi (Chapter 5.1), and Imhausen (Chapter 9.1) in this volume.
5. Moreover Archimedes himself departed from the Euclidean model in much of his work, especially, for example, in the area we would call combinatorics; cf. Saito (Chapter 9.2) in this volume and Netz (forthcoming).

some writers claimed for it. Nevertheless, to revert to the fundamental point, mathematics, in the view both of some mathematicians and of outsiders, was superior to most other disciplines, precisely in that it could outdo the merely persuasive arguments that were common in most other fields of inquiry.

It is particularly striking that Archimedes, the most original, ingenious, and multifaceted mathematician of Greek antiquity, insisted on such strict standards of demonstration that he was at one point led to consider as merely heuristic the method that he invented and set out in his treatise of that name. He there describes how he discovered the truth of the theorem that any segment of a parabola is four-thirds of the triangle that has the same base and equal height. The method relies on two assumptions: first that plane figures may be imagined as balanced against one another around a fulcrum and second that such figures may be thought of as composed of a set of line segments indefinitely close together. Both ideas breached common Greek presuppositions. It is true that there were precedents both for applying some quasi-mechanical notions to geometrical issues—as when figures are imagined as set in motion—and for objections to such procedures, as when in the *Republic* 527ab Plato says that the language of mathematicians is absurd when they speak of 'squaring' figures and the like, as if they were doing things with mathematical objects. But in Archimedes' case, the first objection to his reasoning would be that it involved a category confusion, in that geometrical objects are not the types of item that could be said to have centres of gravity. Moreover, Archimedes' second assumption, that a plane figure is composed of its indivisible line segments, clearly breached the Greek geometrical notion of the continuum. The upshot was that he categorized his method as one of discovery only, and he explicitly claimed that its results had thereafter to be demonstrated by the usual method of exhaustion. At this point, there appears to be some tension between the preoccupation with the strictest criteria of proof that dominated one tradition of Greek mathematics (though only one) and the other important aim of pushing ahead with the business of discovery.

The issues of the canon of proof, and of whether and how to provide an axiomatic base for work in the various parts of 'mathematics', were not the only subjects of dispute. Let me now illustrate the range of controversy first in harmonics and then in the study of the heavens.

'Music', or rather *mousikē*, was a generic term, used of any art over which one or other of the nine Muses presided. The person who was *mousikos* was one who was well-educated and cultured generally. To specify what we mean by 'music' the Greeks usually used the term *harmonikē*, the study of harmonies or musical scales. Once again the variety of ways that study was construed is remarkable and it is worth exploring this in some detail straight away as a classic illustration of the tension between mathematical analysis and perceptible phenomena. There were those whose interests were in music-making, practical musicians who were interested in producing pleasing sounds. But there were also plenty of theorists who attempted analyses involving, however, quite different starting assumptions. One approach, exemplified by Aristoxenus, insisted that the unit of measurement should be something identifiable to perception. Here, a tone is defined as the difference between the fifth and the fourth, and in principle the whole of music theory can be built up from these perceptible intervals, namely by ascending and descending fifths and fourths.

But if this approach accepted that musical intervals could be construed on the model of line segments and investigated quasi-geometrically, a rival mode of analysis adopted a more exclusively arithmetical view, where the tone is defined as the difference between sounds whose 'speeds' stand in a ratio of 9:8. In this, the so-called Pythagorean tradition, represented in the work called the *Sectio canonis* in the Euclidean corpus, musical relations are understood as essentially ratios between numbers, and the task of the harmonic theorist becomes that of deducing various propositions in the mathematics of ratios.

Moreover, these quite contrasting modes of analysis were associated with quite different answers to particular musical questions. Are the octave, fifth, and fourth exactly six tones, three and a half, and two and a half tones respectively? If the tone is identified as the ratio of 9 to 8, then you do not get an octave by taking six such intervals. The excess of a fifth over three tones, and of a fourth over two, has to be expressed by the ratio 256 to 243, not by the square root of 9/8.

This dispute in turn spilled over into a fundamental epistemological disagreement. Is perception to be the criterion, or reason, or some combination of the two? Some thought that numbers and reason ruled. If what we heard appeared to conflict with what the mathematics yielded by way of an analysis, then too bad for our hearing. We find some theorists who denied that the interval of an octave plus a fourth can be a harmony precisely because the ratio in question (8:3) does not conform to the mathematical patterns that constitute the main concords. Those all have the form of either a multiplicate ratio as, for example, 2:1 (expressing the octave) or a superparticular one as, for example, 3:2 and 4:3, both of which meet the criterion for a superparticular ratio, namely n+1 : n.

It was one of the most notable achievements of the *Harmonics* written by Ptolemy in the second century AD to show how the competing criteria could be combined and reconciled (cf. Barker 2000). First, the analysis had to derive what is perceived as tuneful from rational mathematical principles. Why should there be any connection between sounds and ratios, and with the particular ratios that the concords were held to express? What hypotheses should be adopted to give the mathematical underpinning to the analysis? But just to select some principles that would do so was, by itself, not enough. The second task the music theorist must complete is to bring those principles to an empirical test, to confirm that the results arrived at on the basis of the mathematical theory did indeed tally with what was perceived by the ear in practice to be concordant—or discordant—as the case might be.

The study of the heavens was equally contentious. Hesiod is supposed to have written a work entitled *Astronomia*, though to judge from his *Works and days* his interest in the stars related rather to how they tell the passing of the seasons and can help to regulate the farmer's year. In the *Epinomis* 990a (whether or not this is an authentic work of Plato) Hesiod is associated with the study of the stars' risings and settings—an investigation that is *contrasted* with the study of the planets, sun, and moon. *Gorgias* 451c is one typical text in which the task of the astronomer is said to be to determine the relative speeds of the stars, sun and moon.

Both *astronomia* and *astrologia* are attested in the fifth century BC and are often used interchangeably, though the second element in the first has *nemo* as its root and that relates to distribution, while *logos*, in the second term, is rather a matter of giving an account. Although genethlialogy, the casting of horoscopes based on geometrical calculations of the positions of the planets at birth, does not become prominent until the fourth century BC, the stars were already associated with auspicious and inauspicious phenomena in, for example, Plato's *Symposium* 188b. Certainly by Ptolemy's time (second century AD) an explicit distinction was drawn between predicting the movements of the heavenly bodies themselves (astronomy, in our terms, the subject-matter of the *Syntaxis*), and predicting events on earth on their basis (astrology, as we should say, the topic he tackled in the *Tetrabiblos*, which he explicitly contrasts with the other branch of the study of the heavens). Yet both Greek terms themselves continued to be used for either. Indeed, in the Hellenistic period the term *mathēmatikos* was regularly used of the astrologer as well as of the astronomer.

Both studies remained controversial. The arguments about the validity of astrological prediction are outlined in Cicero's *De divinatione* for instance, but the Epicureans also dismissed astronomy as speculative. On the other hand, there were those who saw it rather as one of the most important and successful of the branches of mathematics—not that they agreed on how it was to be pursued. We may leave to one side Plato's provocative remarks in the *Republic* 530ab that the *astronomikos* should pay no attention to the empirical phenomena—he should 'leave the things in the heavens alone'—and engage in a study of 'quickness and slowness' themselves (529d), since at that point Plato is concerned with what the study of the heavens can contribute to abstract thought. If we want to find out how Plato himself (no practising astronomer, to be sure) viewed the study of the heavens, the *Timaeus* is a surer guide, where indeed the contemplation of the heavenly bodies is again given philosophical importance—such a vision encourages the soul to philosophize—but where the different problems posed by the varying speeds and trajectories of the planets, sun, and moon are recognized each to need its own solution (*Timaeus* 40b–d).

Quite how the chief problems for theoretical astronomy were defined in the fourth century BC has become controversial in modern scholarship (Bowen 2001). But it remains clear first that the problem of the planets' 'wandering', as their Greek name ('wanderer') implied, was one that exercised Plato. In his *Timaeus*, 39cd, their movements are said to be of wondrous complexity, although in his last work, the *Laws* 822a, he came to insist that each of the heavenly bodies moves with a *single* circular motion. The model of concentric spheres that Aristotle in *Metaphysics lambda (Λ)* ascribes to Eudoxus, and in a modified form to Callippus, was designed to explain *some* anomalies in the apparent movements of the sun, moon, and planets. Some *geometrical* model was thereafter common ground to much Greek astronomical theorizing, though disputes continued over *which* model was to be preferred (concentric spheres came to be replaced by eccentrics and epicycles). Moreover, some studies were purely geometrical in character, offering no comments on how (if at all) the models proposed were to be applied to the physical phenomena. That applies to the books that Autolycus of Pitane wrote *On the moving sphere*, and *On risings and settings*. Even Aristarchus in the one treatise of his that is extant, *On the sizes and distances of the sun and moon*, engaged (in the view I favour) in a purely geometrical analysis of how those results could be obtained, without committing himself to concrete conclusions, although in the work in which he adumbrated his famous heliocentric hypothesis, there are no good grounds to believe he was *not* committed to that as a physical solution.

Yet if we ask *why* prominent Greek theorists adopted *geometrical* models to explain the apparent irregularities in the movements of the heavenly bodies, when most other astronomical traditions were content with purely numerical solutions to the patterns of their appearances, the answer takes us back to the ideal of a demonstration that can carry explanatory, deductive force, and to the demands of a teleological account of the universe, that can show that the movements of the heavenly bodies are supremely orderly.

We may note once again that the history of Greek astronomy is not one of uniform or agreed goals, ideals, and methods. It is striking how influential the contrasts that the philosophers had insisted on, between proof and persuasion or between demonstration and conjecture, proved to be. In the second century AD, Ptolemy uses those contrasts twice over. He first does so in the *Syntaxis* in order to contrast 'mathematics', which here clearly includes the mathematical astronomy that he is about to embark on in that work, with 'physics' and with 'theology'. Both of those studies are merely conjectural, the first because of the instability of physical objects, the second because of the obscurity of the subject. 'Mathematics', on the other hand, can secure certainty, thanks to the fact that it uses—so he says—the incontrovertible methods of arithmetic and geometry. In practice, of course, Ptolemy has to admit the difficulties he faces when tackling such subjects as the movements of the planets in latitude (that is, north and south of the ecliptic): and his actual workings are full of approximations. Yet that is not allowed to diminish the claim he wishes to make for his theoretical study.

Then, the second context in which he redeploys the contrast is in the opening chapters of the *Tetrabiblos*, which I have already mentioned for the distinction it draws between two types of prediction. Those that relate to the movements of the heavenly bodies themselves can be shown demonstratively, *apodeiktikōs*, he says, but those that relate to the fortunes of human beings are an *eikastikē*, conjectural, study. Yet, while some had used 'conjecture' to undermine an investigation's credibility totally, Ptolemy insists that astrology is founded on assumptions that are tried and tested. Like medicine and navigation, it cannot deliver certainty, but it can yield probable conclusions.

Many more illustrations of Greek ideas and practices could be given, but enough has been said for one important and obvious point to emerge in relation to our principal question of what mathematics was in Greece, namely that generalization is especially difficult in the face of the widespread disagreements and divergences that we find at all periods and in every department of inquiry. Some investigators, to be sure, got on with pursuing their own particular study after their own manner. But the questions of the status and goals of different parts of the study, and of the proper methods by which it should be conducted, were frequently raised both within and outside the circles of those who styled themselves mathematicians. But if no single univocal answer can be given to our question, we can at least remark on the intensity with which the Greeks themselves debated it.

Chinese perspectives

The situation in ancient China is, in some respects, very different. The key point is that two common stereotypes about Chinese work are seriously flawed: the first that their concern for practicalities blocked any interest in theoretical issues, and the second that while they were able calculators and arithmeticians, they were weak geometers.

It is true that while the Greek materials we have reviewed may suffer from a deceptive air of familiarity, Chinese ideas and practices are liable to seem exotic. Their map or maps of the relevant intellectual disciplines, theoretical or practical and applied, are very different both from those of the Greeks and from our own. One of the two general terms for number or counting, *shu* 數, has meanings that include 'scolding', 'fate', or 'destiny', 'art' as in 'the art of', and 'deliberations' (Ho 1991). The second general term, *suan* 算, is used of 'planning', 'scheming', and 'inferring', as well as 'reckoning' or 'counting'. The two major treatises that deal with broadly mathematical subjects that date from between around 100 BC and 100 AD, both have *suan* in their title: we shall have more to say on each in due course. The *Zhou bi suan jing* 周 髀 算 經 is conventionally translated 'Arithmetic classic of the gnomon of Zhou'. The second treatise is the *Jiu zhang suan shu* 九 章 算 術, the 'Nine chapters on mathematical procedures'. This draws on an earlier text recently excavated from a tomb sealed in 186 BC, which has both general terms in its title, namely *Suan shu shu* 算 數 書, the 'Book of mathematical procedures', as Chemla and Guo (2004) render it, or more simply, 'Writings on reckoning' (Cullen 2004). But the 'Nine chapters' goes beyond that treatise, both in presenting the problems it deals with more systematically, and in extending the range of those it tackles, notably by including discussing *gou gu* 句股, the properties of right-angled triangles (a first indication of those Chinese interests in geometrical questions that have so often been neglected or dismissed). Indeed, thanks to the existence of the *Suan shu shu* we are in a better position to trace early developments in Chinese mathematics than we are in reconstructing what Euclid's *Elements* owed to its predecessors.

When, in the first centuries BC and AD the Han bibliographers, Liu Xiang and Liu Xin, catalogued all the books in the imperial library under six generic headings, *shu shu* 數 術 'calculations and methods' appears as one of these. Its six subspecies comprise two that deal with the study of the heavens, namely *tian wen* 天 文 'the patterns in the heavens' and *li pu* 曆譜 'calendars and tables', as well as *wu xing* 五 行 'the five phases', and a variety of types of divinatory studies. The five phases provided the main framework within which change was discussed. They are named fire, earth, metal, water, and wood, but these are not elements in the sense of the basic physical constituents of things, so much as processes. 'Water' picks out not so much the substance, as the process of 'soaking downwards', as one text (the Great plan) puts it, just as 'fire' is not a substance but 'flaming upwards'.

This already indicates that the Chinese did not generally recognize a fundamental contrast between what we call the study of nature (or the Greeks called *phusike*) on the one hand and mathematics on the other. Rather, each discipline dealt with the quantitative aspects of the phenomena it covered as and when the need arose. We can illustrate this with harmonic theory, included along with calendar studies in the category *li pu*.

Music was certainly of profound cultural importance in China. We hear of different types of music in different states or kingdoms before China was unified under Qin Shi Huang Di in 221 BC, some the subject of uniform approval and appreciation, some the topic of critical comment as leading to licentiousness and immorality—very much in the way in which the Greeks saw different modes of their music as conducive to courage or to self-indulgence. Confucius is said to have not tasted meat for three months once he had heard the music of *shao* in the kingdom of *Qi* (*Lun yu* 7 14).

But musical sounds were also the subject of theoretical analysis, indeed of several different kinds. We have extensive extant texts dealing with this, starting with the *Huai nan zi*, a cosmological summa compiled under the auspices of Liu An, King of Huainan, in 136 BC, and continuing in the musical treatises contained in the first great Chinese universal history, the *Shi ji* written by Sima Tan and his son Sima Qian around 90 BC. Thus *Huai nan zi*, ch 3, sets out a schema correlating the twelve pitchpipes, that give what we would call the 12-tone scale, with the five notes of the pentatonic scale. Starting from the first pitchpipe, named Yellow Bell (identified with the first pentatonic note, *gong*), the second and subsequent pitchpipes are generated by alternate ascents of a fifth and descents of a fourth—very much in the manner in which in Greece the Aristoxenians thought that all musical concords should be so generated. Moreover, *Huai nan zi* assigns a number to each pitchpipe. Yellow Bell starts at 81, the second pitchpipe, Forest Bell, is 54 —that is 81 times 2/3, the next is 72, that is 54 times 4/3, and so on. The system works perfectly for the first five notes, but then complications arise. The number of the sixth note is rounded from 42 2/3 to 42, and at the next note the sequence of alternate ascents and descents is interrupted by two consecutive descents of a fourth—a necessary adjustment to stay within a single octave.

On the one hand it is clear that a numerical analysis is sought and achieved, but on the other a price has to be paid. Either approximations must be allowed, or alternatively very large numbers have to be tolerated. The second option is the one taken in a passage in the *Shi ji* 25, where the convention of staying within a single octave is abandoned, but at the cost of having to cope with complex ratios such as 32,768 to 59,049. Indeed *Huai nan zi* itself in another passage, 3. 21a, generates the twelve pitchpipes by successive multiplications by 3 from unity, which yields the number 177,147 (that is 3^{11}) as the 'Great Number of Yellow Bell'. That section associates harmonics with the creation of the 'myriad things' from the primal unity. The *Dao* 道 is one, and this subdivides into *yin* 陰 and *yang* 陽, which between them generate everything else. Since *yin* and *yang* themselves are correlated with even and with odd numbers respectively, the greater and the lesser *yin* being identified as six and eight respectively, and the greater and lesser *yang* nine and seven, the common method of divination, based on the hexagrams set out in such texts as the *Yi jing* 易 經 'Book of changes', is also given a numerical basis. But, interestingly enough, the 'Book of changes' was not classified by Liu Xiang and Liu Xin under *shu shu*. Rather it was placed in the group of disciplines that dealt with classic, or canonical, texts. Indeed the patterns of *yin* and *yang* lines generated by the hexagrams were regularly mined for insight into every aspect of human behaviour, as well as into the cosmos as a whole.

Similarly complicated numbers are also required in the Chinese studies of the heavens. One division dealt with 'the patterns of the heavens', *tian wen*, and was chiefly concerned with the interpretation of omens. But the other *li fa* included the quantitative analysis of periodic cycles, both to establish the calendar and to enable eclipses to be predicted. In one calendrical schema, called the Triple Concordance System, a lunation is 29 43/81 days, a solar year 365 385/1539 days, and in the concordance cycle 1539 years equals 19,035 lunations and 562,120 days (cf. Sivin 1995). On the one hand, considerable efforts were expended on carrying out the observations needed to establish the data on which eclipse cycles could be based. On the other, the figures for the concordances were also manipulated mathematically, giving in some cases a spurious air of precision—just as happens in Ptolemy's tables of the movements of the planets in longitude and in anomaly in the *Syntaxis*.

Techniques for handling large-number ratios are common to both Chinese harmonics and to the mathematical aspects of the study of the heavens. But there is also a clear ambition to integrate these two investigations—which both form part of the Han category *li pu*. Thus, each pitchpipe is correlated with one of the twelve positions of the handle of the constellation 'Big Dipper' as it circles the celestial pole during the course of the seasons. Indeed, it was claimed that each pitchpipe resonates spontaneously with the *qi* of the corresponding season and that that effect could be observed empirically by blown ash at the top of a half-buried pipe, a view that later came to be criticized as mere fantasy (Huang Yilong and Chang Chih-Ch'eng 1996).

While the calendar and eclipse cycles figure prominently in the work of Chinese astronomers, the study of the heavens was not limited to those subjects. In the *Zhou bi suan jing,* the Master Chenzi is asked by his pupil Rong Fang what his *Dao* achieves, and this provides us with one of the clearest early statements acknowledging the power and scope of mathematics.[6] The *Dao*, Chenzi replies, is able to determine the height and size of the sun, the area illuminated by its light, the figures for its greatest and least distances, and the length and breadth of heaven, solutions to each of which are then set out. That the earth is flat is assumed throughout, but one key technique on which the results depend is the geometrical analysis of gnomon shadow differences. Among the observational techniques is sighting the sun down a bamboo tube. Using the figure for the distance of the sun obtained in an earlier study, the dimension of the sun can be gained from those of the tube by similar triangles. Such a result was just one impressive proof of the power of mathematics (here *suan shu*) to arrive at an understanding of apparently obscure phenomena. But it should be noted that although Chenzi eventually explains his methods to his pupil on the whole quite clearly, he first expects him to go away and work out how to get these results on

6. The term *Dao*, conventionally translated 'the Way', can be used of many different kinds of skills, and here the primary reference is to Chenzi's ability in mathematics. But those skills are thought of as subordinate to the supreme principle at work in the universe, which it is the goal of the sage to cultivate, indeed to embody (Lloyd and Sivin 2002).

his own. Instead of overwhelming the student with the incontrovertibility of the conclusion '*quod erat demonstrandum*', the Chinese master does not rate knowledge unless it has been internalized by the pupil.

The major classical Chinese mathematical treatise, the 'Nine chapters', indicates both the range of topics covered and the ambitions of the coverage. Furthermore the first of the many commentators on that text, Liu Hui in the third century AD, provides precious evidence of how he saw the strategic aims of that treatise and of Chinese mathematics as a whole. The 'Nine chapters' deals with such subjects as field measurement, the addition, subtraction, multiplication, and division of fractions, the extraction of square roots, the solutions to linear equations with multiple unknowns (by the rule of double false position), the calculation of the volumes of pyramids, cones, and the like.

The problems are invariably expressed in concrete terms. The text deals with the construction of city-walls, trenches, moats, and canals, with the fair distribution of taxes across different counties, the conversion of different quantities of grain of different types, and much else besides. But to represent the work as just focused on practicalities would be a travesty. A problem about the number of workmen needed to dig a trench of particular dimensions, for instance, gives the answer as 7 427/3064ths labourers. The interest is quite clearly in the exact solution to the equation rather than in the practicalities of the situation. Moreover the discussion of the circle–circumference ratio (what we call π) provides a further illustration of the point. For practical purposes, a value of 3 or 3 1/7 is perfectly adequate, and such values were indeed often used. But the commentary tradition on the 'Nine chapters' engages in the calculation of the area of inscribed regular polygons with 192 sides, and even 3072-sided ones are contemplated (the larger the number of sides, the closer the approximation to the circle itself of course): by Zhao Youqin's day, in the thirteenth century, we are up to 16384-sided polygons (Volkov 1997).

Liu Hui's comments on the chapter discussing the volume of a pyramid illustrate the sophistication of his geometrical reasoning (cf. Wagner 1979). The figure he has to determine is a pyramid with rectangular base and one lateral edge perpendicular to the base, called a *yang ma* 陽馬. To arrive at the formula setting out its volume (namely one third length, times breadth, times height) he has to determine the proportions between it and two other figures, the *qian du* 塹堵 (right prism with right triangular base) and the *bie nao* 鱉臑 (a pyramid with right triangular base and one lateral edge perpendicular to the base). A *yang ma* and a *bie nao* together go to make up a *qian du*, and its volume is simple: it is half its length, times breadth, times depth. That leaves Liu Hui with the problem of finding the ratio between the *yang ma* and the *bie nao*. He proceeds by first decomposing a *yang ma* into a combination of smaller figures, a box, two smaller *qian du*, and two smaller *yang ma*. A *bie nao* similarly can be decomposed into two smaller *qian du* and two smaller *bie nao*. But once so decomposed it can be seen that the box plus two smaller *qian du* in the original *yang ma* are twice the two smaller *qian du* in the original *bie nao*. The parts thus determined stand in a relation of 2:1. The remaining problem is, of course, to determine the ratios of the smaller *yang ma* and the smaller *bie nao*: but an exactly similar procedure can be applied to them. At each stage more of the original figure has been determined, always yielding a 2:1 ratio for the *yang ma* to the *bie nao*. If the process is continued, the series converges on the formula one *yang ma* equals two *bie nao*, and so a *yang ma* is two-thirds of a *qian du*, which yields the requisite formula for the volume of the *yang ma*, namely one third length, times breadth, times height (Fig. 1.1.1).

Two points of particular interest in this stretch of argument are first that Liu Hui explicitly remarks on the uselessness of one of the figures he uses in his decomposition. The *bie nao*, he says, is an object that 'has no practical use'. Yet without it the volume of the *yang ma* cannot be calculated. At this point we have yet another clear indication that the interest in the exact geometrical result takes precedence over questions of practical utility.

Second, we may observe both a similarity and a difference between the procedure adopted by Liu Hui and some Greek methods. In such cases (as in Euclid's determination of the pyramid at *Elements* 12 3) the Greeks used an indirect proof, showing that the volume to be determined cannot be either greater or less than the result, and so must equal it. Liu Hui by contrast uses a direct proof, the technique of decomposition which I have described, yielding increasingly accurate approximations to the volume, a procedure similar to that used in the Chinese determination of the circle by inscribing regular polygons, mentioned above. Such a technique bears an obvious resemblance to the Greek method of exhaustion, though I remarked that in that method the area or volume to be determined was precisely not exhausted. Liu Hui sees that his process of decomposition can be

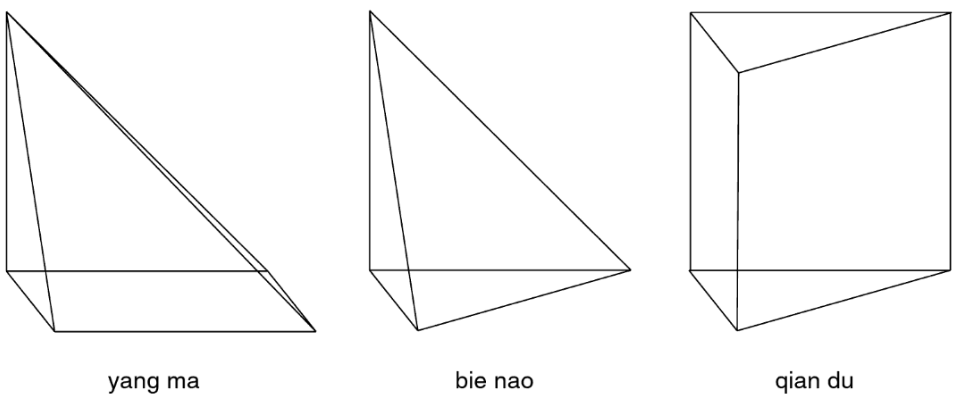

Figure 1.1.1 the *yang ma*, *bie nao*, **and** *qian du*

continued indefinitely, and he remarks on the progressively smaller remainders that this yields. We are dealing evidently with what we would call a converging series, but although Liu Hui has no explicit concept for the limit of such, he ends his investigation with the rhetorical question 'how can there be any remainder?'.

There is no suggestion, however, in any of the texts we have been considering, of giving mathematics an axiomatic base. The notion of axiom is absent from Chinese mathematics until the arrival of the Jesuits in the sixteenth century. Rather the chief aims of Chinese mathematicians were to explore the unity of mathematics and to extend its range. Liu Hui, especially, comments that it is the *same* procedures that provide the solutions to problems in different subject-areas. What he looks for, and finds, in such procedures as those he calls *qi* 齊 'homogenizing' and *tong* 同 'equalizing', is what he calls the *gang ji* 綱 紀 'guiding principles' of *suan* 'mathematics'. In his account of how, from childhood, he studied the 'Nine chapters', he speaks of the different branches of the study, but insists that they all have the same *ben* 本 'trunk'. They come from a single *duan* 端 'source'. The realizations and their *lei* 類 'categories', are elaborated mutually. Over and over again the aim is to find and show the connections between the different parts of *suan shu*, extending procedures across different categories, making the whole 'simple but precise, open to communication but not obscure'. Describing how he identified the technique of double difference, he says (92.2) he looked for the *zhi qu* 指 趣 'essential characteristics' to be able to extend it to other problems.

While Liu Hui is more explicit in all of this than the 'Nine chapters', the other great Han classic, the *Zhou bi*, represents the goal in very similar terms. We are not dealing with some isolated, maybe idiosyncratic, point of view, but with one that represents an important, maybe even the dominant, tradition. 'It is the ability to distinguish categories in order to unite categories' which is the key according to the *Zhou bi* (25.5). Again, among the methods that comprise the *Dao* 'Way', it is 'those which are concisely worded but of broad application which are the most illuminating of the categories of understanding. If one asks about one category and applies [this knowledge] to a myriad affairs, one is said to know the Way' (24.12ff., Cullen 1996, 177).

Conclusions

To sum up what our very rapid survey of two ancient mathematical traditions suggests, let me focus on just two fundamental points. We found many of the Greeks (not all) engaged in basic methodological and epistemological disagreements, where what was at stake was the ability to deliver certainty—to be able to do better than the merely persuasive or conjectural arguments that many downgraded as inadequate. The Chinese, by contrast, were far more concerned to explore the connections and the unity between different studies, including between those we consider to be mathematics and others we class as physics or cosmology. Their aim was not to establish the subject on a self-evident axiomatic basis, but to expand it by extrapolation and analogy.

Each of those two aims we have picked out has its strengths and its weaknesses. The advantages of axiomatization are that it makes explicit what assumptions are needed to get to which results. But the chief problem was that of identifying self-evident axioms that were not trivial. The advantage of the Chinese focus on guiding principles and connections was to encourage extrapolation and analogy, but the corresponding weakness was that everything depended on perceiving the analogies, since no attempt is made to give them axiomatic foundations. It is apparent that there is no one route that the development of mathematics had to take, or should have taken. We find good evidence in these two ancient civilizations for a variety of views of its unity and its diversity, its usefulness for practical purposes and for understanding. The value of asking the question 'what is mathematics?' is that it reveals so clearly, already where just two ancient mathematical traditions are concerned, the fruitful heterogeneity in the answers that were given.

Bibliography

Barker, A D, *Scientific method in Ptolemy's harmonics*, Cambridge University Press, 2000.

Bowen, A C, 'La scienza del cielo nel periodo pretolemaico', in S Petruccioli (ed), *Storia della scienza*, vol 1, Enciclopedia Italiana, 2001, 806–839.

Chemla, K and Guo Shuchun, *Les Neuf chapitres. Le Classique mathématique de la Chine ancienne et ses commentaires*, Dunod, 2004.

Cullen, C, *Astronomy and mathematics in ancient China: the Zhoubi Suanjing,* Cambridge University Press, 1996.

Cullen, C, *The Suan Shu Shu: writings on reckoning* (Needham Research Institute Working Papers, 1), Needham Research Institute, 2004.

Cuomo, S, *Ancient mathematics*, Routledge, 2001.

Ho Peng-Yoke, 'Chinese science: the traditional Chinese view', *Bulletin of the School of Oriental and African Studies*, 54 (1991), 506–519.

Huang Yilong and Chang Chih-Ch'eng, 'The evolution and decline of the ancient Chinese practice of watching for the ethers', *Chinese Science*, 13 (1996), 82–106.

Knorr, W, *The ancient tradition of geometric problems,* Birkhäuser, 1986.

Lloyd, G E R, and Sivin, N, *The way and the word*, Yale University Press, 2002.

Netz, R, *The shaping of deduction in Greek mathematics*, Cambridge University Press, 1999.

Netz, R, *Ludic proof*, Cambridge University Press, forthcoming.

Sivin, N, 'Cosmos and computation in early Chinese mathematical astronomy', in *Researches and Reflections vol 1: Science in Ancient China*, Variorum, 1995, ch. II.

Volkov, A, 'Zhao Youqin and his calculation of π', *Historia Mathematica*, 24 (1997), 301–331.

Wagner, D B, 'An early Chinese derivation of the volume of a pyramid: Liu Hui, third century AD', *Historia Mathematica*, 6 (1979), 164–188.

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CHAPTER 1.2

Mathematics and authority: a case study in Old and New World accounting

Gary Urton

The title of an article published by Alan Bishop, 'Western mathematics: the secret weapon of cultural imperialism' (1990), must surely be one of the most provocative in the recent literature concerning the history of mathematics and the nature and status of mathematical practice.[1] There are several surprises in this title, beginning with the adjective 'western'. According to Platonism, the grounding philosophy that informs the thinking of most mathematicians, mathematical truths lie beyond human experience, in an abstract realm set apart from language, culture, and history. In what sense, then, could mathematics be conceived of as preferentially linked to one or the other of the earthly hemispheres? And how could mathematics—the supposed dispassionate and logical investigation of arrangement, quantity, and related concepts in algebra, analysis, and geometry—be implicated in any meaningful way with such socially and politically loaded objects and concepts as 'weapons', 'culture', and 'imperialism'? Conveniently, Bishop's title provides an answer to this puzzle in the assertion that the association of mathematics with this disturbing set of modifiers is (or was) a 'secret'.

1. Thanks to Carrie Brezine and Julia Meyerson for their critical readings of drafts of this work. I alone am responsible for any errors of fact or logic that remain.

In the article in question, Bishop argues that western European colonizing societies of the fifteenth to nineteenth centuries carried with them to various exotic locales the gifts of rationalism and 'objectism' (that is, a way of conceiving of the world as composed of discrete objects that could be abstracted from their contexts), as well as a number of clearly formulated ways of employing mathematical ideas and procedures, all of which combined to promote western control over the physical and social environments in the colonies. Such regimes of power and control constituted what Bishop (1990, 59) terms a 'mathematico-technological cultural force' embedded in the colonies in institutions related to accounting, trade, administration, and education:

> Mathematics with its clear rationalism, and cold logic, its precision, its so-called 'objective' facts (seemingly culture and value free), its lack of human frailty, its power to predict and to control, its encouragement to challenge and to question, and its thrust towards yet more secure knowledge, was a most powerful weapon indeed. (Bishop 1990, 59)

When we look more broadly at the uses to which mathematics has been put, especially in accounting systems and in other administrative projects in ancient and modern states, it becomes clear that what is ideally conceived of as the fine, elegant, and dispassionate art of mathematics has in many times and places been intimately linked to systems and relations of authority in a wide range of ideological, philosophical, and political programs and productions. The central questions that we will address here in relation to this history are: how has the linkage between mathematics and authority come about? And how and why has this relationship evolved in the particular ways it has in different historical settings?

To speak of a relationship between mathematics and authority is by no means to limit the issues to imperialist administrative regimes. It also arises in other settings, from the authority that emerges among mathematicians as a result of the successful execution of mathematical proofs, to the attempt by those steeped in the measurement and quantification of social behaviors to adopt math-based paradigms for ordering society (see Mazzotti, Chapter 3.3 in this volume). In short, what we will be concerned with here are a number of problems connected with the manipulation of numbers by arithmetical procedures and mathematical operations and the ways these activities enhance authority and underlie differences in power between different individuals and/or groups or classes in society—for example, between bureaucrats and commoners, or, as in the particular setting to be discussed below, between conquerors and conquered.

We will address the questions raised above in three different but historically related cultural and social historical contexts. The first concerns mathematical philosophies and concepts of authority in the West in the centuries leading up to the European invasion of the New World. This section will include an overview of the rise of double entry bookkeeping in European mercantile capitalism. Next, we will examine the practice of *khipu* (knotted-string) record-keeping in the Inka empire of the Pre-Columbian Andes. And, finally, we will examine the encounter between Spanish written (alphanumeric) record-keeping practices and Inka knotted-string record-keeping that occurred in the Andes following the European invasion and conquest of the Inka empire, in the sixteenth century.

Accounting, authority, power, and legitimacy

A wealth of literature produced by critical accounting historians over the past several decades has elucidated the role of accounting as a technology of, and a rationality for, governance in state societies. Accounting and its specialized notational techniques are some of the principal instruments employed by states in their attempts to control and manage subjects (Hoskin and Macve 1986; Miller and O'Leary 1987; Miller 1990). As Miller has argued:

> Rather than two independent entities, accounting and the state can be viewed as interdependent and mutually supportive sets of practices, whose linkages and boundaries were constructed at least in their early stages out of concerns to elaborate the art of statecraft. (Miller 1990, 332)

A focus on accounting is one of the most relevant approaches to take in examining Andean and European (Spanish) mathematical practices, as this was the context of the production of most of the documentation deriving from mathematical activities in these two societies that is preserved in archives and museums. The *khipu* was, first and foremost, a device used for recording information pertaining to state activities, such as census-taking and the assessment of tribute; this was also true of the information recorded by Spanish bureaucrats in written documents in the administration of the crown's overseas holdings. For instance, among the some 34,000 *legajos* (bundles of documents) deriving from Spanish colonial administration in the New World, preserved today in the Archivo de Indias in Seville, the largest collections—other than those labeled *Indiferente* 'miscellaneous/unclassified'—are those categorized under the headings *Contaduría* 'accountancy' (1953 *legajos*) and *Contratación* 'trade contracts' (5873 *legajos*; Gómez Cañedo 1961, 12–13). Focusing on accounting will, therefore, provide us with the best opportunity for investigating the relative complexity of arithmetic and mathematical practices employed in the records of these two states, as well as similarities and differences in their principles of quantification.

Although the focus of this essay is on the relationship between mathematics and authority in the context of accounting, we will not get far in our examination of these concepts and domains of human intellectual activity without first developing a clear sense of the meaning of 'authority' and discussing how this concept relates to the wider field of social and political relations that includes legitimacy, power, and social norms. The principal figure whose work must be engaged on these topics is, of course, Max Weber (1964). Insofar as the question of power is concerned, Weber famously defined this concept as '... the probability that one actor within a social relationship will be in a position to carry out his[/her] own will despite resistance, regardless of the basis on which this probability rests' (cited in Uphoff 1989, 299). It is clear from this definition that power is inextricably linked to authority and legitimacy. Uphoff makes a forceful argument to the effect that authority should be understood as a *claim* for compliance, while legitimacy should be understood as an *acceptance* of such a claim. Thus, different persons are involved in such power relationships; on the one hand there are 'the authorities' and on the other there are those who are subject to and accept the claims of the authorities (Uphoff 1989, 303). Thus, the three central concepts we are concerned with are linked causally in the sense that the power associated with authority depends on the legitimacy accorded to it.

Weber identified three principal types of authority, each having a particular relationship to norms. One type, referred to as 'charismatic authority', which may be embodied by the prophet or the revolutionary, Weber considered the purest form of authority in that, in coming into being, it breaks down all existing normative structures. In 'traditional authority', the leader comes into power by heredity or some other customary route, and the actions of the leader are in turn limited by custom. Thus, in traditional systems of authority, norms generate the leader, and one who comes into such a position of authority—the king, chief, or other hereditary leader—depends on traditional norms for his/her authority. Finally, in what Weber termed 'legal-rational authority', the leader occupies the highest position in a bureaucratic structure and derives authority from the legal norms that define the duties and the jurisdiction of the office he/she occupies (Spencer 1970, 124–5).

In terms of the relationship between types of authority and forms of political rule relevant to our study, both the Inka state under its (possibly dual) dynastic rulers, as well as the Spanish kings of the Hapsburg dynasty, experienced processes of increasing regularization of bureaucratic procedures from traditional to rational-legal authority structures during the century or so leading up to the European invasion of the Andes. Our study will examine ways in which mathematical activities linked to accounting practices in pre-modern states in the Old and New Worlds served to legitimize or empower particular individuals or classes in their claims for compliance of the exercise of their will. Our task will be particularly challenging because we will examine these matters in the context of the Spanish conquest of the Inka empire, a historical conjuncture that brought two formerly completely unrelated world traditions of mathematics and authority into confrontation with each other.

Two almost simultaneous developments in European mathematics and commercialism during the fourteenth and fifteenth centuries are critical to the picture we are sketching here of accounting and record-keeping practices of

Spanish colonial administrators in the sixteenth century. These developments were the invention of double-entry bookkeeping and the replacement of Roman numerals by Hindu-Arabic numerals.

The earliest evidence for double-entry bookkeeping dates from the thirteenth century when the method was put to use by merchants in northern Italy (Yamey 1956; Carruthers and Espeland 1991). The first extended explanation of double-entry bookkeeping appeared in a treatise on arithmetic and mathematics written by the Franciscan monk Luca Pacioli in 1494 (Brown and Johnston 1984). In the double-entry method, all transactions are entered twice, once as a debit and again as a credit (Fig. 1.2.1). Daily entries are posted to a journal, which are later

Hypothetical Medieval Ledger Postings
based on Luca Pacioli's Directions

In the Name of God

+Jesus MCDIII On this day, Cash shall give to Capital CLI lire in the form of coin. CLI lire Cr. ref. page	+Jesus MCDIII On this day, Capital shall have from Cash in the form of coin CLI lire. CLI lire Dr. ref. page
+Jesus MCDLXXX Giovanni Bessini shall give, on This day, CC lire, which he promised to pay to us at our pleasure, for the debt which Lorenzo Vincenti owes us. CC lire Cr. ref. page	+Jesus MCDLXXX Giovanni Bessini shall have back on Nov. II, the CC lire, which he deposited with us in cash. CC lire Dr. ref. page
+Jesus MCDLXXIV On this day, Jewels with a value DLXX lire, shall give to Capital DLXX lire Cr. ref. page	+Jesus MCDLXXIV On this day, Capital shall have of from Jewels, a value of DLXX lire. DLXX lire Dr. ref. page
+Jesus MCDXXX On this day, Business Expense for office material worth CCC lire Shall give to Cash CCC lire Cr. ref. page	+Jesus MCDXXX On this day, Cash shall have from Business Expense CCC lire. CCC lire Dr. ref. page

Figure 1.2.1 Double entry book-keeping ledger postings based on Luca Pacioli's (1494) directions (Aho 2005, 71, Table 7.2)

transferred to a ledger. The ledger books provide the material for the process of accounting, which relies on the equation: assets = liabilities + equity. For books to remain in balance, a change in one account (a debit or credit) must be matched by an equal change in the other. In the rhetorical form in which Pacioli presented the method, the balancing of accounts by double-entry was constructed as an undertaking that had deep religious and moral implications.

The invention and implementation of double-entry went hand-in-hand with the replacement of Roman numerals by Hindu-Arabic numerals, which had been introduced into western Europe almost five hundred years before their eventual acceptance into accounting practice in the fifteenth century. Ellerman (1985, 232) argues that what is distinctive about double-entry is not that it relates two or more accounts, as that is a characteristic of the transaction itself; rather, the distinction of double-entry is that this is a new system of *recording* transactions. Double-entry required complex mathematics based on an efficient system of numbers—like Hindu-Arabic numerals, rather than the cumbersome Roman numerals. There are extensive literatures documenting (Swetz 1989, 11–13; Durham 1992, 48–49) and demonstrating (Donoso Anes 1994, 106) that the coupling of Hindu-Arabic numerals and double-entry in accounting had a powerful affect in promoting increasing rationality in business, society, and politics. There is controversy over whether capitalism was nurtured initially and primarily by Catholicism, with its emphasis on penance and confession constituting a form of accounting (Sombert 1967; Aho 2005), or by Protestantism (Weber 1958). However, those arguing on both sides of this question agree that the spread of double-entry bookkeeping throughout western Europe was a central component of the increasing rationalization and standardization associated with the rise of mercantile capitalism (Carruthers and Espeland 1991, 32; Aho 2005).

While the centers of development of double-entry bookkeeping were the burgeoning mercantile city-states of northern Italy, the method soon spread to other regions of western Europe, including the Iberian peninsula. From detailed study of accounts pertaining to the sale of gold and silver brought from the Americas kept in the *Casa de Contratación* 'Treasury House', in Seville, Donoso Anes (1994) has shown convincingly that the double-entry method was employed in the central accounts of the Royal Treasury of Castille from as early as 1555. In fact, Spain was the first European country to issue laws (in 1549 and 1552) compelling merchants to apply the double-entry method, as well as the first country in which the method was implemented by a public institution—the *Casa de Contratación* (Donoso Anes 1994, 115). Furthermore, Spanish merchants appear to have taught the method to English traders (Reitzer 1960, 216), and they were instrumental in developing and passing on to French merchants the practice of drawing bills of exchange (Lapeyre 1955, 22; cited in Reitzer 1960, 216). While double-entry was used in Spanish accounting for the sale and minting of gold and silver by the Royal Treasury, single-entry accounts were kept at the same time, primarily as the official accounting procedure controlling the activities of the treasurer of the *Casa de Contratación* (Donoso Anes 1994, 115; cf. Klein and Barbier 1988, 54; Hoffman 1970, 733).

The cities of northern Italy that were the centers of commercial activities from the fourteenth to the sixteenth centuries also became centers of learning in arithmetic and mathematics. It was in these cities—Venice, Bologna, Milan—that Hindu-Arabic numerals were first linked with double-entry to form the basis of modern accounting science. It was here as well that abacus or 'reckoning' schools grew up that were patronized by the sons and apprentices of merchants throughout Europe. The masters of those schools, the *maestri d'abbaco*, taught the new arithmetic, or *arte dela mercadanta*, 'the mercantile art' (Swetz 1989, 10–16). It was in northern Italy as well where, a couple of decades prior to the publication of Pacioli's exposition of double-entry bookkeeping, the first arithmetic textbook, the so-called *Treviso arithmetic*, was published in 1478 (Swetz 1989). While not discussing the double-entry method itself, the *Treviso arithmetic* proclaimed itself from the opening passage as intended for study by those with an interest in commercial pursuits (Swetz 1989, 40).

This, then, was a new kind of authority in mathematics, one that was grounded not in theoretical considerations, but rather with a mathematics that served the practical needs and interests of the merchant. The efficacy of this new mathematics was determined not by how closely it cleaved to some body of theoretical principles or philosophical values, but rather by how well it tracked the debits, credits, and profit fluctuations of merchant capitalists, how well it served in arbitrating disputes, and its overall contribution to the well-being of those who put the methods into practice. This new mathematics of the fifteenth century both stimulated and reflected the development of mercantilism and economic accounting and administration throughout Europe, and it was this mathematical practice that was transplanted to the New World in the fifteenth and sixteenth centuries as the basis of accounting for trade, tribute, and the growth of wealth in the American colonies.

From virtually the earliest years following the invasion of the Andes, European administrators—toting accounting ledgers filled with columns of Hindu-Arabic numerals and alphabetically-written words and organized in complex formats—came into contact with Inka administrative officials wielding bundles of colorful knotted cords. These local administrators—known as *khipukamayuqs* 'knot-keepers/makers/organizers'—were, oddly enough, speaking the language (in Quechua) of decimal numeration and practicing what may have looked for all the world, to any Spaniard trained by the reckoning masters of northern Italy, like double-entry bookkeeping.

A new world of knotted-cord record keeping

Khipus are knotted-string devices made of spun and plied cotton or camelid fibers (Figure 1.2.2).[2] The colors displayed in *khipus* are the result of the natural colors of cotton or camelid fibers or of the dyeing of these materials with natural dyes. The 'backbone' of a *khipu* is the so-called primary cord—usually around 0.5 cm in diameter—to which are attached a variable number of thinner strings, called pendant cords. *Khipus* contain from as few as one up to as many as 1500 pendants (the average of some 450+/- samples studied by the Harvard *Khipu* Database project is 84 cords). Top cords are pendant-like strings that leave the primary cord opposite the pendants, often after being passed through the attachments of a group of pendant strings. Top cords often contain the sum of values knotted on the set of pendant cords to which they are attached. About one-quarter of all pendant cords have second-order cords attached to them; these are called subsidiaries. Subsidiaries may themselves bear subsidiaries, and there are examples of *khipus* that contain up to thirteen levels of subsidiaries, making

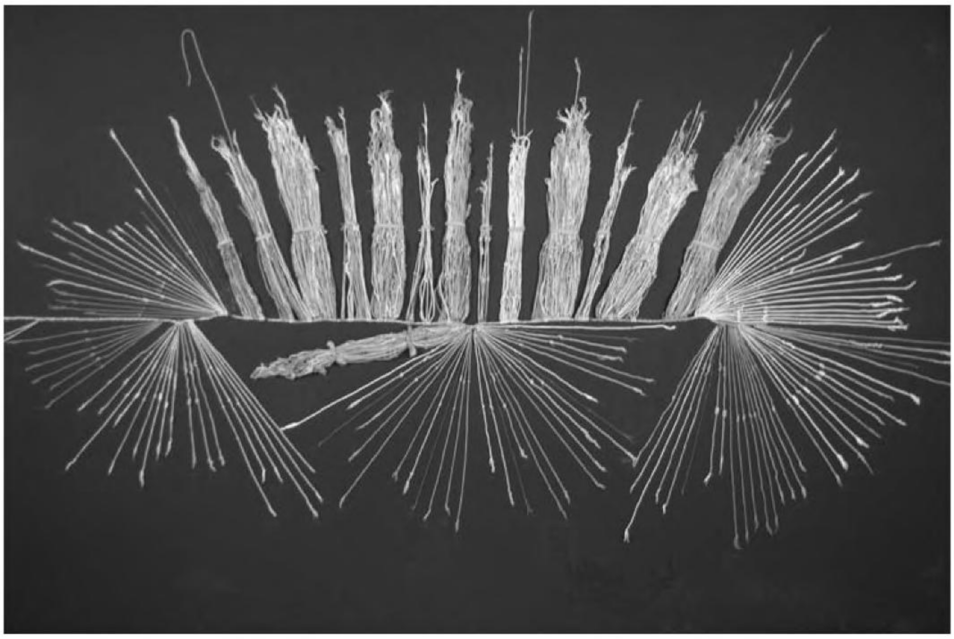

Figure 1.2.2 A *khipu* from Museum for World Culture, Göteborg, Sweden (#1931.37.0001 [UR113])

2. According to my own inventory, there are some 780+/- *khipu* samples in museums and private collections in Europe, North America, and South America. While many samples are too fragile to permit study, almost 450 samples have been closely studied to date. Observations on a few hundred *khipus* may be viewed at <http://khipukamayuq.fas.harvard.edu/> and <http://instruct1.cit.cornell.edu/resear4ch/quipu~ascher/>.

the *khipu* a highly efficient device for the display of hierarchically organized information.[3]

The majority of *khipus* have knots tied into their pendant, subsidiary, and top strings (Locke 1923; Pereyra 2001). The most common knots are of three different types, which are usually tied in clusters at different levels in a decimal place system of numerical registry (Fig. 1.2.3).[4] The most thorough treatment to date of the numerical, arithmetic, and mathematical properties of the *khipus* is Ascher and Ascher's *Mathematics of the Incas: code of the quipus* (1997; see also Urton 1997; 2003). The Aschers have shown that the arithmetic and mathematical operations used by Inka accountants included, at a minimum, addition, subtraction, multiplication, and division; division into unequal fractional parts and into proportional parts; and multiplication of integers by fractions (Ascher and Ascher 1997, 151–2).

What kinds of information were registered on the *khipus*? In addressing this question, it is important to stress that, although we are able to interpret the quantitative data recorded in knots on the *khipus*, we are not yet able to read the accompanying nominative labels, which appear to have been encoded in the colors, twist, and other structural features of the cords. The latter would, were we able to read them, presumably inform us as to the identities of the items that were being enumerated by the knots. Thus, in discussing the identities of objects accounted for in the *khipus*, we are forced to rely on the Spanish documents from the early years following the European invasion.

According to the Spanish accounts, records were kept of censuses, tribute assessed and performed, goods stored in the Inka storehouses, astronomical periodicities and calendrical calculations, royal genealogies, historical events, and so on (see Murra 1975; Zuidema 1982; Julien 1988; Urton 2001; 2002; 2006). The overriding interest in the recording, manipulation and eventual archiving of quantitative data in the *khipus* was the attempt to control subject peoples throughout the empire. This meant being able to enumerate, classify, and retain records on each subject group. The most immediate use to which this information was put was the implementation of the labor-based system of tribute. Tribute in the Inka state took the form of a labor tax, which was levied on all married, able-bodied men (and some chroniclers say women as well) between the ages of 18 and 50. In its conception and application to society, Inka mathematics appears to have taken a form remarkably like the political arithmetic of seventeenth-century Europeans.[5] In sum, the decimal place system of recording values—including zero (Urton 1997, 48–50)—of the

3. For general works on *khipu* structures and recording principles, see Urton (1994; 2003); Ascher and Ascher (1997); Arellano (1999); Conklin (2002); Radicati di Primeglio (2006).

4. Approximately one-third of *khipu* studied to date do not have knots tied in (decimal-based) tiered arrangements. I have referred to these as 'anomalous *khipu*' and have suggested that their contents may be more narrative than statistical in nature (Urton 2003).

5. See the discussions of Inka arithmetic and mathematics in Ascher (1992); Ascher and Ascher (1997); Pereyra (2001); and Urton (1997).

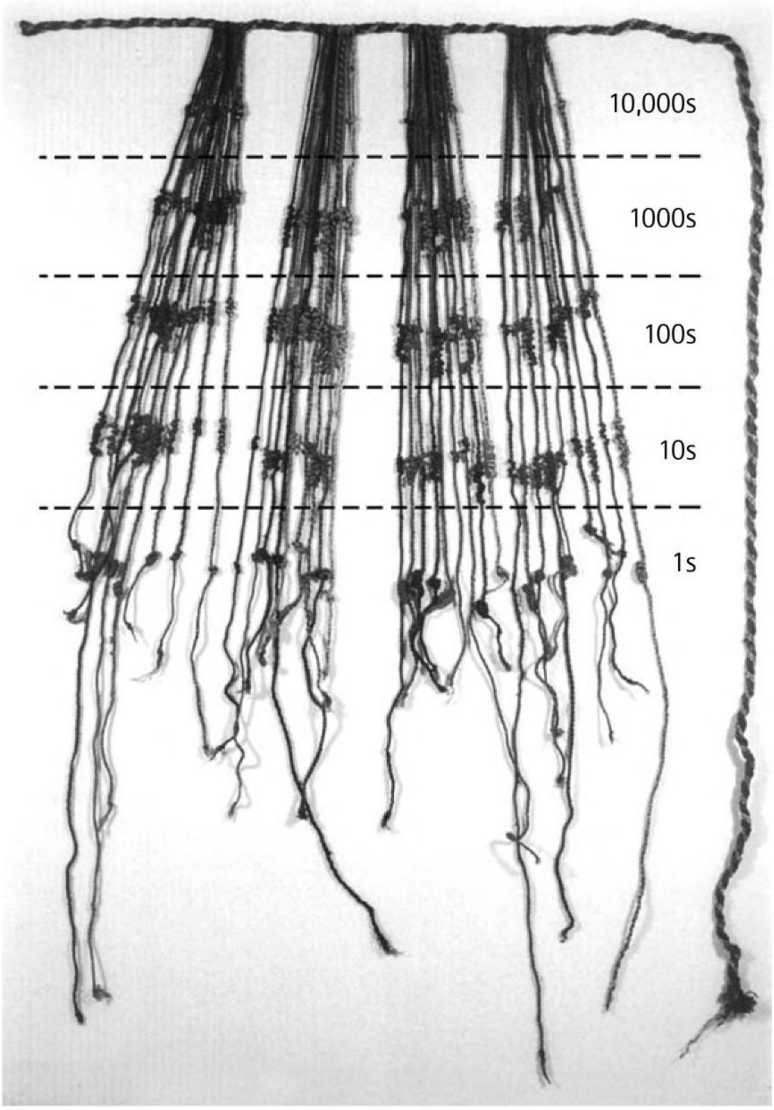

Figure 1.2.3 A clustering of knots on a *khipu* in decimal hierarchy

Inka knotted-cords was as precise and complex a system of recording quantitative data as the written Hindu-Arabic numeral-based recording system of Europeans at the time of the conquest, although the records of the former were not as rapidly produced, nor as easily changeable, as those of the latter.

Richardson (1987, 341) has argued that accounting has long been one of the principal institutions and administrative practices involved in maintaining and legitimizing the status quo in western European nation-states. Can this be said of *khipu* accounting in the pre-Hispanic Andes as well? We gain a perspective on this question by looking at two accounts of how censuses were carried out in the

Inka state. As in other ancient societies, census-taking was a vital practice in the Inka strategy of population control, as well as serving as the basis for the assessment and eventual assignment of laborers in the *mit'a* (taxation by labor) system (Murra 1982; Julien 1988). The first account of census-taking is from the famed mid-sixteenth century soldier and traveller, Cieza de León:

the nobles in Cuzco told me that in olden times, in the time of the Inka kings, it was ordained of all the towns and provinces of Peru that the head men [*señores principales*] and their delegates should [record] every year the men and women who had died and those who had been born; they agreed to make this count for the payment of tribute, as well as in order to know the quantity of people available to go to war and the number that could remain for the defense of the town; they could know this easily because each province, at the end of the year, was ordered to put down in their *quipos*, in the count of its knots, all the people who had died that year in the province, and all those that had been born.[6] (Cieza de León 1967 [1551], 62; my translation)

Some forty years after Cieza wrote down the information cited above, Martín de Murúa gave an account of Inka census-taking that varies somewhat from Cieza's understanding of this process and that contains interesting details concerning the actual procedures involved in local population counts.

They sent every five years *quipucamayos* [*khipu*-keepers], who are accountants and overseers, whom they call *tucuyricuc*. These came to the provinces as governors and visitors, each one to the province for which he was responsible and, upon arriving at the town he had all the people brought together, from the decrepit old people to the newborn nursing babies, in a field outside town, or within the town, if there was a plaza large enough to accommodate all of them; the *tucuyricuc* organized them into ten rows ['streets'] for the men and another ten for the women. They were seated by ages, and in this way they proceeded [with the count]...[7] (Murúa 2004 [1590], 204; my translation)

Late sixteenth-century drawings—what we could term 're-imaginings'—of these male and female accounting events from the chronicle of Martín de Murúa, are shown in Figs. 1.2.4 and 1.2.5.

One would be hard put to find better examples than the two quotations cited above, and the images of census events in Figs. 1.2.4 and 1.2.5, of what Michel

6. ...concuerdan los orejones que en el Cuzco me dieron la relación, que antiguamente, en tiempo de los reyes Incas, se mandaba por todos los pueblos y provincias del Perú que los señores principales y sus delegados supiesen cada año los hombres y mugeres que habían sido muertos y todos los que habían nacido; porque, así para la paga de los tributes como para saber la gente que había para la Guerra y la que podia quedar por defensa del pueblo, convenía que se tuviese ésta [cuenta]; la cual fácilmente podían saber porque cada provincia, en fin del año, mandaba asentar en los quipos por la cuenta de sus nudos todos los hombres que habían muerto en ella en aquel año, y por el [con]siguiente los que habían nacido.

7. Enviaba de cinco a cinco años *quipucamayos*, que son contadores y veedores, que ellos llaman *Tucuyricuc*. Estos venían por sus provincias como gobernadores y visitadores, cada uno en las que le cabía, y llegado al pueblo hacía juntar toda la gente, desde los viejos decrépitos hasta los indios niños de teta y en una pampa o plaza, si la había, hacían estos gobernadores, llamados *Tucuyricuc*, señalar diez calles para los indios y otras diez para las indias, con mucho orden y concierto, en que por las edades ponían los dichos indios con mucha curiosidad y concierto...

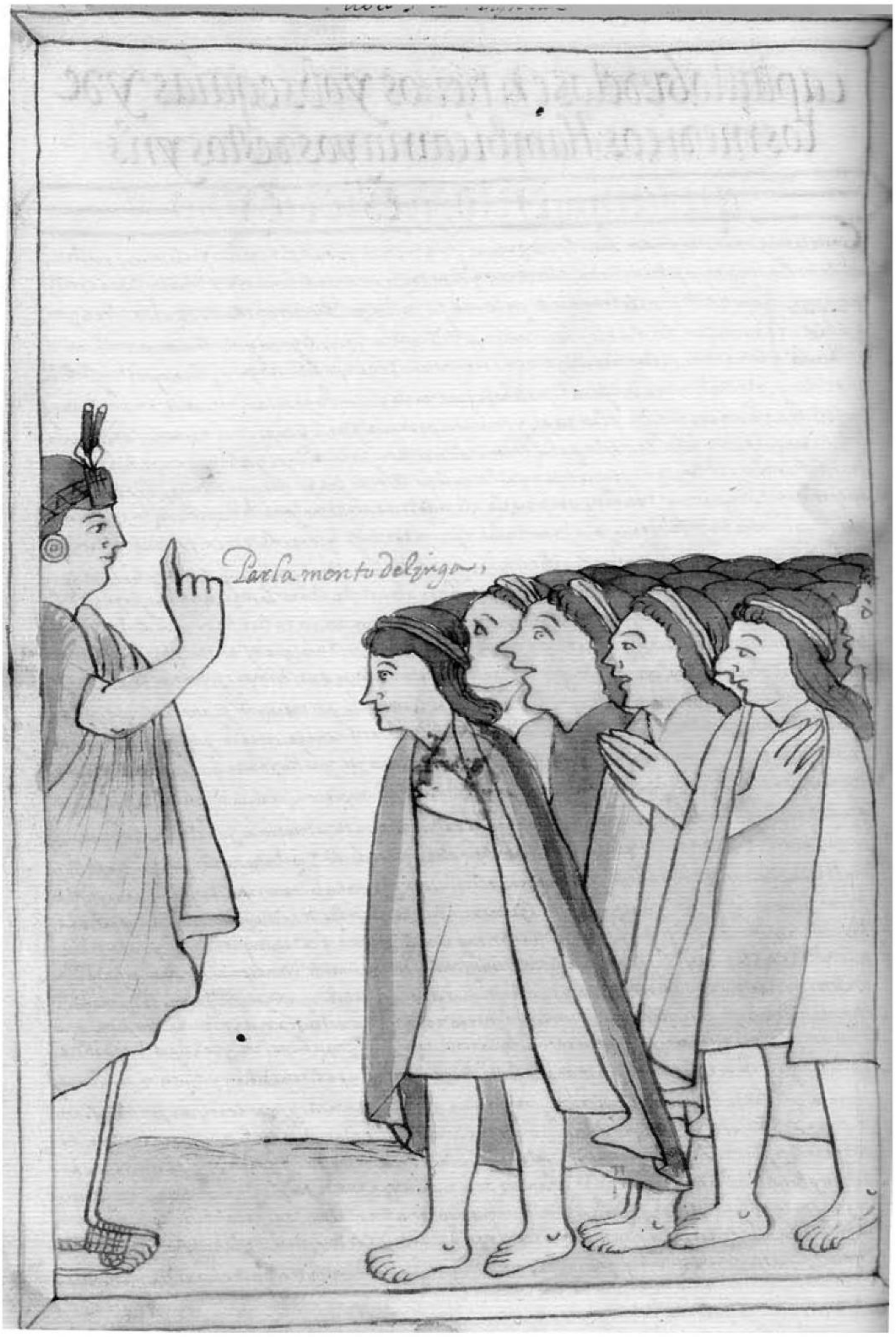

Figure 1.2.4 Conducting a census count of men, by age-grade (Murúa 2004, 114v)

Figure 1.2.5 Conducting a census count of women, by age-grade (Murúa 2004, 116v)

Foucault characterized as the disciplinary power of state institutions—what he termed power/knowledge structures—as they attend to the work of social surveillance and the control of the bodies of subjects. The result of such procedures was, as Foucault noted, the production of subjects that cooperate and connive in their own subjection (Foucault 1977, 184–187; Hoskin and Macve 1986, 106; Stewart 1992). In Inka census-taking, people were ordered into public spaces to be counted and classified. Although resistance and evasion may have been common in such proceedings, from what the Spanish chroniclers and administrative officials tell us, Inka censuses were accomplished using non-coercive measures—that is, local people apparently were compliant with the claims of authority coming from local officials and state administrators. Such surveillance, reporting, and social control procedures are examples of what Foucault termed a disciplinary, as opposed to sovereign, form of power.

> Sovereign power is identified as a diminished form of power. Its ultimate recourse is seizure—of things, of bodies and ultimately of life. Disciplinary power is much richer and entails penetrating into the very web of social life through a vast series of regulations and tools for the administration of entire populations and of the minutiae of people's lives. (cited in Miller and O'Leary 1987, 238)

Thus, as much as an accounting tool, the census *khipu* was an instrument for the performance and display of state authority and power within local communities.[8] The census data collected by local record-keepers were knotted into *khipus*, copies were made of each record, and the data were subsequently reported to higher-level accountants in regional and provincial administrative centers (see Urton and Brezine 2005). Two issues arise with respect to these procedures: one concerns the practice of making one or more copies of *khipu* records, the other concerns the training and education of state record-keepers.

While there are a number of references in the Spanish chronicles to *khipu* copies, the study of such copies in the corpus of extant *khipus* has proceeded slowly. Recent advances have come about, however, following the development of a searchable database—the *Khipu* Database (KDB).[9] From searches of the 450 or so samples included in the KDB, some 12–15 examples of copies of accounts have been identified (Urton 2005). While referred to as duplicate, or 'matching' *khipus*, we could also consider 'pairs' of *khipus* to represent an original and a copy.

Copies (or matching) *khipus* occur in three different forms. First, there are examples in which the numerical values on a sequence of strings on one sample

8. Guevara-Gil and Salomon (1994) have discussed what were similar procedures, and results, in the censuses undertaken by Spanish *visitadores* (administrative 'visitors') who were responsible for counting, classifying, and (re-)organizing local populations in the early colonial Andes.

9. The *Khipu* Database project (KDB), located in the Department of Anthropology, Harvard University, is described fully on the project website <http://khipukamayuq.fas.harvard.edu/>. I gratefully acknowledge the following research grants from the National Science Foundation, which made the creation of the KDB possible: #SBR-9221737, BCS-0228038, and BCS-0408324. Thanks also to Carrie J Brezine, who served as *Khipu* Database Manager from 2002 to 2005.

are repeated exactly on another *khipu*. In some samples of this type, we find that while the pair of *khipus* bears the same knot values, the colors of the strings may vary (see Urton 2005, 150–151). The second type of matching *khipus*, which I have termed 'close matches', involves instances in which two different samples contain not exactly matching sequences of numbers, but rather ones in which the values are similar (for example, those of one sample varying a small amount from those on another sample). And, finally, we have examples in which a numerical sequence recorded on one cord section of a *khipu* are repeated exactly, or closely, on another section of cords of that same *khipu*.

I argued elsewhere (Urton 2005) that duplicate *khipus* may have been produced as a part of a system of 'checks and balances'. However, duplicates seem also to possess most of the requisite elements of double-entry bookkeeping in which 'all transactions were entered twice, once as a debit and once as a credit…The debit side pertained to debtors, while the credit side pertained to creditors' (Carruthers and Espeland 1991, 37). Close matches would be accounts in which the debits and credits sides of the ledger were not in balance. On pairs of *khipus* having identical numerical values on sequences of strings, but in which string colors vary (Urton 2005, 150–151), color could have been used to signal the statuses of credits and debits in the matching accounts.[10] In the Inka state, debit/credit accounting would have been employed primarily in relation to the levying of labor tribute on subject populations.

The principal information that we lack in order to be able to confirm whether or not duplicate *khipus* might have been produced and used as double-entry-like accounts are the identities of the objects recorded on the *khipus*. Since we still cannot read the code of the *khipus*, we are unable to determine whether paired accounts were simply copies made for the purposes of checks-and-balances or if they might represent a relationship between a debit for an item on one account and the credit for that same item on another account. Research into this matter is on-going.[11]

What can we say about the individuals who became *khipu*-keepers for the state? How were these individuals recruited and trained? What role did they play in exercising authority and maintaining social and political control in the Inka state? The late sixteenth-century chronicler Martín de Murúa provided the

10. It is interesting to note that in early Chinese bookkeeping, red rods signified positive numbers while black rods were used for negative numbers. As Boyer noted, '[f]or commercial purposes, red rods were used to record what others owed to you and black rods recorded what you owed to others' (cited in Peters and Emery 1978, 425).

11. Three articles published in the 1960s and 1970s by economists and accounting historians contain a lively debate not only about whether or not the *khipus* contained double-entry bookkeeping, but about the claim made by one of the disputants (Jacobsen) to the effect that the Inkas may in fact have invented the technique (Jacobsen 1964; Forrester 1968; Buckmaster 1974). There is not space here to review the arguments made in these three articles. Suffice it to say that, while interesting for historical purposes, these articles are all poorly informed about the nature of the *khipus*, about what the Spanish documents say about their use, as well as about Inka political and economic organizations.

following account of a school that was set up in the Inka capital of Cusco for the training of *khipu*-keepers.

The Inca... he set up in his house [palace] a school, in which there presided a wise old man, who was among the most discreet among the nobility, over four teachers who were put in charge of the students for different subjects and at different times. The first teacher taught the language of the Inca... and upon gaining facility and the ability to speak and understand it, they entered under the instruction of the next [second] teacher who taught them to worship the idols and the sacred objects [*huacas*]... In the third year the next teacher entered and taught them, by use of *quipus*, the business of good government and authority, and the laws and the obedience they had to have for the Inca and his governors... The fourth and last year, they learned from the other [fourth] teacher on the cords and *quipus* many histories and deeds of the past.[12] (Murúa 2001, 364; my translation).

The curriculum of the young administrators aimed at engendering loyalty to the Inka and adherence to state values, policies, and institutions. The *khipu* studies component of the administrative curriculum fulfilled what Miller and O'Leary (1987) have referred to as accounting education's objective of producing 'governable persons' who themselves went on to administer for the state in the provinces. The curriculum also incorporated what has been described as a process whereby examination, discipline, and accounting are bound together to empower texts, rationalize institutional arrangements for state interests and, ultimately, to transform the bodies of the persons subjected to training (Hoskin and Macve 1986, 107).

The situation outlined above was not to last for long, as less than half a century after the school of administration was set up, a cataclysmic event brought the school, not to mention the entire imperial infrastructure, crashing down; this event was the Spanish conquest.

Conquest, colonization, and the confrontation between knot- and script-based texts

The story of the conquest of the Inka empire by the Spaniards, which was undertaken by Francisco Pizarro and his small force of around 164 battle-hardened *conquistadores*, beginning in 1532, has been told too many times—in all its astonishing

12. Dijo el Ynga... puso en su casa una escuela, en la cual presidía un Viejo anciano, de los más discretos *orejones*, sobre cuatro maestros que había para diferentes cosas y diferentes tiempos de los discípulos. El primer maestro enseñaba al principio la lengua del Ynga... Acabado el tiempo, que salían en ella fáciles, y la hablaban y entendían, entraban a la sujeción y doctrina de otro maestro, el cual les enseñaba a adorar los ídolos y sus *huacas*... Al tercer año entraban a otro maestro, que les declaraba en sus *quipus* los negocios pertenecientes al buen gobierno y autoridad suya, y a las leyes y la obediencia que se había de tener al Ynga y a sus gobernadores... El cuarto y postrero año, con otro maestro aprendían en los mismos cordeles y *quipus* muchas historias y sucesos antiguos...

and entrancing/appalling details—for me to add much to the telling in the space available here (see Hemming 1970). The events of the conquest and the processes of colonization that are relevant for our discussion here are the following. The initial battle of conquest, which occurred in November 1532 in the Inka provincial center of Cajamarca, in the northern highlands of what is today Peru, resulted in the defeat of the Inka army and the capture and execution of Atahualpa, one of two contenders for succession to the Inka throne. Pizarro then led his small force southward, arriving in the Inka capital city of Cuzco in 1534. The Spaniards and their native allies were soon forced to defend Cuzco against a rebellion led by the Spanish-installed puppet-king, Manco Inca. This gave rise to a decades-long war of pacification of the rebels, which finally came to an end in 1572 with the execution of the then rebel leader, Tupac Amaru (Hemming 1970).

Three years prior to the capture and execution of Tupac Amaru, a new Viceroy of Peru (the fourth), Francisco de Toledo, had arrived in Peru with a mandate to put down the rebellion and to transform the war- and disease-ravaged land of the former Inka empire into an orderly and productive colony for the benefit of the king of Spain, Philip II. Viceroy Toledo instituted a set of reforms that were in some respects a continuation of certain of the processes of pacification, reorganization, and transformation that had been on-going since the earliest days following the initial conquest. In other ways, Toledo's reforms represented something completely new, different, and profoundly transformative in their effects on Andean ways of life (Stern 1993, 51–79).

The end result of the Toledan reforms, the clear shape of which became manifest by the mid-to-late 1570s, included, most centrally, the following institutions: *encomiendas*—grants of groups of Indians to Spanish *encomenderos* 'overseers' who were charged with the care and religious indoctrination of the natives and who, in exchange, had the right to direct native labor for their personal benefit but without the right (after the Toledan reforms) to levy tribute demands on them; *corregimientos*—territorial divisions for the management and control of civil affairs, including (theoretically) oversight of the *encomenderos*; *reducciones*—newly-formed towns that were laid out in grid-like ground plans to which the formerly dispersed natives were transferred for their surveillance, control, and indoctrination; *doctrinas*—parish districts staffed by clergy who attended to the religious indoctrination of the natives within the reducciones and who received a portion of the tribute for their own maintenance; and *mita*—a form of labor tax based on the Inka-era *mit'a*, which supplemented what was, for Andeans, a new kind of tribute imposed on them by Toledo: specified quantities of agricultural produce, manufactured goods (textiles, sandals, blankets), and coinage (Rowe 1957; Ramírez 1996, 87–102).

The census was a critical institution for reorganizing Andean communities. Spanish censuses were carried out by administrative *visitadores* 'visitors' who produced documents, known as a *visitas*, which were detailed enumerations of the population in the *reducciones* broken down (usually) into household groupings. Each household member was identified by name, age and—in the case of adult males—*ayllu* 'social group' affiliation (Guevara-Gil and Salomon 1994; Urton 2006). The *visitadores* were usually joined in their rounds by the *kurakas* 'local lords' and often by the local *khipukamayuqs*. The *khipu*-keepers could supply historical, corroborating information on population figures and household composition (Loza 1998). It is important to stress that participation by the native record-keepers was not primarily for the benefit of the Spaniards, rather, it was to ensure that the natives would have their own, *khipu*-based accounts of the enumeration in the event—which seems always and everywhere to have come to pass—that a dispute arose over the population count, the amount of tribute levied, or other administrative questions.

There are two contexts in which I will explore native Andean encounters with Old World mathematical principles and practices, each of which was linked to a wholly new relation of authority and power: the manner of collecting information pertaining to the censuses, and the striking and circulation of coinage. These practices were closely linked to new forms of tribute, as well as to what was, for Andean peoples, a completely new form of communication: writing—that is, the inscribing of marks in ordered, linear arrangements on paper, parchment, or some other two-dimensional surface. Such a medium and associated recording technology were unprecedented in the Andean world.

There have been numerous important works published in recent years on the confrontation between *khipu* records and alphabetic texts in the early colonial Andes (Rappaport and Cummins 1994, 1998; Mignolo 1995; Brokaw 1999; Quilter and Urton 2002; Fossa 2006; Quispe-Agnoli 2006). That this body of works responds to what was, in fact, an area of intense interest and concern on both sides of an initially starkly drawn dual—native/Spaniard—world of social interactions and power relations is confirmed by the documentation detailing initial efforts by the Spaniards to establish an orderly colony in the former Inka territories. Central to this process from the 1540s through the 1570s was a program of enumerating the native population, investigating its form(s) of organization, and beginning to sketch out its history. One form that this process took was to call the *khipu*-keepers before colonial officials and have them read the contents of their cords (Loza 1998; Urton 1998). These recitations were made before a *lengua* 'translator'; the Spanish words spoken by the translator were written down by a scribe. This activity resulted in the production of written transcriptions in Spanish alphanumeric script of the census data and other information previously jealously guarded by the *khipu*-keepers in their cords.

Many of the *khipu* transcriptions discovered to date have been assembled in an important collection, entitled *Textos Andinos* (Pärssinen and Kiviharju 2004).

While these documents have been studied in terms of the displacement, and eventual replacement, of *khipu* 'literacy' by alphabetic literacy, what has received virtually no attention to date is the equally striking information they contain with respect to the confrontation between Inka knot-based numeration and Spanish grapheme-based written numerals and mathematics. How and what did individuals on either side of this confrontation think about the translation of quantitative values from knotted-cords to written texts?

Fig. 1.2.6a shows an image of a *khipu* juxtaposed to an *unrelated khipu* transcription, in Fig. 1.2.6b.[13] It is important to stress that we do not have an actual match—such as that suggested in Figs. 1.2.6a and b—between an extant *khipu* and a transcription of that same sample. As for the *khipu* in Fig. 1.2.6a, we are able to read the knot values of this sample and thereby interpret the numerical information encoded on this sample. We assume that the identities of the objects accounted for in this *khipu* were represented in a constellation of elements, including color, structure, and perhaps numbers interpreted as labels (Urton 2003). In the (unrelated) *khipu* transcription in Fig. 1.2.6b, the text is organized line by

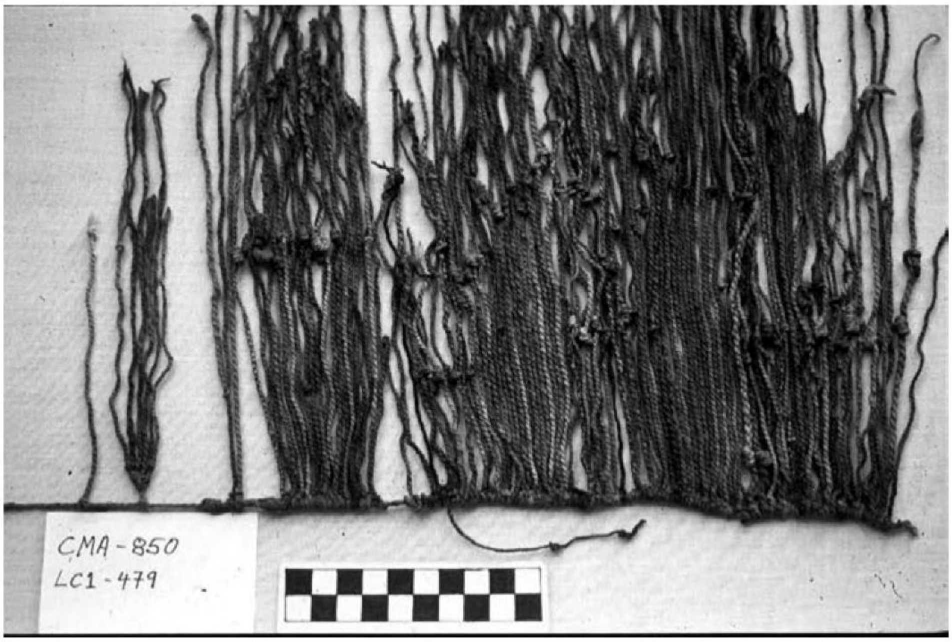

Figure 1.2.6 a) A *khipu* from Centro Mallqui, Leymebamba, Amazonas, Peru (#CMA 850/LC1–479 [UR9])

13. The *khipu* sample shown in Fig.1.2.6a is from the site of Laguna de los Cóndores, in the area of Chachapoyas, northern Peru (#CMA 850/LC1–479; in the 'Data table' page of the KDB website, this is sample UR9). The *khipu* transcription shown in Fig. 1.2.6b is from a tribute *khipu* from Xauxa, in the central Peruvian highlands, dating to 1558 (AGI, Lima 205, no. 16 folio 10r; see Pärssinen and Kiviharju 2004, 172–173).

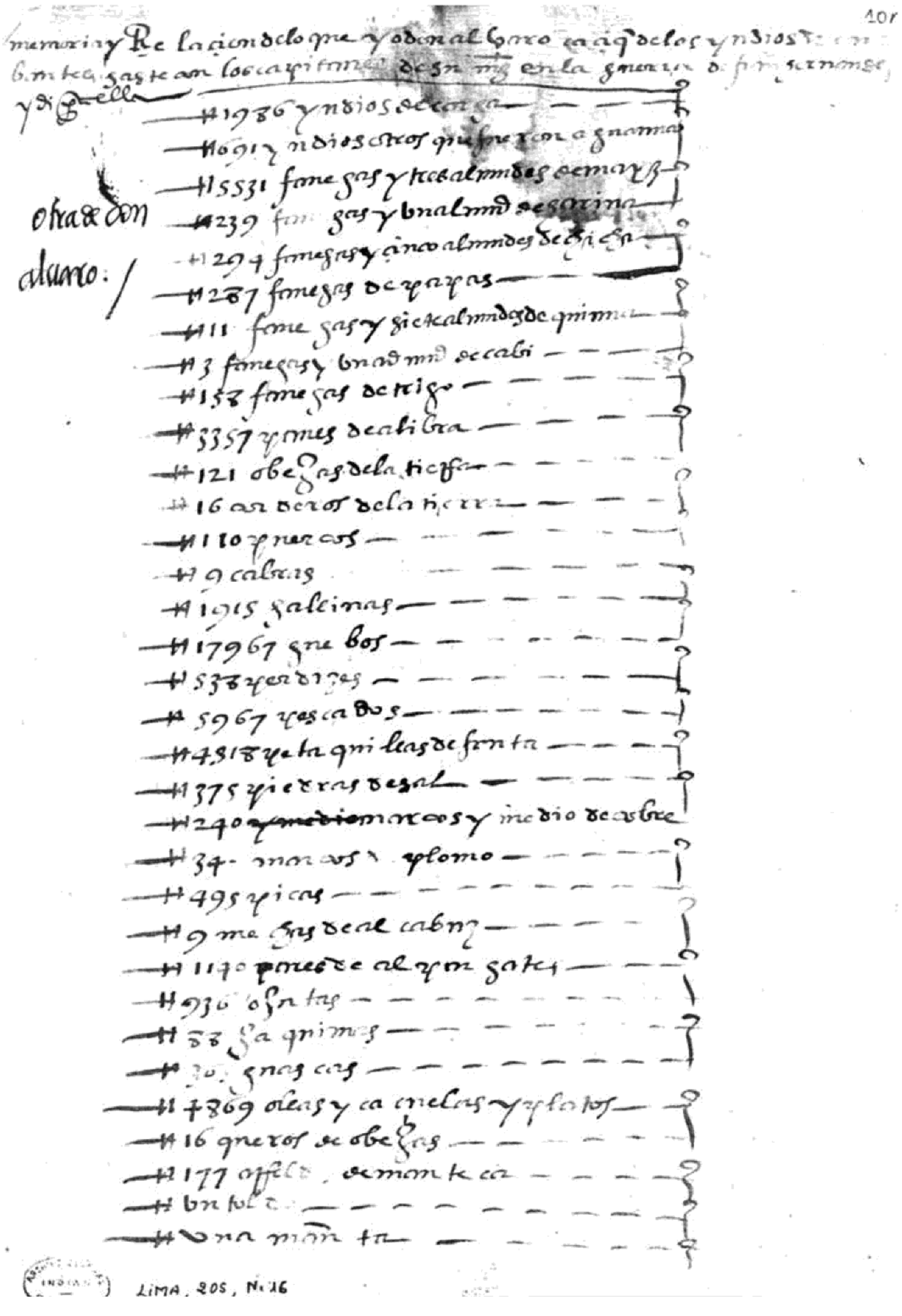

Figure 1.2.6 b) A *khipu* transcription (AGI, Lima 205, no. 16, folio 10r)

line, as the *khipu* itself is organized string by string; each line in the transcription contains a number followed by the name/identity of the object enumerated on that string in the *khipu* from which the transcription was derived.

The numerical values recorded in the *khipu* shown in Fig. 1.2.6a are similar in their range of magnitudes and distribution to those found in early Spanish census accounts in the Andes (Urton 2006). The *khipu* transcription illustrated in Fig. 1.2.6b is a tribute account recorded in the valley of Xauxa, in 1558 (Pärssinen and Kiviharju 2004, 172–173). If Fig. 1.2.6a were the *khipu* from which the transcription in Fig. 1.2.6b was drawn up (which it is not), we assume (but do not know for certain) that there would be a parallelism between number signs and object identity signs that would form a bridge across the semiotic—nominative and quantitative—divide separating these two species of texts.

Not surprisingly, almost all of the information we have in order to address the question of how Andean people thought about *khipus* and their translation and transcription into written texts comes to us from the Spanish side of the equation. The Spaniards were at least initially respectful of the *khipus* and their keepers, as the *khipus* were the primary sources of information on the basis of which Spanish officials began to erect the colonial administration. The most important point that should be made for our interest here concerning the juxtaposition of documents in Figs. 1.2.6a and b is that not only information, but *authority* as well, was located initially in the *khipu* member of the *khipu*/transcription pair juxtaposed in Fig. 1.2.6. However, once the information was transferred from *khipus* to written texts, the locus of textual authority, legitimacy, and power began to shift toward the written documents.

While many native Andeans learnt how to read and write alphabetic script and how to manipulate Hindu-Arabic number signs, only a handful of Spaniards appear to have achieved any degree of familiarity with the *khipus* (Pärssinen 1992, 36–50); it appears that no Spaniard became truly proficient at manipulating and interpreting the cords (Urton 2003, 18–19). What this meant was that, rather than contests over interpretations of information contained in the two sets of documents coming down to reciprocal readings of the two sets of texts, what emerged between the 1540s and the 1570s were separate, contested readings by the keepers of the two different text types before a Spanish judge. As disputes intensified, and as more and more original data were recorded uniquely in the written documents, the *khipu* texts became both redundant and increasingly troublesome for the Spaniards (Platt 2002). By the end of the tumultuous sixteenth century, *khipus* had been declared to be idolatrous objects—instruments of the devil—and were all but banned from official use.[14]

14. The *khipus* were declared idolatrous objects and their use was severely proscribed by the Third Council of Lima, in 1583 (Vargas Ugarte 1959). However, the *khipus* continued to be used for local record-keeping purposes—in some cases down to the present day (see Mackey 1970; Salomon 2004).

The circulation of coins is another area in which Andeans were confronted with a completely new and unfamiliar terrain of political relations, economic activity, and shifting relations of authority over the course of the early colonial period. The first mint in South America was formally established in Lima in 1568, just 36 years after the events of Cajamarca. The royal decree that controlled the weights, fineness, and the fractional components of the coins to be struck in Lima—the *real* and the *escudo*—were issued by Ferdinand and Isabella in 1479 and amended by Charles V in 1537 (Craig 1989, 2). The first coins struck in Lima bore a rendering of the Hapsburg coat of arms on the obverse and a cross with castles and lions on the quartered face on the reverse (Craig 1989, 6).

As noted, the two coin types were the *real*, a silver coin, and the gold *escudo*. Each of these coin types was broken down into subunits, each of which was valued in relation to a general, unified standard of valuation known as the *maravedí*. The latter was not a coin but rather it was what Moreyra Paz Soldán (1980, 66) terms *moneda imaginaria y de cuenta* 'imaginary money of account'. The *maravedí* was used to coordinate values between different types of coins as determined by material differences and subdivisions of standard units (for example, the silver *real* = 34 *maravedís*; the gold *escudo* = 350 [from 1537–1566] or 400 [after 1566] *maravedís*). From this primary coordinating function, the *maravedí* served as a common denominator that permitted the interrelating of heterogeneous monetary values pertaining to gold and silver. For example, until 1566, the *maravedí* coordinated the value of silver to gold at 11.5 to 1; after 1566 the ratio was 12.12 to 1 (Moreyra Paz Soldán 1980, 66–67; Craig 1989, 2).

What did any of the above have to do with Andean peoples? How were they to understand the meaning of these words and concepts? To understand the force of these questions, we can begin by imagining how one might go about translating the previous two paragraphs into a language like those spoken by large numbers of people throughout the Andes in the first few decades following the conquest, such as Quechua, Aymara, Puquina, or Yunga. They did not have terms for money or coinage, much less a term like *maravedí*, and had formed such concepts as 'value', 'heterogeneity', and 'account' in the absence of markets and a monetary economy (Murra 1995). It is clear in this case where authority would quickly come to reside in any dispute that might arise over the exchange value of any one of the several coin types in this system that would have begun to circulate through Andean communities by the 1570s. But we are getting ahead of ourselves.

From almost the earliest years following the conquest, Spanish officials in the countryside (the *encomenderos*) had been levying tribute in kind, which in some places included a demand for plates of silver and bars of gold, and translating the value of these items into Spanish currency values (Ramírez 1996, 92–112). Spanish officials regularly produced documents translating the quantities of items of tribute in kind into values in *pesos ensayados* (a unit of value in silver currency). This was the main context within which the *kurakas* 'local lords' in communities would have begun to encounter translations of the use-value of objects, which they were familiar with in their local non-monetized economies, into exchange-values stated in terms of currency equivalents (Spalding 1973). Furthermore, the Viceroy Francisco de Toledo introduced in the mid-1570s a new tribute system, which included not only produce and manufactured goods but also coins; the sum to be given yearly by each tributary was four-to-five *pesos ensayados* (that is, coinage in *plata ensayada* 'assayed silver'). Tribute payers were designated as male heads of households between the ages of 18–50. The native chronicler, Guaman Poma de Ayala, drew several images of native people paying their tribute using what appears to be coinage bearing the quartered reverse face of the *cuatro reales* (Fig. 1.2.7).[15]

People in communities—the newly-built *reducciones*—were able to acquire coins to pay their tribute from forced work in the mines (another component of the Toledan tributary system), as well as from marketing and wage labor. The engagements with currency that resulted from these activities required people to begin to think about the different units of coinage, shifting equivalencies between coinage units, as well as to accommodate themselves to fluctuations in currency values in the periodic currency devaluations and the debasement of coinage that took place during the colonial period. The act of 'devaluing' currency is a claim of authority on the part of some entity (such as the state) over the exchange-value of the coinage one holds in one's own purse. One's subsequent use of that same coinage according to the newly announced rate of exchange represents compliance with the claim by the entity in question to control the value of one's currency. Although we have almost no data on the basis of which to consider how Andean peoples responded to such changes (see Salomon 1991), these were some of the processes that were transpiring on the front lines of the confrontation between Old and New World mathematics and notions and relations of authority in the early colonial Andes.

Conclusions

We began this exploration by asking about the relevance and salience of a characterization of mathematics as 'the secret weapon of cultural imperialism' (Bishop

15. See the study by Salomon (1991) of one of the few references in the colonial literature to the engagement with coinage (*la moneda de cuatro reales*) by a native Andean during the colonial period. Salomon argues that the story, which appears in a well-known manuscript from Huarochirí (Salomon and Urioste 1991), is concerned with the internal conflicts of a man due to the competing religious sentiments he experiences over loyalty to a local deity (*huaca*) and the Christian deity. The narrative plays on the precise symbolism of images, as well as the lettering, on a quartered Spanish coin.

Figure 1.2.7 Paying tribute with coin bearing a quartered design (Guaman Poma de Ayala 1980, 521[525])

1990). Having now looked at several aspects of arithmetic, mathematics, and accounting in western Europe and the Andes during the period leading up to, and a century or so beyond, the fateful encounter between Pizarro and Atahualpa in Cajamarca in 1532, we return to ask: in what sense was mathematics linked to authority, power, and legitimacy in this historical conjuncture?

I argue that the answer to the question posed above is found in the same rationale and set of explanations that explain who writes history and who determines truth in history. The answer to both of these questions is: the conqueror. This is not because the conqueror knows what is, in fact, true; rather, it is because the conqueror possesses the power to speak, and to represent and establish the rules of the game as it is to be played from that moment forward. This is the case not only in terms of narrating and writing the events of history and explaining their causes (Urton 1990), but also in taking the measure of the world and accounting for those measurements—geographic, demographic, economic, and so on—for as long as the dominant group holds power.

Power, which is intimately linked to the exercise of authority, takes many forms. In its most extreme and, paradoxically, weakest form, power is maintained by force. As Foucault has shown more clearly than any recent political theorist, the most effective species of power is that which takes shape as individuals and groups become complicit with and participate in institutions of the state, such as in censuses, regulatory and corrective institutions, and accounting (Foucault 1979, 140–141; Stewart 1992; Smart 2002, 102–103). What is the place of mathematics in this Foucauldian, 'genealogical' conception of power and authority? I think that here we must return to the question of the certainty of mathematics, and of how that certainty relates to truth and, ultimately, to power. I suggest that the critical observation on these matters for our purposes here is that mathematics may be made to serve, although it itself is not responsible for giving rise to, regimes of power. A 'regime of power' may be manifested in the trappings of a king's court, in the ministrations of a priestly hierarchy, or in complex 'book-keeping' procedures—such as bundles of knotted cords in the hands of individuals authorized to record information (numerical and otherwise) in the interests of the state.

Bibliography

Aho, James, *Confession and bookkeeping: the religious, moral and rhetorical roots of modern accounting*, State University of New York, 2005.

Arellano, Carmen, 'Quipu y tocapu: sistemas de comunicación incas', in Franklin Pease (ed), *Los incas: arte y símbolos*, Lima: Banco de Crédito del Perú, 1999, 215–261.

Ascher, Marcia, 'Before the conquest', *Mathematics Magazine,* 65 (1992), 211–218.

Ascher, Marcia, and Ascher, Robert, *Mathematics of the Incas: code of the quipu*, Dover, 1997.

Bishop, Alan J, 'Western mathematics: the secret weapon of cultural imperialism', *Race and Class,* 32/2 (1990), 51–65.

Brokaw, Galen, Transcultural intertextuality and quipu literacy in Felipe Guaman Poma de Ayala's *Nueva corónica y buen gobierno*, PhD Dissertation, Indiana University, Ann Arbor: UMI Dissertation Services, 1999.

Brown, R Gene, and Johnston, Kenneth S, *Paciolo on accounting*, McGraw-Hill Book Co, 1984.

Buckmaster, Dale, 'The Incan quipu and the Jacobsen hypothesis', *Journal of Accounting Research,* 12 (1974), 178–181.

Carruthers, Bruce G, and Espeland, Wendy N, 'Accounting for rationality: double-entry bookkeeping and the rhetoric of economic rationality', *The Journal of American Sociology,* 97 (1991), 31–69.

Cieza de León, Pedro de, *El señorio de los Incas*, Lima: Instituto de Estudios Peruanos, 1967.

Conklin, William J, 'A *khipu* information string theory', in Jeffrey Quilter and Gary Urton (eds), *Narrative threads: accounting and recounting in Andean khipu*, University of Texas Press, 2002, 53–86.

Craig, Freeman, Jr, 'Coinage of the viceroyalty of el Perú', in William I Bischoff (ed), *The coinage of el Perú*, The American Numismatic Society, 1989.

Donoso Anes, Rafael, 'The double-entry bookkeeping method applied in Spain to account for transactions related to the minting process of gold and silver in the sixteenth century', *The Accounting Historians Journal,* 21 (1994), 97–116.

Durham, John W, 'The introduction of "Arabic" numerals in European accounting', *The Accounting Historians Journal,* 19 (1992), 25–55.

Ellerman, David P, 'The mathematics of double entry bookkeeping', *Mathematics Magazine,* 58 (1985), 226–233.

Forrester, D A R, 'The Incan contribution to double-entry accounting', *Journal of Accounting Research,* 6 (1968), 283.

Fossa, Lydia, *Narrativas problemáticas: los Inkas bajo la pluma española*, Lima: Instituto de Estudios Peruanos, 2006.

Foucault, Michel, *Discipline and punish*, Allen Lane, 1977.

Foucault, Michel, *The history of sexuality,* vol 1, Allen Lane, 1979.

Gómez Cañedo, Lino, *Los archivos de la historia de America: período colonial español,* Mexico, DF, 1961.

Guaman Poma de Ayala, Felipe, *El primer nueva corónica y buen gobierno*, (trans J L Urioste, eds John V Murra and Rolena Adorno), 3 vols, Siglo Veintiuno, 1980.

Guevara-Gil, Armando, and Salomon, Frank, 'A "personal visit": colonial political ritual and the making of Indians in the Andes', *Colonial Latin American Review,* 3 (1994), 3–36.

Hemming, John, *The conquest of the Incas*, Harcourt Brace Jovanovich, 1970.

Hoffman, Paul E, 'The computer and the colonial treasury accounts: a proposal for a methodology', *The Hispanic American Historical Review,* 50 (1970), 731–740.

Hoskin, Keith W, and Macve, Richard H, 'Accounting and the examination: a genealogy of disciplinary power', *Accounting, Organizations and Society,* 11 (1986), 105–136.

Jacobsen, Lyle E, 'The ancient Inca empire of Peru and the double entry accounting concept', *Journal of Accounting Research,* 2 (1964), 221–228.

Julien, Catherine, 'How Inca decimal administration worked', *Ethnohistory,* 35 (1988), 257–279.

Klein, Herbert S, and Barbier, Jacques A, 'Recent trends in the study of Spanish American colonial public finance', *Latin American Research Review,* 23 (1988), 35–62.

Lapeyre, Henri, *Une famille de marchands, les Ruiz*, Paris, 1955.

Locke, L Leland, *The ancient quipu or Peruvian knot record*, American Museum of Natural History, 1923.

Loza, Carmen Beatriz, 'Du bon usage des quipus face a l'administration coloniale Espagnole, 1553–1599', *Population,* 53 (1998), 139–160.

Mackey, Carol, *Knot records of ancient and modern Peru*, PhD Dissertation, University of California, Berkeley, Ann Arbor: University Microfilms, 1970.

Mignolo, Walter D, *The darker side of the Renaissance: literacy, territoriality, and colonization,* University of Michigan Press, 1995.

Miller, Peter, 'On the Interrelations between Accounting and the State', *Accounting, Organizations and Society,* 15 (1990), 315–338.

Miller, P, and O'Leary, T, 'Accounting and the construction of the governable person', *Accounting, Organizations and Society,* 12 (1987), 235–265.

Moreyra Paz Soldán, Manuel, *La moneda colonial en el Perú*, Lima: Banco Central de la Reserva, 1980.

Murra, John V, 'Las etno-categorías de un *khipu* estatal', in *Formaciones económicas y políticas en el mundo andino*, Lima: Instituto de Estudios Peruanos, 1975, 243–254.

Murra, John V, 'The *mit'a* obligations of ethnic groups to the Inka state', in George A Collier, Rosaldo Renato, and John D Wirth (eds), *The Inca and Aztec states, 1400–1800: anthropology and history*, Academic Press, 1982, 237–262.

Murra, John V, 'Did tribute and markets prevail in the Andes before the European invasion?' in Brook Larson and Olivia Harris (eds), *Ethnicity, markets, and migration in the Andes: at the crossroads of history and anthropology*, Duke University Press, 1995, 57–72.

Murúa, Fray Martín de, *Historia general del Perú*, (ed Ballesteros Gaibrois), Dastin Historia, 2001.

Murúa, Fray Martín de, *Códice Murúa - historia y genealogía de los reyes Incas del Perú del Padre Mercenario Fray Martín de Murúa (Códice Galvin)*, (ed Juan Ossio), Testimonio Compañía Editorial, SA, 2004.

Pärssinen, Martti, *Tawantinsuyu: the Inca state and its political organization* (Studia Historica, 43), Societas Historica Finlandiae, 1992.

Pärssinen, Martti, and Kiviharju, Jukka (eds), *Textos Andinos: corpus de textos khipu Incaicos y Coloniales*, vol I, Instituto Iberoamericano de Finlandia and Universidad Complutense de Madrid, 2004.

Pereyra, Hugo, 'Notas sobre el descubrimiento de la clave numeral de los quipus incaicos', *Boletín del Museo de Arqueología y Antropología,* 4 (2001), 115–123.

Peters, Richard M, and Emery, Douglas R, 'The role of negative numbers in the development of double entry bookkeeping', *Journal of Accounting Research,* 16 (1978), 424–426.

Platt, Tristan, '"Without deceit or lies": variable *chinu* readings during a sixteenth-century tribute-restitution trial', in Jeffrey Quilter and Gary Urton (eds), *Narrative threads: accounting and recounting in Andean khipu*, University of Texas Press, 2002, 225–265.

Quilter, Jeffrey, and Urton, Gary, (eds), *Narrative threads: accounting and recounting in Andean khipu*, University of Texas Press, 2002.

Quispe-Agnoli, Rocío, *La fe andina en la escritura: resistencia e identidad en la obra de Guaman Poma de Ayala*, Lima: Universidad Nacional Mayor de San Marcos, 2006.

Radicati di Primeglio, Carlos, *Estudios sobre los quipus*, (ed and intro Gary Urton), Fondo Editorial Universidad Nacional Mayor de San Marcos, 2006.

Ramírez, Susan E, *The world upside down: cross-cultural contact and conflict in sixteenth century Peru*, Stanford University Press, 1996.

Rappaport, Joanne, and Cummins, Thomas, 'Literacy and power in colonial Latin America', in George C Bond and Angela Gilliam (eds), *Social construction of the past: representation as power*, London and Routledge, 1994, 89–109.

Rappaport, Joanne, and Cummins, Thomas, 'Between images and writing: the ritual of the king's *quillca*', *Colonial Latin American Review,* 7/1 (1998), 7–32.

Reitzer, Ladislas, 'Some observations on Castilian commerce and finance in the sixteenth century', *The Journal of Modern History,* 32 (1960), 213–231.
Richardson, Alan J, 'Accounting as a legitimating institution', *Accounting, Organizations and Society,* 12 (1987), 341–355.
Rowe, John H, 'The Incas under Spanish colonial institutions', *Hispanic American Historical Review,* 37 (1957), 155–199.
Salomon, Frank, 'La moneda que a don Cristóbal se le cayó: el dinero como elemento simbólico en el texto *Runa Yndio Ñiscap Machoncuna*', in Segundo Moreno and Frank Salomon (eds), *Reproducción y transformación de las sociedades andinas, siglos XVI–XX*, Quito: Ediciones ABYA-YALA, 1991, 481–586.
Salomon, Frank, *The cord keepers: khipus and cultural life in a Peruvian village*, Duke University Press, 2004.
Salomon, Frank, and Urioste, J L, *The Huarochirí manuscript: a testament of ancient and colonial Andean religion*, University of Texas Press, 1991.
Smart, Barry, *Michel Foucault*, Routledge, 2002.
Sombart, Werner, *The quintessence of capitalism* (trans M Epstein), Howard Fertig, 1967.
Spalding, Karen, 'Kurakas and commerce: a chapter in the evolution of Andean society', *The Hispanic American Historical Review,* 53 (1973), 581–599.
Spencer, Martin E, 'Weber on legitimate norms and authority', *The British Journal of Sociology,* 21 (1970), 123–134.
Stern, Steve J, *Peru's Indian peoples and the challenge of Spanish conquest*, University of Wisconsin Press, 1993.
Stewart, Ross E, 'Pluralizing our past: Foucault in accounting history', *Accounting, Organizations and Society,* 5 (1992), 53–73.
Swetz, Frank J, *Capitalism and arithmetic: the new math of the 15th century*, Open Court, 1989.
Uphoff, Norman, 'Distinguishing power, authority and legitimacy: taking Max Weber at his word by using resources-exchange analysis', *Polity,* 22 (1989), 295–322.
Urton, Gary, *The history of a myth: Pacariqtambo and the origin of the Inkas*, University of Texas Press, 1990.
Urton, Gary, 'A new twist in an old yarn: variation in knot directionality in the Inka *khipus*', *Baessler-archiv* Neue Folge, 42 (1994), 271–305.
Urton, Gary, *The social life of numbers: a Quechua ontology of numbers and philosophy of arithmetic*, University of Texas Press, 1997.
Urton, Gary, 'From knots to narratives: reconstructing the art of historical record-keeping in the Andes from Spanish transcriptions of Inka *khipus*', *Ethnohistory,* 45 (1998), 409–438.
Urton, Gary, 'A calendrical and demographic tomb text from northern Peru', *Latin American Antiquity,* 12 (2001), 127–147.
Urton, Gary, 'Recording signs in narrative-accounting *khipu*', in Jeffrey Quilter and Gary Urton (eds), *Narrative threads: accounting and recounting in Andean khipu*, University of Texas Press, 2002, 171–196.
Urton, Gary, *Signs of the Inka khipu: binary coding in the Andean knotted-string records*, University of Texas Press, 2003.
Urton, Gary, '*Khipu* archives: duplicate accounts and identity labels in the Inka knotted string records', *Latin American Antiquity,* 16 (2005), 147–167.
Urton, Gary, 'Censos registrados en cordeles con "amarres". Padrones poblacionales pre-hispánicos y coloniales tempranos en los *khipu* Inka', *Revista Andina,* 42 (2006), 153–196.
Urton, Gary, and Brezine, Carrie J, '*Khipu* accounting in ancient Peru', *Science,* 309 (2005), 1065–1067.
Vargas Ugarte, Rubén, *Historia de la iglesia en el Perú*, vol 2 (1570–1640), Aldecoa, 1959.

Weber, Max, *The Protestant ethic and the spirit of capitalism* (trans Talcott Parsons), Scribner, 1958.

Weber, Max, *The theory of economic and social organization* (trans A M Henderson and Talcott Parsons), The Free Press, 1964.

Yamey, B S, 'Introduction', in A C Littleton and B S Yamey (eds), *Studies in the history of accounting*, Sweet and Maxwell, 1956, 1–13.

Zuidema, R Tom, 'Bureaucracy and systematic knowledge in Andean civilization', in George A Colliedr, R I Rosaldo, and John D Wirth (eds), *The Inca and Aztec states, 1400–1800,* Academic Press, 1982, 419–458.

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CHAPTER 1.3

Heavenly learning, statecraft, and scholarship: the Jesuits and their mathematics in China

Catherine Jami

The story of the transmission of mathematics from Europe to China in the early modern age is closely linked to that of the Jesuit mission from 1582 to 1773, which spanned the last decades of the Ming dynasty (1368–1644) and the Qing dynasty (1644–1911) from its advent to its apogee in the mid-eighteenth century.[1] For almost two centuries, the Jesuits put the sciences in the service of evangelization: their knowledge enhanced the prestige of their religion and opened the way first to the patronage of individual officials, and then to that of the state. This emphasis on the sciences as a tool for proselytization seems to have been unique at the time, both among the missionary orders present in China, and among Jesuit missions around the world (Standaert 2001, 309–354; Romano 2002). Even within the China mission, most Jesuits devoted their time and effort solely to evangelization, while only a few 'specialists' among them taught and practiced the sciences (Brockey 2007). However, it could be argued that Jesuits' science had a much more pervasive influence than their religion. Christianity remained a minority religion, even a marginal one.[2] On the other hand all scholars interested

1. For names and dates of people and dynasties, see Table 2 on page 80.
2. For the period under discussion, there were no more than about 200,000 Chinese Christians, with this maximum reached around 1700, when the Chinese population is estimated to have been about 150 million (Standaert 2001, 380–386).

in the mathematical sciences knew about *xi xue* 西學 'western learning' whatever their attitude towards it might have been.

Two factors contributed to shaping Jesuit science in China: on the one hand, the importance that the Society of Jesus gave to mathematics (in the usual sense of this term in early modern Europe)—what we might call the offer; on the other hand, the renewal of interest in *shi xue* 實學 'solid learning' among late Ming scholars—what we might call the demand. Accordingly, I shall first outline the place of mathematics in Jesuit education and briefly describe the state of mathematics in China around 1600. I will then go on to discuss translations of works on the mathematical sciences during the first decades of the mission, and recount how participation in the Calendar Reform of 1629 and integration in the Chinese civil service in 1644 shaped the Jesuits' practice and teaching of mathematics. Chinese responses to western learning entailed competing propositions for structuring the discipline. Mei Wending's 梅文鼎 integration of Chinese and western mathematics was the most elaborate reconfiguration of the field. His synthesis and the Kangxi emperor's appropriation of western science were instrumental in reshaping the landscape of mathematics in China.

Mathematics in the Jesuit curriculum

Founded in 1540, the Society of Jesus soon started setting up colleges across Europe. The sons of the elites of Catholic countries were educated in them, as were most members of the Society. The latter often trained to be teachers, and for some of them this remained their main occupation. The content and structure of the education provided by the Society were crucial in shaping Jesuit culture, in Europe as well as in China. Having previously studied the *trivium* (grammar, logic, and rhetoric), students entering a Jesuit college would typically begin with further training in rhetoric. This was followed by three years devoted to logic, philosophy, and metaphysics. Early in the Order's history, natural philosophy (or physics) and mathematics were both grouped under philosophy. According to the Aristotelian classification, physics and mathematics addressed two of the ten categories, respectively quality and quantity. Physics provided a qualitative explanation of natural phenomena; it was based on the four-elements theory, according to which all matter was composed of earth, air, fire, and wind, and the earth lay motionless at the centre of concentric crystalline spheres. In the scholastic tradition, mathematics consisted of the four disciplines of the *quadrivium*, namely arithmetic, music, geometry, and astronomy.

However, it was somewhat redefined in the Jesuit curriculum. The Roman College, founded in 1551, set the standards for the Society's educational network. The *Ratio studiorum* (final version 1599), which defined the Jesuit system of teaching, gave a new importance to mathematics (*Ratio studiorum* 1997). Christoph Clavius, architect of the Gregorian Calendar Reform of 1582, had taught mathematics at the Roman College since 1565, and was the first to hold the chair of mathematics there. He was instrumental in establishing it as a subject independent from philosophy and in asserting its status as a science (Baldini 1992; Romano 1999, 133–178). This was the outcome of a debate within the Society that was mainly epistemological. However the new importance of the mathematical arts in sixteenth-century Italy must also have played a role in the inclusion of mathematics in the subjects in which the Jesuits strove to be eminent (Gorman 1999, 172).

While establishing mathematics as an independent discipline in the Jesuit curriculum, Clavius redefined its structure and produced textbooks for its teaching. Following Proclus, he divided mathematics into 'pure' and 'mixed', the former consisting of arithmetic and geometry, the latter comprising six major branches (which were further divided into subordinate disciplines): natural astrology (astronomy), perspective, geodesy, music, practical arithmetic, and mechanics. This structure broadened the scope of mathematics and extended its fields of application (Engelfriet 1998, 30–32; Feldhay 1999, 110). The works authored by Clavius, first and foremost his editions of and commentaries on Euclid's *Elements* and Sacrobosco's *Sphaera* 'Sphere' (a thirteenth-century treatise on astronomy), as well as his textbooks on arithmetic and algebra, formed the basis of mathematical education as he defined it for the Society (Feldhay 1999, 109).

Jesuit education was not uniform: there were local variants in the mathematics taught,[3] and, as with any school curriculum, a number of updates occurred. Thus, after the 1620s, the Ptolemaic system defended and taught by Clavius was gradually replaced by the Tychonic system, in which the sun, while revolving around the earth, was the centre of the orbit of the planets (Baldini 2000, 77). By and large, the tradition Clavius had established was continued in the sense that many teachers produced their own textbooks that were conceived as continuations of his, though departing from them in their approach (Feldhay 1999, 114). Two examples are relevant to the mathematics transmitted to China. First, the number of textbooks entitled *Elements of geometry* produced in the seventeenth century, within and without the Society, was such that the phrase, and even the name of Euclid, came to refer to a genre—that of geometry textbooks—rather than merely to editions of the Greek classic. Second, whereas Clavius' *Algebra* was one of the last representatives of the medieval tradition of cossic algebra, in which the unknown and its powers are denoted by abbreviations of their names (Reich 1994), Viète's new notation, with vowels denoting the unknowns and consonants the given quantities, was introduced into Jesuit teaching in the 1620s

3. On Portugal see Leitão (2002); on France see Romano (1999, 183–354; 2006).

(Feldhay 1999, 116–126). Matteo Ricci, the first Jesuit to enter China at the end of the sixteenth century, had studied with Clavius at the Roman College and brought with him Clavius' mathematics, but some of his successors in the China mission would present mathematics as it evolved in Jesuit colleges later in the century.

Mathematics in Late Ming China: the 'Unified lineage of mathematical methods'

It is widely admitted that by 1600, the most significant achievements of the Chinese mathematical tradition had fallen into oblivion. The *Jiu zhang suan shu* 九章算術 'Nine chapters on mathematical procedures' (first century AD), regarded as the founding work of the Chinese mathematical tradition (Chemla and Guo 2004) and included in the *Suan jing shi shu* 算經十書 'Ten mathematical classics' (656), had effectively been lost. Furthermore the sophisticated *tian yuan* 天元 'celestial element' algebra developed in the thirteenth century had been forgotten.[4] The calculating device on which both were based, the counting rods, had fallen into disuse; the abacus had become the universally used calculating device.[5]

By contrast with this picture of decline in mathematics, some historians describe the sixteenth century as a 'second Chinese Renaissance'. In reaction against Wang Yangming's 王陽明 philosophy of the mind, which, around 1500, gave priority to introspection over concern with the outside world, as well as in response to a more and more perceptible political crisis, the last decades of the sixteenth century witnessed a strong renewal of interest in technical learning and statecraft (Cheng 1997, 496–530). The advocates of 'solid learning' emphasized the social role of literati, underlining that scholarship was of value only if it contributed to welfare and social harmony, while being grounded in verifiable evidence. At the same time, the lowering of the cost of printing resulted in a significant broadening of the book market, which facilitated the circulation of knowledge. The renewal in many fields of scholarship is exemplified by such major works as *Ben cao gang mu* 本草綱目, 'Compendium of medical material' (1593) by Li Shizhen 李時珍, *Lü lü jing yi* 律呂精義 'Essential meaning of pitchpipes' (1596) by Zhu Zaiyu 朱載堉, and *Tian gong kai wu* 天工開物 'Exploitation of the works of nature' (1637) by Song Yingxing 宋應星. The mathematical treatise *Suan fa tong zong* 算法統宗 'Unified lineage of mathematical methods' (1592) by Cheng Dawei 程大位 can be regarded as belonging to this trend and is representative of

4. General accounts of the history of Chinese mathematics in western languages include Li and Du (1987), Martzloff (1997), Yabuuti (2000).

5. On counting rods see Volkov (1998); Lam and Ang (2004, 43–112). On the abacus see Jami (1998a).

the state of mathematics in China by 1600. It was to remain a bestseller to the end of the imperial age (1911).

In mathematics as in many other fields, late Ming scholars blamed the perceived loss of ancient traditions on their more recent predecessors' indulgence in self-centred and esoteric pursuits. Far from claiming to innovate radically, Cheng Dawei aimed at providing a compilation of earlier treatises that he had spent decades collecting. This is reflected in the book's title:[6] it is not unlikely that Cheng saw himself as the heir of a lineage of scholars versed in mathematics. In his work he gave a bibliography of all earlier works on the subject that he knew of. *Suan* 算 was the usual term to refer to mathematics, *fa* 法 referred to the methods by which each problem was solved; *suan fa* occurred in the title of many of the works known to Cheng Dawei. The 'Unified lineage of mathematical methods' represents a synthesis of a tradition of popular mathematics based on the abacus that can be traced back to Yang Hui 楊輝 (fl. 1261), in the Southern Song dynasty (Lam 1977; Yabuuti 2000, 103–121). This tradition is usually contrasted with the learned tradition of the Song and Yuan dynasty that culminated with celestial element algebra.

Like most of the predecessors known to him, Cheng Dawei referred to a canonical nine-fold classification of mathematics that can be traced back to the 'Nine chapters on mathematical procedures', although the book itself was evidently unavailable to him. In fact, during the late Ming and early Qing period, the phrase *jiu zhang* 九章 'nine chapters' mostly referred to that classification rather than to the classic work itself. But like most if not all authors before and after him, Cheng Dawei failed to fit all the mathematical knowledge at his command into the headings of the nine chapters: his work is divided into seventeen chapters. It opens with a general discussion of some ancient diagrams then thought to represent the origins of mathematics. Chapter 1 contains some general prescriptions for the study of mathematics, a list of the nine chapters, concise glosses of more than seventy terms used thereafter (*yong zi fan li* 用字凡例, 'guide to characters used'), lists of powers of tens and units, tables of addition, subtraction, multiplication, and division for the abacus, and brief explanations of some terms referring to common operations such as the simplification of fractions or the extraction of cube roots. Chapter 2 focuses on abacus calculation; it opens with an illustrated description of the instrument. The following fifteen chapters contain 595 problems presented in the traditional form: question, answer, and method of solution. Chapters 3 to 6 and 8 to 12 take up the headings of the nine chapters in the traditional order, whereas Chapter 7 introduces a particular type of problem, which involves *Fen tian jie ji fa* 分田截積法 'Methods for dividing

6. This is the reason why I prefer to translate *tong zong* 統宗 literally as 'unified lineage' rather than to use the most common translation: 'systematic treatise'.

fields by cutting off their areas': they deal with the dimensions and areas of figures obtained by cutting off a part of a known figure. Chapters 13 to 16 contain *Nan ti* 難題 'Difficult problems', often stated as rhymes; these problems are again classified according to the headings of the nine chapters. The last chapter gives *Za fa* 雜法 'Miscellaneous methods'; it includes various diagrams such as magic squares and depictions of hand calculation mnemonics. The chapter closes with a bibliography of 52 titles, from the Song edition (1084) of the 'Ten mathematical classics' to works published in Cheng's lifetime, spanning five centuries (Guo 1993, II 1217–1453). As this brief description of the work suggests, while the 'Nine chapters on mathematical procedures' was not accessible to Cheng, and while abacus calculation underlies the whole of his mathematics, his work belongs to a lineage that can be traced back to the first-century AD classic. In this as well as in other respects, the 'Unified lineage of mathematical methods' can be regarded as representative of mathematics as practised in China at the time of the first Jesuits' arrival.

Teaching and translating: Jesuit mathematics in Ming China

The China mission was part of the Portuguese assistancy of the Society of Jesus: since the end of the fifteenth century, all Asian missions were under the patronage of the Portuguese crown. The port of Macao, founded by the Portuguese in 1557, served as their Eastern base. While their Japanese mission flourished, the Jesuits' attempts to settle in China were unsuccessful until 1582. The first Jesuit residence in China was set up in Zhaoqing 肇慶 (Guangdong province). In establishing contact with local elites, Matteo Ricci used both knowledge and artifacts that he had brought with him from Europe. At the same time, he assessed their knowledge in terms familiar to him:

> They have acquired quite a good mastery not only of moral philosophy, but also of astrology [that is, astronomy] and of several mathematical disciplines. However, in the past they have been better versed in arithmetic and in geometry; but they have acquired all this and dealt with it in a confused way.[7] (Ricci and Trigault 1978, 95)

In line with this emphasis on the shortcomings of the Chinese as regards mathematics, Ricci turned himself into a teacher. His relations with the first literati interested in Christianity were modelled on a master–disciple relationship, which can be interpreted both in the context of Jesuit education and of Chinese lineages

7. Ils ont non seulement acquis assez bonne connaissance de la philosophie morale, mais encore de l'astrologie et de plusieurs disciplines mathématiques. Toutefois il ont autrefois été plus entendus en l'arithmétique et géométrie; mais aussi ils ont acquis ou traité tout ceci confusément.

of scholarship (Jami 2002a). He described the progress of Qu Rukui 瞿汝夔, one of his first sympathizers and advisors, who eventually converted (Standaert 2001, 419–420):

He started with arithmetic, which in method and ease by far surpasses the Chinese one: for the latter all in all consists in a certain wooden instrument in which round beads, strung on copper wire, are changed here and there, to mark numbers. Although in fact it is sure, it is easily subject to misuse, and reduces a broad science to very little. He then heard Christoph Clavius' sphere and Euclid's elements, only what is contained in Book I; towards the end he learnt to paint almost any kind of figures of dials to mark the hours. He also acquired knowledge of the heights of things through the rules and measures of geometry. And being, as I said, a man of wit and well versed in writing, he reduced all this into commentaries in a very neat and elegant language, which he later showed to mandarins. One would hardly believe what reputation this earned to him and to our fathers, from whom he acknowledged having learned it all. For all that he had been taught delighted the Chinese, so that it seemed that he himself could never learn to his heart's content. For he repeated day and night what he had heard, or adorned the beginnings with figures so beautiful that they were by no means inferior to those of our Europe. He also made several instruments, spheres, astrolabes, dials, magnet boxes, mathematical, and other similar instruments very elegantly and artistically set up.[8] (Ricci and Trigault 1978, 308–309)

Ricci's success is evidenced by his student's capacity to produce both instruments and texts that were fit for circulation among literati. The former points to the inclusion of instrument making in mathematics as the Jesuits taught it in China. The latter brings out the fact that the Jesuits needed Chinese scholars' help in order to write in Chinese. During the first decades of the mission, the translation of mathematical texts was the outcome of teaching. After Ricci settled in Beijing in 1601, he taught mathematics to Xu Guangqi 徐光啓 and Li Zhizao 李之藻, two high officials who converted and took on the role of protectors of the Jesuit mission. They collaborated with Ricci in producing works based on some of Clavius' textbooks (Martzloff 1995).

8. Il commença par l'arithmétique qui en méthode et en facilité surpasse de beaucoup la chinoise: car icelle consiste toute en certain instrument de bois auquel des grains ronds enfilés de fil d'archal sont changés çà et là, pour marquer les nombres. Ce qu'encore que véritablement il soit assuré est sujet à recevoir facilement de l'abus et réduit à peu d'espèces d'une science très ample. Il ouït en après la sphère de Christopher *Clavius* et les éléments d'Euclide, ce que seulement est contenu au premier livre; sur la fin il apprit à peindre quasi toutes sortes de figures de cadrans pour marquer les heures. Il acquit aussi la connaissance des hauteurs des choses par les règles et mesures de la géométrie. Et, pour autant, comme je l'ai dit, qu'il était homme d'esprit et fort expert en l'écriture, il réduisit tout ceci en commentaires d'un langage fort net et élégant, lesquels venant par après à montrer aux mandarins ses amis, à peine pourrait-on croire quelle réputation cela acquit tant à lui qu'à nos Pères, desquels il confessait avoir tout appris. Car tout ce qui lui avait été enseigné ravissait par sa nouveauté tous les Chinois en admiration, de façon qu'il semblait que lui même ne pouvait en aucune sorte se saouler et contenter d'apprendre. Car il répétait jour et nuit ce qu'il avait ouï ou ornait ses commencements de figures si belles qu'ils ne cédaient en rien à ceux de notre Europe. Il fit aussi plusieurs instruments, des sphères, astrolabes, cadrans, boîtes d'aimants, instruments de mathématiques et autres semblables fort élégamment et artistement dressés.

Table 1 Chinese translations and adaptations of Clavius' works

Clavius' work	Date of Chinese work	Title of Chinese work	Authors/ translators
Euclidis elementorum, 1574	1607	Elements of geometry (*Ji he yuan ben* 幾何原本)	Matteo Ricci Xu Guangqi
Astrolabium, 1593	1607	Illustrated explanation of cosmographical patterns (*Hun gai tong xian tu shuo* 渾蓋通憲圖說)	Matteo Ricci Li Zhizao
Geometria practica, 1604	1608	Meaning of measurement methods (*Ce liang fa yi* 測量法義)	Matteo Ricci Xu Guangqi
In sphaeram Ioannis de Sacro Bosco commentarius, 1570	1608	On the structure of heaven and earth (*Qian kun ti yi* 乾坤體義)	Matteo Ricci Li Zhizao
Epitome arithmeticae practicae, 1583	1614	Instructions for calculation in common script (*Tong wen suan zhi* 同文算指)	Matteo Ricci Li Zhizao
In sphaeram Ioannis de Sacro Bosco commentarius, 1570	1614	Meaning of compared [figures] inscribed in circles (*Yuan rong jiao yi* 圜容較義)	Matteo Ricci Li Zhizao
Geometria practica, 1604	1631	Complete meaning of measurement (*Ce liang quan yi* 測量全義)	Giacomo Rho

The relationship between the Chinese works and their Latin sources varies. The 'Meaning of measurement methods', a brief treatise on surveying, completed by Ricci and Xu Guangqi at the same time as their translation of the first six books of Euclid's *Elements* in 'Elements of geometry', is not a direct translation from the *Geometrica practica*; it is probably based on Ricci's lecture notes (Engelfriet 1998, 297). The 'Instructions for calculation in common script' takes up a number of problems found in earlier Chinese works such as Cheng Dawei's 'Unified lineage of mathematical methods' and applies to them written arithmetic and the methods given by Clavius in his *Epitome arithmeticæ practicæ* (Jami 1992; Pan 2006).

Collaboration seems to have followed a pattern common to all translations, religious or secular: the Jesuit gave an oral explanation of the meaning of some original text, which the Chinese scholar then wrote down in classical Chinese. New terms were coined when there was no obvious equivalent for a Latin term in Chinese. For example, terms like definition, axiom, postulate, proposition, proof, had to be created during the translation of Euclid's *Elements*. These new Chinese terms were not explicitly defined before being used: whereas their Latin originals were part of the cultural background of those who studied geometry in Europe, such notions would have been entirely alien to a Chinese reader, and would probably remain somewhat of a mystery unless this reader was taught by someone familiar with them. But the vast majority of Chinese scholars who read the 1607 translation did so without the help of such teaching. It is little surprise, therefore, that while there was much interest in the content of the *Elements*, the Euclidean style on the whole aroused more perplexity than enthusiasm (Martzloff 1980; Engelfriet 1998, 147–154; Jami 1996).

Defining and situating mathematics

Whereas Euclidean geometry was presented as a radical innovation, in arithmetic western learning was introduced as an improvement on the Chinese mathematical tradition. The dichotomy between number and magnitude was made explicit in the structure of mathematics described in Ricci's preface to the 'Elements of geometry':

> The school of quantity (*ji he jia* 幾何家) consists of those who concentrate on examining the parts (*fen* 分) and boundaries (*xian* 限) of things. As for the parts, if [things] are cut so that there are a number (*shu* 數) [of them], then they clarify how many (*ji he zhong* 幾何衆) the things are; if [things] are whole so as to have a measure (*du* 度), then they point out how large (*ji he da* 幾何大) the things are. These number and measure may be discussed (*lun* 論) in the abstract, casting off material objects. Then those who [deal with] number form the school of calculators (*suan fa jia* 算法家); those who [deal with] measure form the school of mensurators (*liang fa jia* 量法家). Both [number and measure] may also be opined on with reference to objects. Then those who opine on number, as in the case of harmony produced by sounds properly matched, form the school of specialists of pitchpipes and music (*lü lü yue jia* 律呂樂家); those who opine on measure, in the case of celestial motions and alternate rotations producing time, form the school of astronomers (*tian wen li jia* 天文曆家).[9]

This is a description of the *quadrivium*, which, in Chinese terms, proposes to subsume four well-known technical fields under a broader, hitherto unknown discipline: the 'study of quantity'. *Ji he* 幾何 renders the Latin *quantitas*. The title chosen by Ricci and Xu for their translation was apparently intended to encompass not just geometry, but rather the whole *quadrivium*. The claim here is also that the 'Elements of geometry' provides foundations for a discipline that includes

9. 幾何家者專察物之分限者也其分者若截以為數則顯物幾何衆也若完以為度則指物幾何大也其數與度或脫於物體而空論之則數者立算法家度者立量法家也或二者在物體而偕其物議之則議數者如在音相濟為和而立律呂樂家議度者如在動天迭運為時而立天文曆家也 Guo (1993, V 1151–1152, cf. Engelfriet (1998, 139); Hashimoto and Jami (2001, 269–270).

the Chinese tradition of *suan fa* as one of its parts. *Ji he* means 'how much' in classical Chinese: it occurred in every problem of the 'Nine chapters on mathematical procedures'. In the 'Unified lineage of mathematical methods', however, *ruo gan* 若干 (a synonym) is used in the question stated in problems; *ji he* appears in the 'Guide to characters used': it is glossed by 'same as *ruo gan*' (Guo 1993, II 1230). Later in the seventeenth century, *ji he* came to refer to the content of the 'Elements of geometry', that is, to Euclidean geometry.[10]

The dichotomy between the two instances of quantity rendered as *shu* 數 'number' and *du* 度 'magnitude' respectively would have been new to a Chinese reader at the time: *shu* was more evocative of numerology and the study of the *Yi jing* 易經 'Classic of change', than of procedures of *suan fa*. By using this last term to refer to procedures, Ricci and Xu again implied that mathematics as hitherto practised in China was to be embedded into a broader discipline.

No matter how unfamiliar Ricci's distinction between *shu* and *du* might have appeared, the translations based on Clavius' works, made in response to the curiosity of a few Chinese scholars, aroused enduring interest among a wider audience. Moreover, bringing together mathematics, surveying, astronomy, and musical harmony was not foreign to their tradition (Lloyd, Chapter 1.1; Cullen, Chapter 7.1 in this volume): surveying was one of the main themes of mathematical problems; mathematical astronomy and musical harmony were discussed in the same section of quite a few dynastic histories. Also, one finds many examples of scholars known both as mathematicians and astronomers: Zhu Zaiyu, mentioned above as the author of the 'Essential meaning of pitchpipes' (1596, the earliest known discussion of equal temperament), strove to unify musical harmony and astronomy (Needham 1962, 220–228).

The translations mentioned above were part of the Jesuits' larger enterprise of 'apostolate through books': their publications merged into the thriving book market of the late Ming (Standaert 2001, 600–631). Their teachings were first presented as a whole in a compendium edited by Li Zhizao, the *Tian xue chu han* 天學初函 'First collection of heavenly learning' (1626). It was divided into two parts: *li* 理 'Principles' (nine works) and *qi* 器 'Tools' (ten works). The first part opens with a description of the European educational system, entitled *Xi xue fan* 西學凡 'Outline of Western learning' (1621). Like Ricci, its author, Giulio Aleni, had been a student of Clavius at the Roman College. It presents the structure of disciplines that was then most common, mathematics consisting of the *quadrivium* and being one subdivision of philosophy (Standaert 2001, 606). The next six works of the collection discuss mainly ethics and religion. The first part closes with an introduction to world geography, also written by Aleni. Illustrated by several maps, including an elliptical world map, the *Zhi fang wai ji* 職方外紀, 'Areas

10. The modern Chinese term for geometry is *ji he xue* 幾何學, literally 'the study of *ji he*'.

outside the concern of the imperial geographer' (1623) describes the earth as part of the universe created by God, and Europe as the ideal realm where Christianity has brought long-lasting peace.

The second part of the 'First collection' contains five of Ricci's six works based on Clavius. It also includes three works by another former student of Clavius, Sabatino de Ursis, dealing respectively with hydraulics, the altazimuth quadrant, and the gnomon. A short treatise entitled *Gou gu yi* 句股義 'The meaning of base-and-altitude', written by Xu Guangqi after he had completed the translation of the *Elements* with Ricci, is also included. This was the first attempt to interpret the traditional approach to right triangles (*gou* 句, base, refers to the shorter side of the right angle, and *gu* 股, altitude, to the longer one) in terms of Euclidean geometry (Engelfriet 1998, 301–313; Engelfriet and Siu 2001, 294–303).

In this compilation, Ricci's treatise on the sphere based on Clavius was substituted by another one, the *Tian wen lüe* 天問略 'Epitome of questions on the heavens'. This is the only work pertaining to 'tools' that does not stem from the student lineage of Clavius: its author, Manuel Dias Jr, never left the Portuguese Assistancy of the Society of Jesus. Due to the importance of navigation in Portugal, the study of the sphere was emphasized in Jesuit colleges there (Leitão 2002). Clavius' treatise was one of a genre; it seems that the 'Epitome of questions on the heavens' was an original composition within that genre rather than a translation of a Latin text. It gave a description of Ptolemy's system; in an appendix, it reported Galileo's invention of the telescope and the observations he had made with it (Leitão 2008). This was in keeping with the Society's policy in Europe, where innovations were usually incorporated into teaching. As a whole, the works on instruments in the 'First collection' were part of the mathematical sciences construed and constructed by Clavius for Jesuit colleges.

For converted officials like Xu Guangqi and Li Zhizao, Jesuit teaching met essential concerns of their own agenda. The 'Principles' and the 'Tools' of the 'First collection' formed a coherent whole: whereas the latter could better the material life of the people, the former could contribute to their moral improvement and therefore to social harmony. Heavenly learning was a fitting response to the concerns of 'solid learning', to which *jing shi* 經世 'statecraft' was central. Xu Guangqi's list of the applications of mathematics is revealing in this respect: astrological prediction (for the state), surveying and water control, music (harmony and instruments), military technology, book-keeping and management for the civil service, civil engineering, mechanical devices, cartography, medicine, and clockwork (Wang 1984, 339–342). All these were fields in which any progress would be socially useful. This list includes not only the topics in which western learning proposed innovations but also, more importantly, some of the main fields that 'solid learning' scholars strove to study. The latter's agenda thus oriented the choice of topics for translation; only the subjects that met their concerns had a significant influence. Mathematics in its broader sense was among those subjects.

Mathematics and calendar reform

The field in which converted officials were most successful in promoting the Jesuits and their learning was astronomy. The calendar had always been of utmost symbolical and political importance in China; issued in the emperor's name, it ensured that human activity followed the cycles of the cosmos. The need for calendar reform had been felt before the Jesuits' arrival (Peterson 1968), and Ricci had recommended the Society to send missionaries versed in this matter. In 1613, Li Zhizao proposed that three Jesuits be commissioned to reform the calendar (Hashimoto 1988, 16–17). This may well have fostered opposition to Christianity (Dudink 2001). In 1629 a new proposal put forward by Xu Guangqi was finally approved. Under his supervision, a special *Li ju* 曆局 'Calendar Office' was created (Hashimoto 1988, 34–39). This meant that officials rather than private literati became the main recipients of European science.

The first output of this newly created office was a series of twenty-two works (a few of which had actually been written before 1629). They were presented to the emperor between 1631 and 1634, and formed the *Chong zhen li shu* 崇禎曆書 'Books on calendrical astronomy of the Chongzhen reign'. The knowledge they contained was very different in content and structure from that of the 'First collection': reference was no longer made to an overarching system of knowledge, nor to the Catholic religion. The Ptolemaic system was discarded in favor of the Tychonic system. Thus institutionalized, western learning had become a technical subject organized according to official astronomers' needs.

Three Jesuits, Johann Schreck, Johann Adam Schall von Bell, and Giacomo Rho, were in charge of the work; in 1633 Li Tianjing 李天經 succeeded Xu Guangqi as supervisor. More than twenty Chinese collaborated in this task. Some of these were converts, as were many Chinese who worked at the *Qin tian jian* 欽天監 'Astronomical Bureau' thereafter. In late Ming officials' eyes, calendar reform was to contribute to the restoration of social order and the dynasty's strength, at a time when the military situation in particular was getting worse. However the result of the work done at the Calendar Office ultimately benefited the newly established Qing dynasty, to which Schall offered his service on the fall of the Ming; the calendar he had calculated was promulgated in 1644. The compendium's title was changed to *Xi yang xin fa li shu* 西洋新法曆書 'Books on calendrical astronomy according to the new Western method' and a few works were added to it. This marked the Jesuits' entry into officialdom at the Astronomical Bureau.

According to Xu Guangqi's classification, which he proposed while the 'Books on calendrical astronomy' were being composed, those books should fall into five categories: *fa yuan* 法原 'fundamentals', *fa shu* 法数 'numbers', *fa suan* 法算 'calculations', *fa qi* 法器 'instruments', and *hui tong* 會通 'intercommunication', or correspondence between Chinese and western units. None of these categories correspond to specifically mathematical subjects as opposed to astronomical ones. Once the works were completed, it was not always specified which of these categories they belonged to; the 'Calculation' category remained empty. 'Fundamentals' include practical geometry and trigonometry; *Bi li gui jie* 比例規解 'Explanation of the proportional compass' is among the 'instruments'; trigonometric tables and Napier's rods are included in 'numbers'; this suggests that the latter aid to calculation was understood as a kind of moveable table (Jami 1998b). Neither Euclidean geometry nor the basics of written calculation were deemed necessary for the purposes of calendar reform. On the other hand Ricci's mathematics had to be supplemented, mainly by trigonometry. On the whole, the 'Books on calendrical astronomy' do not bring out astronomy and mathematics as two separate disciplines.

In the 1644 version of the 'Books on calendrical astronomy', a geometry treatise was added, which was not allotted into any of these categories: the *Ji he yao fa* 幾何要法 'Essential methods of geometry' (1631). It was composed of extracts from the 'Elements of geometry', focusing on constructions and leaving out proofs. The work was the result of collaboration between Aleni and Qu Shigu 瞿式穀, Qu Rukui's son, and a Christian like his father (Jami 1997). To paraphrase Xu Guangqi, a recasting of western knowledge into the 'Chinese mould'[11] had occurred between the translation of Euclid's *Elements* and the calendar reform. At the time of the former, astronomy was a branch of 'the study of quantity'. During the latter, mathematics was conversely subsumed under calendrical astronomy for which it provided a series of tools and methods.

Integrating Chinese and Western mathematics: the work of Mei Wending

Whereas conversion to Catholicism remained a marginal phenomenon in officialdom and literati circles, a number of scholars during the late Ming and early Qing period were interested in the Jesuits' mathematics. While the calendar reform took place in Beijing, it was mostly in the Lower Yangzi region, which had been of foremost economic and cultural importance since the tenth century AD, that some scholars read the Jesuits' works. The most thorough and systematic

11. For a discussion of this phrase and its posterity, see Han Qi (2001, 367–373).

of them was Mei Wending 梅文鼎, who is the best known mathematician and astronomer of the period.

Mei's syncretistic approach is suggested by the title of a collection of nine of his works that he put together in 1680: *Zhong xi suan xue tong* 中西算學通 'Integration of Chinese and western mathematics'. Only six of these nine works were eventually printed: this reflects the limits of Chinese literati's interest in the mathematical sciences at the time. Mei, however, argued that they were a key to understanding the world: in his view, *li* 理 'principles', a key concept of Neo-Confucian philosophy, could only be fathomed through *shu* 數 'numbers', and the principles thus uncovered were universally valid. For him numbers encompassed the whole of mathematics, which he divided into *suan shu* 算術 'calculation procedures' and *liang fa* 量法 'measurement methods' (*SKQS* 794, 64); accordingly, he proposed to reorganize the traditional nine chapters into two groups. Unlike the Jesuits and Chinese scholars before him, however, he also argued that calculation had primacy over measurement, as only the former could deal with invisible objects; however, the fashion of Euclidean geometry resulted in the neglect of this primordial field. In his view the great contribution of Western learning to both mathematics and astronomy was that it explained *suo yi ran* 所以然 'why it is so', whereas the Chinese tradition stated only *suo dang ran* 所當然 'what must be so' (Engelfriet 1998, 430–431; Jami 2004, 708 and 719). Acknowledging the excellence of Westerners in measurement methods, Mei proposed alternative proofs for some propositions of the 'Elements of geometry', and went on to explore solids (Martzloff 1981, 260–290). In calculation, however, he emphasized the shortcomings of the Westerners. This did not prevent him from adopting and adapting written calculation: in his lengthy *Bi suan* 筆算 'Brush calculation' he transposed the four basic operations by writing all numbers in place-value notation vertically, with the aim of making the orientation of the layout of calculations consistent with writing in China, as it was in the West.

On the other hand, Mei set out to restore what had been lost of the Chinese mathematical tradition. Thus he proposed a reconstruction of the method of *fang cheng* 方程 'rectangular arrays', equivalent to systems of linear equations in several unknowns. The method had been handed down from the eighth of the 'Nine chapters on mathematical procedures' through works like the 'Unified lineage of mathematical methods', in which problems were classified according to the number of unknowns; he regarded it as the acme of calculation. In his *Fang cheng lun* 方程論 'Discussion of the comparison of arrays' (1672),[12] Mei criticized this classification, and also chastised the authors of the 'Instructions for calculation in common script' for failing to recognize the specificity and powerfulness of the *fang cheng* method. Against both works, from which he took up a number

12. Unlike today's historians of mathematics, Mei interpreted *fang cheng* as 'comparison of arrays' (*SKQS* 795, 67; cf. Martzloff 1981, 166–168).

of problems, and corrected several mistakes, he proposed an entirely new classification of problems according to the operations involved in their resolution rather than to the number of unknowns (Martzloff 1981, 161–231; Jami 2004, 706–714). Further, he clarified how the arrays were to be laid out according to the way the problem was stated, as regarded both the place where each number was to be and its *ming* 名 'denomination', that is to say, the sign assigned to it for the purpose of solving the problem. 'Denominations' had been transmitted from the 'Nine chapters on mathematical procedures', so in this respect Mei was indeed restoring an ancient method rather than innovating.[13] After explaining the 'comparison of arrays' in all its technicalities, he went on to use it in order to solve a number of problems that pertained to other 'chapters' of the traditional nine-fold classification, and to astronomy. By showing that his reconstructed method was a generic tool that could solve problems traditionally associated with more specific methods, he substantiated the claim that it was the acme of calculation. By applying it to astronomical problems, he also exemplified why he gave primacy to calculation over measurement.

In several respects the style of the 'Discussion of the comparison of arrays' is in rupture with that dominant in mathematical works by Chinese authors of the time. Indeed, the work contained a series of problems, followed by their solution and the *fa* 法 'method' used to solve them, which included the array associated to each problem. However, the author warns us, these problems only occupy 30% of the work, and play the role of *li* 例 'examples', to illustrate *lun* 論 'discussion', which occupies 70% of the work. Indeed the examples always follow a general discussion and in turn each of them is followed by further lengthy discussion, for the purpose of *ming suan li* 明算理 'clarifying the principles of calculation' (*SKQS* 795, 68). Thus after a general discussion of positive and negative denominations, one particular problem, borrowed from the 'Unified lineage of mathematical methods', is rephrased four times; four corresponding arrays are given, in order to illustrate the rule that the first number given in the problem should be laid out in the top right place of the array, and should always be assigned a *zheng ming* 正名 'positive denomination' (*SKQS* 795, 76–78).

Mei's choice of the term *lun* 'discussion' to designate the discursive parts of his text is significant: whereas it was not a term traditionally used in mathematical texts, he knew at least two precedents. In the 'Unified lineage of mathematics', the method for solving a problem was sometimes followed by a discussion in the form of a poem, most likely with a mnemonic function. *Lun* also rendered 'proof' in the 1607 translation of the 'Elements of geometry' but it is difficult to tell whether this was independent of its use for 'discussion'. As mentioned before,

13. Following the earlier Chinese tradition, Mei considered signs associated to numbers only in the context of *fang cheng* problems. No concept of negative numbers occurs in his works.

no explanation of the deductive structure of Euclidean proofs was given by the Jesuits; on the other hand, the latter themselves frequently used *lun* in the broader sense of 'discussion' or 'discuss', as Ricci did in the passage of his preface of the 'Elements' quoted above. It is not unlikely, therefore, that Mei Wending saw the portions of the 'Elements' entitled *lun* as discussions that clarified the 'why it is so' of each proposition. The presence of lengthy 'discussions' in his own work can be understood as his appropriation of what he felt was a strong point of the Westerners' mathematical style for writing on a subject anchored in the Chinese tradition. Thus the integration of Western learning was not simply a matter of adding a new field, like Euclidean geometry, or choosing, among the methods proposed by the Jesuits and those found in earlier works, the most relevant one. The craft of writing mathematics itself shows signs of hybridization. In discourse on mathematics Chinese and western were often opposed, but in practice they were combined at every possible level.

The Kangxi emperor's appropriation of mathematics

After he was put in charge of the Astronomical Bureau, Schall successfully cultivated the favour of the young Shunzhi emperor. After the death of the latter in 1661, however, the conflicts around Schall culminated in the Calendar Case (1664–1669). Choosing dates and locations for rituals was part of his tasks as the head of the Astronomical Bureau. Therefore, when it was found out that the time of an imperial prince's funeral had been miscalculated, this mistake was added to the charge of promoting heterodox ideas that had previously been brought against him. This brought about his downfall: he was sentenced to death—a sentence soon commuted to house arrest—and all the missionaries who worked in the provinces were expelled to Macao. In 1669, in the process of assuming personal rule at the end of the regency that had followed the death of his father, the young Kangxi emperor had the case reexamined. Ferdinand Verbiest, who succeeded Schall as the main specialist in the sciences after the latter's death, turned out to be more accurate than his Chinese adversaries in predicting the length of the shadow of a gnomon at noon, and the verdict was reversed (Chu 1997). Following this, Kangxi undertook the study of western science, which he was to continue throughout his reign. Verbiest, who was his first tutor, listed the mathematical sciences which thus 'entered the imperial Court' in the wake of astronomy, each presenting to the Emperor some achievement in the form of one or several technical objects: gnomonics, ballistics, hydragogics, mechanics, optics, catoptrics, perspective, statics, hydrostatics, hydraulics, pneumatics, music, horologic technology, and meteorology (Golvers 1993, 101–129). Thus from the early years of the reign, the two-fold pattern of the Jesuits' role at court was settled. On the one hand they were court savants, who built and maintained various machines and instruments and took part in imperial projects. In line with the late Ming trend of 'solid learning', the emperor regarded most of their skills as tools for statecraft. On the other hand the Jesuits were imperial tutors, who wrote textbooks in both Chinese and Manchu. Kangxi's motivations for studying western science were two fold: genuine curiosity was combined with eagerness to be in a position to control all issues and arbitrate all controversies, and to display his abilities to higher officials. The mathematical sciences within western learning were thus integrated into the body of Confucian learning mastered by the emperor—who emulated the Sages of antiquity (Jami 2002b; 2007).

The Jesuits' tutoring of Kangxi in mathematics is best documented for the 1690s, when it seems to have been at its most intensive. There were two different teams of tutors. Geometry was mostly taught by two French Jesuits, Jean-François Gerbillon and Joachim Bouvet, in Manchu; meanwhile, Antoine Thomas was in charge of calculation and he used the Chinese language, with Tomé Pereira as his interpreter. Both teams of tutors produced textbooks that have been preserved as manuscripts (Jami and Han 2003). Kangxi also had his sons trained in the mathematical sciences; Thomas was their tutor. His most talented pupil was prince Yinzhi 胤祉, Kangxi's third son. In 1702 tutor and student were sent on an expedition to measure the length of a degree of a meridian (Bosmans 1926). This was a preliminary to the general survey of the empire that Kangxi commissioned in 1708. A number of Jesuits took part in it, applying the methods used by the Paris Academicians in their survey of France a few years earlier. The outcome of this was the famous *Huang yu quan lan tu* 皇輿全覽圖 'Complete maps of the Empire' (1718) known in Europe as the 'Kangxi Atlas' (Standaert 2001, 760–763).

The tutoring reflected Jesuit mathematical education at the time in Europe. Thus the geometry treatise that the two Frenchmen composed for the emperor was based on one of the many handbooks produced in Europe under the title 'Elements of geometry' in the seventeenth century. Their choice of *Elemens de geometrie* (1671) by Ignace Gaston Pardies for tutoring the emperor—a choice that Kangxi approved—echoed the success of the work in Europe, where it underwent several editions and reprints up to 1724, and was translated into Latin, Dutch, and English (Ziggelaar 1971, 64–68). This work fitted in with Gerbillon and Bouvet's specific agenda in teaching Kangxi. As they were among the five Jesuits sent to China in 1685 by Louis XIV, they saw themselves as representatives of French science as practised under the auspices of the Paris Académie Royale des Sciences. They were in China not only to contribute to its evangelization but also to further French interests in Asia. The latter entailed gathering data for the Académie (Landry-Deron 2001). In their tutoring, which also included medicine and other aspects of philosophy, they claimed that they wrote 'in the briefest and clearest way that [they] could, removing all there is of complicated terms and of pure chicanery, following the style of the moderns'.[14] Pardies, who had dedicated his geometry textbook to the Paris Academicians, discarded the axiomatic and deductive style that characterized Euclid as edited by Clavius, in favour of shortness and ease. This was an adjustment to the widening audience of Jesuit colleges in Europe; it also reflects the idea, common among seventeenth-century mathematicians, that clarity is an intrinsic quality of mathematics (Jami 1996; 2005, 217–221). Both the Manchu and the Chinese versions of the treatise, which are abridged translations, were written under the emperor's personal supervision: some corrections and comments in his hand are found on two copies of the treatise. Like its European counterpart, this new treatise took up the title of the translation of Euclid's *Elements*: in Chinese it was called *Ji he yuan ben* 幾何原本, like the 1607 translation.

Meanwhile, Antoine Thomas composed two lengthy treatises. Before setting sail for Asia, he had taught mathematics in Coimbra, Portugal. For this purpose he had written a kind of *vademecum*, the *Synopsis mathematica*, a work explicitly designed for candidates to the China mission as well as for novices. The first of his Chinese treatises was called *Suan fa zuan yao zong gang* 算法纂要總綱 'Outline of the essentials of calculation', possibly a translation of the title of his Latin treatise. The structure of the former work followed that of the chapters devoted to arithmetic in the latter (Han and Jami 2003, 150–152). However, while the Latin work only gave one example to illustrate each rule of calculation, the Chinese treatise contained a wealth of problems for each of these rules. Some problems were drawn from the 'Instructions for calculation in common script' by Ricci and Li Zhizao. Others evoked subjects that Kangxi discussed with the Jesuits during the tutoring sessions, such as astronomy or the speed of sound (Jami 2007). Another treatise written by Thomas presented a branch of mathematics never before taught by the Jesuits in China, namely algebra. The term was transcribed as *aerrebala* 阿爾熱巴拉 in the foreword of the treatise; however, it was the title of the treatise, *Jie gen fang suan fa* 借根方算法 'Calculation by borrowed root and powers', that gave its name to the mathematical method described in it. Seventy years after some of the Jesuit colleges started to teach Viète's notation, the Kangxi emperor was still being taught cossic algebra. In Chinese, full names in characters were used rather than abbreviations as in European treatises. Thus, for instance, the equation $x^3 + 44\,x^2 + 363\,x = 1950048$ appears in the *Jie gen fang suan fa* 借根方算法 'Calculation by borrowed root and powers' as:

一立方	┼	四四平方	┼	三六三根	═	一九五〇〇四八
1 cube	+	44 square	+	363 root	=	1950048

(Bibliothèque Municipale de Lyon, Manuscript 39–43, V 135)

14. [...] de la maniere la plus brieve et la plus claire qu'il nous a esté possible, en retranchant tout ce qu'il y a de termes embrouillés et de pure chicane, conformément au style des modernes (Archivum Romanum Societatis Iesu, Jap Sin 165, f. 101r).

Thomas's textbook may well be an original composition, but the algebra in it is similar to that found in, among other works, Clavius' *Algebra* (1608). Like Clavius, Thomas included some first degree problems in several unknowns, in which he represented the unknowns by the cyclical characters (*jia* 甲, *yi* 乙, *bing* 丙, *ding* 丁...), in a manner equivalent to that in which one would use letters. Coefficients, on the other hand, were always numerical. Thus, more than three decades after Mei Wending's 'Discussion on the comparison of arrays', a Jesuit produced two treatises that appear as refutations of Mei's criticism of Westerners as incompetent in calculation; moreover one of these treatises contained a possible alternative to the *fang cheng* method as reconstructed by Mei. At the time, algebra was not part of elementary mathematical education in Europe. Thomas had not included it in his Latin mathematical treatise, but he was familiar with symbolic algebra. That he nonetheless taught the emperor cossic algebra may reflect his wish to perpetuate the mathematics taught by Clavius and the Jesuits working in China during the late Ming period. It may also simply be due to the fact that symbolic algebra was regarded as more difficult. In 1713, that is, less than fifteen years after Thomas completed his treatise on cossic algebra, another Jesuit, Jean-François Foucquet, attempted to present symbolic algebra to Kangxi; for this purpose, he set out to write a treatise that he entitled *Aerrebala xin fa* 阿巴拉新法 'New method of algebra'. A section on first-degree problems in several unknowns was completed and explained to the emperor; however the tutoring happened to stop just as Foucquet was starting on second-degree equations, so that the emperor did not have the chance to grasp the meaning of the juxtaposition of two unknowns as a representation of their product. The 'New method of algebra' was rejected, and, given the fact that Kangxi actually arbitrated matters to do with mathematics personally, symbolic algebra did not find its way into Chinese mathematical textbooks until the second half of the nineteenth century (Jami 1986).

The emperor strove to integrate the mathematical sciences into imperial scholarship. In 1713 he created a *Suan xue guan* 算學館 'Office of Mathematics' staffed by Chinese, Manchus, and some Mongols. It was modelled on various offices of the same kind for literary or historical projects and headed by his son Yinzhi. The staff of this office compiled a three-part compendium, the *Yu zhi lü li yuan yuan* 御製律曆淵源 'Origins of musical harmony and calendrical astronomy, imperially composed', which was printed at the beginning of the Yongzheng reign (1723–1735). Western learning was dominant in the astronomical part, the *Li xiang kao cheng* 曆象考成 'Thorough investigation of calendrical astronomy' (42 chapters). It was expounded in a separate appendix in the *Lü lü zheng yi* 律呂正義 'Exact meaning of pitchpipes' (5 chapters). It was interspersed with Chinese learning in the mathematical part, entitled *Shu li jing yun* 數理精蘊 'Essence of numbers and their principles' (53 chapters), which set the standard for the study of the subject. The association of the three fields of astronomy, mathematics, and music points to the influence of the *quadrivium*, all the more so as mathematics was constructed on the dual foundations of geometry and calculation. However, the link between calendrical astronomy and the pitchpipes was a traditional one: both were about measuring and setting norms for the cosmos. Since number, that is mathematics, was used in both, putting the three disciplines together would not seem strange to Chinese readers. The rationale put forward to justify it was borrowed from the Classics, the origin of all this learning being said to be the same as that of the *Yi jing* 易經 'Classic of change' (Kawahara 1995).

Most of the content of the 'Essence of numbers and their principles' can be traced back to the Jesuits' tutoring of the 1690s. Some chapters, however, resulted from Chinese scholars' work inspired by the nine chapters tradition. The 'Essence of numbers and their principles' is divided into two parts of very unequal length, followed by some tables. The first five chapters are devoted to *li gang ming ti* 立綱明體 'Establishing the structure to clarify the substance'. After a discussion of the foundations of mathematics, which roots it into Chinese antiquity, three chapters are devoted to the 'Elements of geometry', a revised version of Gerbillon and Bouvet's textbook. This part closes on a chapter on the *Suan fa yuan ben* 算法原本 'Elements of calculation', a revised version of one of the textbooks produced in the 1690s, probably authored by Thomas and mostly based on books VII and VIII of Euclid's *Elements*. Thus, while imperial mathematics was asserted to have its origins in ancient China, its foundations stemmed from Western learning, and more precisely from the early modern European appropriation of the Euclidean tradition. The second part, comprising forty chapters, is on *fen tiao zhi yong* 分條致用 'dividing items to convey their use'. It is divided into five sections: *shou* 首 'initial', *xian* 線 'line', *mian* 面 'area', *ti* 體 'solid', and *mo* 末 'final'. The content is presented in the traditional form, that is, as a sequence of problems and solutions. After basic instruction on the four operations and fractions has been given in the 'beginning section', the three middle sections organize problems according to their dimension. A great part of the material in these first four sections can be traced back to Thomas's 'Outlines of the essentials of calculation', while some material was drawn from Chinese authors as well. Six of the ten chapters in the end section, devoted to cossic algebra, are derived from his 'Calculation by root and powers', with slightly modified vocabulary and notations; three chapters are devoted to a general presentation of the notation and of the techniques for solving equations; the three next chapters give problems that fall respectively in the 'line', 'area', and 'volume' categories. After cossic algebra, there follows a chapter of *Nan ti* 難題 'Difficult problems'; this chapter is one among several clues that suggest that Cheng Dawei's 'Unified lineage of mathematical methods', among other Chinese works, were used to compile the 'Essence of the principles of numbers'. The last three chapters are devoted to the principles of logarithms and to the proportional compass (Guo 1993, III 1143–1235). Logarithms and trigonometric tables were appended. Imperial mathematics, which encompassed most of the knowledge available at the time, integrated an updated version of western learning devised for Kangxi and some revived branches of Chinese learning.

The compilers of 'Essence of numbers and their principles' had at their disposal at least two methods for dealing with problems equivalent to systems of linear equations in several unknowns: Mei Wending's 'comparison of arrays', and Thomas's notation using cyclical characters. Unlike in the case of right triangles, for which they included both the traditional *gou gu* 句股 'base-and-altitude' methods and the techniques of western geometry, they retained only Mei Wending's method, which they presented as an independent chapter of the 'line section'. In the chapter on 'line' problems solved by 'calculation by root and powers', on the other hand, only one root, denoted as usual by *gen* 根, is used. Thus, in the eyes of the compilers, none of the methods proposed by the Jesuits for solving linear problems in several unknowns measured up to the ancient Chinese method as reconstructed by Mei. This can hardly have been the result of a bias in favour of traditional Chinese mathematics on their part: altogether only three chapters of the imperial compendium are titled after the names of the 'Nine chapters'.

In bibliographies compiled during the two centuries that followed its composition, the 'Essence of numbers and their principles' was attributed to Kangxi. The list of editors of the 'Origins of musical harmony and calendrical astronomy, imperially composed', published in 1724, comprises forty-seven names, including Yinzhi and one of his brothers. There is ample evidence that the emperor kept a close eye on the compilation's progress, discussing details such as the layout of numerical tables with Yinzhi (Jami 2002b, 40–41). The compendium was later used for the study of mathematics in imperial institutions (*SKQS* 600, 445). Thus, officials, if not all scholars, were to model their study of mathematics on that of the emperor.

Western learning without the Jesuits

The Rites Controversy, in which the Jesuit policy of accommodation to Chinese customs such as the ritual honouring of ancestors was over-ruled by Rome, brought about a change of imperial policy towards Catholic missionaries. The court Jesuits seem to have lost imperial trust after the visit of a papal legate to Beijing in 1706, bringing the news that Chinese converts must abandon all 'idolatrous' practices. In 1732, all missionaries working in the provinces were expelled to Macao; however, the Beijing Jesuits were allowed to remain and to practise their religion. They continued to be employed as official astronomers and as cartographers, engineers, architects, and artists. Western learning at court remained in the service of imperial magnificence and of control of the expanding Qing territory (Standaert 2001, 358–363, 823–835). During the Qianlong reign (1736–1795), lengthy sequels to the 'Thorough investigation of calendrical astronomy' and to the 'Exact meaning of pitchpipes' were published. By contrast, the 'Essence of numbers and their principles' does not seem to have been regarded as in need of supplementing. Although it never competed with the 'Unified lineage of mathematical methods' for popular readership, the imperial compendium represented the basis of scholarly culture in mathematics.

Eighteenth-century scholars indeed appropriated mathematics and astronomy, but not quite in the way that Kangxi had tried to foster. Instead of becoming an end in itself or a tool for other technical fields, the discipline was integrated into the main intellectual trend of China at the time, *kao zheng xue* 考證學 'evidential scholarship' (Elman 1984, 79–89; Tian 2005, 134–145). The aim was the restoration of the original text of ancient classics, the meaning of which, it was argued, had been distorted, especially by Song dynasty (960–1279) commentators. Scholars who followed this trend developed sophisticated methods in philological disciplines. Mathematics and astronomy were a tool for that purpose: ancient records of astronomical events were used to date documents and events. But they were also an object of study; thus Dai Zhen 戴震, who is regarded as the greatest philologist of the time, reconstructed the text of the 'Nine chapters on mathematical procedures'.

The turn towards ancient texts in the mathematical sciences went together with the development of the idea *xi xue zhong yuan* 西學中源 'western learning originated in China'. While at first he argued for the unity of mathematics East and West, Mei Wending eventually turned to investigating this idea in detail, encouraged by Kangxi (Chu 1994, 184–217; Han 1997). The advantage for the emperor was obvious: if the calendar was based on foreign knowledge, then he could be challenged for applying Barbarian knowledge to regulate the rites that lay at the heart of Chinese civilization. If on the other hand that knowledge had originated in China, he became the personification of the Confucian monarch who retrieved ancient learning for the empire's benefit, which was quite an achievement for a Manchu ruler. For Chinese scholars on the other hand, the Chinese origin of western knowledge neutralized any claim of superiority of the latter. The idea could have some heuristic value as was the case in the field of algebra: identification with calculation by borrowed roots and powers as introduced by Thomas eventually proved instrumental in the rediscovery of thirteenth-century celestial element algebra (Han 2003, 80–81). At the turn of the nineteenth century there were debates over the respective merits of the two methods (Tian 2005, 250–271).

Thus western learning, represented both by late Ming Jesuits' translations and by the 'Essence of numbers and their principles', became an entity opposed to

Chinese learning. Even as eighteenth-century scholars distinguished between these two types of learning, and could side with one against the other, none of them simply ignored western learning; the latter, while keeping its identity, had been appropriated.

Conclusion

Studies of the Jesuits' transmission of mathematics from Europe to China have long focused on Euclid's *Elements of geometry*, arguably to the detriment of other branches of mathematics; this is no doubt a consequence of the role of this work as a supposed embodiment of the essence of either 'western mathematics' or mathematics *tout court*. The story of 'Euclid in China' has been told in terms of European categories, as one of a radical innovation that had universal validity; 'the Chinese understanding' (or misunderstanding) of this innovation supposedly revealed general features of 'Chinese thought'. Writings on geometry by Chinese authors of the seventeenth century have been assessed according to their conformity to the Euclidean model (Martzloff 1980). This fitted in a historiography that modelled Sino-European contacts as (European) 'action' and (Chinese) 'reaction' (Gernet 1982).

Further contextualization of the introduction of Euclidean geometry (Engelfriet 1998; Jami 1996), as well as inclusion of other branches of mathematics into the narrative, have yielded a different picture, one of complex interaction rather than of action and reaction. In introducing European written arithmetic, for example, a synthesis was proposed from the onset between what the Jesuits brought in and what was found in Chinese mathematical works of the time. Looking at the Chinese category *suan* 算, which by 1600 by and large denoted the whole of mathematics, one can trace the restructuring of the field during the hundred and twenty years that followed. The Jesuits first used *suan* as referring to arithmetic, and proposed to embed the Chinese tradition into a broader field, for which their geometry provided a foundation. However, as some Chinese scholars' subsequent interpretations of *suan*, eventually taken up by the Jesuits of the Kangxi court themselves, were broader: a more general category, best rendered by the term 'calculation', was thus constructed, within which a number of competing methods were proposed. In parallel, the term *shu* 數 'number', used by the Jesuits to denote only one of the two instances of quantity, came to name the broader field that encompassed geometry and calculation. Mathematics thus gained a status within scholarship as defined in neo-Confucian philosophy: it was a tool to access *li* 理 'principles' which was the ultimate goal of all learning. Thus the cross-cultural transmission and reception of mathematics entailed its reconstruction at several levels: its methods, branches, the structure of texts, but also the discipline as a whole vis-à-vis other scholarly pursuits, were reshaped by the process of their integration into a different landscape. This conclusion brings out mathematics as a flexible and dynamic system of knowledge and practice, rather than as an immutable body of truths.

Table 2: Names and dates

Song dynasty (960–1279)	Giulio Aleni (1582–1649)
Yuan dynasty (1279–1368)	Joachim Bouvet (1656–1730)
Ming dynasty (1368–1644)	Christoph Clavius (1538–1612)
Qing dynasty (1644–1911)	Manuel Dias Jr (1574–1659)
Chongzhen reign (1628–1644)	Jean-François Foucquet (1665–1741)
Shunzhi reign (1644–1661)	Jean-François Gerbillon (1654–1707)
Kangxi reign (1662–1722)	Ignace Gaston Pardies (1636–1673)
Yongzheng reign (1723–1735)	Giacomo Rho (1592–1638)
Qianlong reign (1736–1795)	Matteo Ricci (1552–1610)
Jesuit mission (1582–1773)	Johann Adam Schall von Bell (1592–1666)
Cheng Dawei 程大位 (1533–1606)	Johann Schreck (1576–1630)
Dai Zhen 戴震 (1724–1777)	Antoine Thomas (1644–1709)
Li Shizhen 李時珍 (1518–1593)	Sabatino de Ursis (1575–1620)
Li Tianjing 李天經 (1579–1659)	Ferdinand Verbiest (1623–1688)
Li Zhizao 李之藻 (1565–1630)	For biographies of the Jesuits who went to China see: http://ricci.rt.usfca.edu/biography/index.aspx
Mei Wending 梅文鼎 (1633–1721)	
Qu Rukui 瞿汝夔 (1549–1611)	
Qu Shigu 瞿式穀 (b. 1593)	
Song Yingxing 宋應星 (1582– after 1665)	
Wang Yangming 王陽明 (1472–1529)	
Xu Guangqi 徐光啓 (1562–1633)	
Yang Hui 楊輝 (fl 1261)	
Yinzhi 胤祉 (1677–1732)	
Zhu Zaiyu 朱載堉 (1536–1611)	

Bibliography

Baldini, Ugo, *Legem impone subactis. Studi su filosofia e scienza dei Gesuiti in Italia, 1540–1632*, Bulzoni Editore, 1992.

Baldini, Ugo, 'The Portuguese Assistancy of the Society and scientific activities in its Asian mission until 1640', in Luis Saraiva ed., in *Història das ciências matematicas: Portugal e o Oriente*, 49–104, Fundação Oriente, 2000.

Bosmans, Henri, 'L'œuvre scientifique d'Antoine Thomas de Namur, S.J. (1644–1709)', *Annales de la société scientifique de Bruxelles,* 44, 46 (1924, 1926), 169–208, 154–181.

Brockey, Liam, *Journey to the East: the Jesuit mission to China, 1579–1724*, Harvard University Press, 2007.

Chemla, Karine, and Shuchun Guo, *Les neuf chapitres: le classique mathématique de la Chine ancienne et ses commentaires*, Dunod, 2004.

Cheng, Anne, *Histoire de la pensée chinoise*, Editions du Seuil, 1997.

Chu, Ping-yi, 'Technical knowledge, cultural practices and social boundaries: Wan-nan scholars and the recasting of Jesuit astronomy, 1600–1800', PhD Dissertation, UCLA, 1994.

Chu, Ping-yi, 'Scientific dispute in the imperial court: the 1664 calendar case', *Chinese Science,* 14 (1997), 7–34.

Cullen, Christopher, *Astronomy and mathematics in ancient China: the Zhoubi suanjing*, Cambridge University Press, 1996.

Cullen, Christopher, *The Suan shu shu 'Writings on reckoning': a draft translation of a Chinese mathematical collection of the second century BC, with explanatory commentary*, Needham Research Institute, 2004.

Dudink, Adrianus, 'Opposition to the introduction of Western science and the Nanjing persecution (1616–1617)', in Catherine Jami, Peter M Engelfriet, and Gregory Blue (eds), *Statecraft and intellectual renewal: the cross-cultural synthesis of Xu Guangqi (1562–1633)*, Brill, 2001, 191–224.

Elman, Benjamin A, *From philosophy to philology: intellectual and social aspects of change in late imperial China*, Harvard University, 1984.

Engelfriet, Peter M, *Euclid in China: The genesis of the first Chinese translation of Euclid's Elements, books I–VI (Jihe yuanben, Beijing, 1607) and its reception up to 1723*, Brill, 1998.

Engelfriet, Peter M, and Man-Keung Siu, 'Xu Guangqi's attempt to integrate Western and Chinese mathematics', in Catherine Jami, Peter M Engelfriet, and Gregory Blue (eds), *Statecraft and intellectual renewal: the cross-cultural synthesis of Xu Guangqi (1562–1633)*, Brill, 2001, 279–310.

Feldhay, Rivka, 'The cultural field of Jesuit science', in John W O'Malley, Gauvin Alexander Bailey, Steven J Harris, and T Frank Kennedy (eds), *The Jesuits: cultures, sciences and the arts 1540–1773*, University of Toronto Press, 1999, 107–130.

Ge Rongjin 葛榮晉 (ed), *Zhongguo shi xue si xiang shi* 中國實學思想史 'History of "solid learning" thought in China', Shoudu shifan daxue chubanshe, 1994, 3 vols.

Gernet, Jacques, *Chine et christianisme: action et réaction*, Gallimard, 1982.

Golvers, Noël, *The Astronomia europaea of Ferdinand Verbiest, S.J. (Dillingen, 1687): text, translation, notes and commentaries*, Steyler Verlag, 1993.

Gorman, Michael John, 'From "the eyes of all" to "usefull Quarries in philosophy and good literature": consuming Jesuit science 1600–1665', in John W O'Malley, Gauvin Alexander Bailey, Steven J Harris, and T Frank Kennedy (eds), *The Jesuits: cultures, sciences and the arts 1540–1773*, University of Toronto Press, 1999, 170–189.

Guo, Shuchun 郭書春 (ed), *Zhong guo ke xue ji shu dian ji tong hui. Shu xue juan* 中國科學技術典籍通彙。 數學卷 'Comprehensive archive Chinese science and technology: mathematics', 5 vols, Henan jiaoyu chubanshe, 1993.

Han, Qi, "Patronage scientifique et carrière politique: Li Guangdi entre Kangxi et Mei Wending", *Etudes chinoises*, 16 (1997), 7–37.

Han, Qi, 'Astronomy, Chinese and Western: the influence of Xu Guangqi's views in the early and mid-Qing', in Catherine Jami, Peter M Engelfriet, and Gregory Blue (eds), *Statecraft and intellectual renewal: the cross-cultural synthesis of Xu Guangqi (1562–1633)*, Brill, 2001, 360–379.

Han, Qi, 'L'enseignement des sciences mathématiques sous le règne de Kangxi (1662–1722) et son contexte social', in Christine Nguyen–Tri and Catherine Despeux (eds), *Education et instruction en Chine. II. Les formations spécialisées*, Centre d'études chinoises and Peeters, 2003, 69–88.

Han, Qi 韓琦, and Catherine Jami 詹嘉玲, 'Kang xi shi dai xi fang shu xue zai gong ting de chuan bo—yi An Duo he "Suan fa zuan yao zong gang" de bian zuan wei li 康熙時代西方數學在宮廷的傳播—以安多和《算法纂要總綱》的編纂為例' 'The transmission of Western mathematics at the Kangxi court—the case of Antoine Thomas' "Outline of the essentials of calculation"', Zi ran ke xue shi yan jiu 自然科學史研究 22 (2003), 145–156.

Hashimoto, Keizo, *Hsü Kuang-ch'i and astronomical reform: The process of the Chinese acceptance of Western astronomy, 1629–1635,* Kansai University Press, 1988.

Hashimoto, Keizo, and Jami, Catherine, 'From the *Elements* to calendar reform: Xu Guangqi's shaping of mathematics and astronomy', in Catherine Jami, Peter M Engelfriet, and Gregory Blue (eds), *Statecraft and intellectual renewal: the cross-cultural synthesis of Xu Guangqi (1562–1633)*, Brill, 2001, 263–279.

Jami, Catherine, 'Jean-François Foucquet et la modernisation de la science en Chine: la "Nouvelle méthode d'algèbre"', Master's Thesis, University of Paris 7, 1986.

Jami, Catherine, 'Rencontre entre arithmétiques chinoise et occidentale au XVII[e] siècle', in Paul Benoit, Karine Chemla, and Jim Ritter (eds), *Histoire de fractions, fractions d'histoire*, Birkhaüser, 1992, 351–373.

Jami, Catherine, 'History of mathematics in Mei Wending's (1633–1721) work', *Historia Scientiarum* 53 (1994), 157–172.

Jami, Catherine, 'From Clavius to Pardies: the geometry transmitted to China by Jesuits (1607–1723)', in Federico Masini (ed), *Western humanistic culture presented to China by Jesuit missionaries (17th–18th centuries)*, Institutum Historicum S.I., 1996, 175–199.

Jami, Catherine, 'Giulio Aleni's contribution to geometry in China: the *Jihe yaofa*', in Tiziana Lippiello and Roman Malek (eds), *'Scholar from the West': Giulio Aleni S.J. and the dialogue between Christianity and China*, Steyler Verlag, 1997, 553–572.

Jami, Catherine, 'Abacus (Eastern)', in Robert Bud and Deborah Jean Warner (eds), *Instruments of science: a historical encyclopedia*, Science Museum; National Museum of American History, Smithsonian Institution, 1998a, 3–5.

Jami, Catherine, 'Mathematical knowledge in the *Chongzhen lishu*', in Roman Malek (ed), *Western learning and christianity in China: The contribution of Johann Adam Schall von Bell, S.J. (1592–1666)*, Sankt Augustin: China-Zentrum and Monumenta Serica Institute, 1998b, 661–674.

Jami, Catherine, 'Teachers of mathematics in China: the Jesuits and their textbooks (1580–1723)', *Archives internationales d'histoire des sciences*, 52 (2002a), 159–175.

Jami, Catherine, 'Western learning and imperial control: the Kangxi Emperor's (r. 1662–1722) Performance', *Late Imperial China*, 23 (2002b), 28–49.

Jami, Catherine, 'Légitimité dynastique et reconstruction des sciences en chine au XVII[e] siècle: Mei Wending (1633–1721)', *Annales* 59 (2004), 701–727.

Jami, Catherine, 'For whose greater glory? Jesuit strategies and science during the Kangxi reign', in Xiaoxin Wu (ed), *Encounters and dialogues: changing perspectives on Chinese-Western exchanges from the sixteenth to the eighteenth centuries*, Steyler Verlag, 2005, 211–226.

Jami, Catherine, 'Western learning and imperial scholarship: the Kangxi emperor's study', *East Asian Science, Technology and Medicine*, 27 (2007), 144–170.

Jami, Catherine, and Qi Han, 'The reconstruction of imperial mathematics in China during the Kangxi reign (1662–1722)', *Early science and medicine,* 8 (2003), 88–110.

Kawahara Hideki 川原秀城, 'Ritsureki engen to Kato Rakusho 律暦淵源と河圖洛書' 'The "Origins of musical harmony and calendrical astronomy" and the "Hetu" and "Luoshu"' *Chūgoku kenkyū shūkan* 中国研究週刊 16 (1995), 1319–1410.

Lam, Lay Yong, *A critical study of the Yang Hui suan fa: a thirteenth-century Chinese mathematical treatise,* Singapore University Press, 1977.

Lam, Lay Yong, and Tian Se Ang, *Fleeting footsteps: tracing the conception of arithmetic and algebra in ancient China*, World Scientific, 2004.

Landry-Deron, Isabelle, 'Les mathématiciens envoyés en Chine par Louis XIV en 1685', *Archive for the History of Exact Sciences*, 55 (2001), 423–463.

Leitão, Henrique, 'Jesuit mathematical practice in Portugal, 1540–1759', in Mordechai Feingold (ed), *The new science and Jesuit science: seventeenth century perspectives,* Kluwer, 2002, 229–247.

Leitão, Henrique, 'The contents and context of Manuel Dias' *Tianwenlüe*', in Luis Saraiva and Catherine Jami (eds), *History of mathematical sciences: Portugal and East Asia III. The Jesuits, the Padroado and East Asian science*, World Scientific Publishing, 2008, 99–121.

Li, Yan, and Shiran Du, *Chinese mathematics: a concise history*, translated John N Crossley and Anthony W–C Lun, Clarendon Press, 1987.

Martzloff, Jean-Claude, 'La compréhension chinoise des méthodes démonstratives euclidiennes au XVII[e] siècle et au début du XVIII[e]', in *Actes du IIe Colloque international de sinologie: les rapports entre l'Europe et la Chine au temps des Lumières. Chantilly, 16–18 septembre 1977,* Les Belles Lettres, 1980.

Martzloff, Jean–Claude, *Recherches sur l'œuvre mathématique de Mei Wending* (1633–1721), Collège de France, 1981.

Martzloff, Jean-Claude, 'Clavius traduit en chinois', in Luce Giard (ed), *Les jésuites à la Renaissance: Système éducatif et production du savoir,* PUF, 1995, 309–332.

Martzloff, Jean-Claude, *A history of Chinese mathematics*, Springer, 1997.

Needham, Joseph, *Science and civilisation in China. 4: Physics and Physical technology, part 1: Physics*, Cambridge University Press, 1962.

Pan, Yining 潘亦寧, 'Zhong xi shu xue hui tong de chang shi: yi *Tong wen suan zhi* (1614 nian) de bian zuan wei li 中西數學會通的嘗試：以"同文算指' (1614年) 的編纂為例.' 'Attempts at integrating Chinese and Western mathematics: a case study of the "instructions for calculation in common script"', *Zi ran ke xue shi yan jiu* 自然科學史研究, 25 (2006), 215–226.

Peterson, Willard, 'Calendar reform prior to the arrival of missionaries at the Ming court', *Ming Studies*, 21 (1968), 45–61.

Peterson, Willard, 'Western natural philosophy published in late Ming China', *Proceedings of the American Philosophical Society*, 117 (1973), 295–322.

Peterson, Willard, 'Learning from heaven: the introduction of Christianity and other Western ideas into late Ming China', in Denis Twitchett and Frederick W Mote (eds), *Cambridge history of China: the Ming dynasty 1368–1644*, Cambridge University Press, 1998, 708–788.

Ratio studiorum: Plan raisonné et institution des études dans la Compagnie de Jésus, Bilingual edition Latin and French. Presented by Adrien Demoustier & Dominique Julia, translated by Léone Albrieux & Dolorès Pralon-Julia, annotated by Marie-Madeleine Compère, Belin, 1997.

Reich, Karen, 'The 'Coss' tradition in algebra', in Ivor Grattan-Guinness (ed), *Companion encyclopedia of the history and philosophy of the mathematical sciences*, Routledge, 1994, 192–199.

Ricci, Matthieu, and Trigault, Nicolas, *Histoire de l'expédition chrétienne au royaume de la Chine 1582–1610*, Desclée de Brouwer, 1978 (annotated reprint; original edition: Lille, 1617).

Romano, Antonella, *La Contre-réforme mathématique: constitution et diffusion d'une culture mathématique jésuite à la Renaissance (1560–1640)*, Ecole Française de Rome, 1999.

Romano, Antonella, 'Arpenter la "vigne du Seigneur"? Note sur l'activité scientifique des jésuites dans les provinces extra-européennes (XVI^e–XVII^e siècles)', *Archives Internationales d'Histoire des Sciences*, 52 (2002), 73–101.

Romano, Antonella, 'Teaching mathematics in Jesuit schools: programs, course content, and classroom practices', in John W O'Malley, Gauvin Alexander Bailey, Steven J Harris, and T Frank Kennedy (eds), *The Jesuits II: cultures, sciences and the arts 1540–1773*, University of Toronto Press, 2006, 355–370.

SKQS: Ying yin Wen yuan ge Si ku quan shu 景印文淵閣四庫全書 'Facsimile of the "Complete library in four sections" kept at the Wenyuan Pavilion' 1500 vols, Taibei shangwu yinshuguan, 1986.

Standaert, Nicolas, (ed), *Handbook of Christianity in China*, Brill, 2001.

Standaert, Nicolas, 'European astrology in early Qing China: Xue Fengzuo's and Smogulecki's translation of Cardano's Commentaries on Ptolemy's *Tetrabiblos*', *Sino-Western cultural relations journal*, 23 (2001b), 50–79.

Tian, Miao 田淼, *Zhong guo shu xue de xi hua li cheng* 中國數學的西化歷程 'The process of Westernization of Chinese mathematics', Shandong jiaoyu chubanshe, 2005.

Volkov Alexei, 'Counting rods', in Robert Bud and Deborah Jean Warner (eds), *Instruments of science: a historical encyclopedia*, Science Museum; National Museum of American History, Smithsonian Institution, 1998, 155–156.

Wang Zhongmin 王重民 (ed), *Xu Guangqi ji* 徐光啓集 'Collected writings of Xu Guangqi', 2 vols, Shanghai guji chubanshe, 1984.

Wardy, Robert, *Aristotle in China: Language, categories, and translation*, Cambridge University Press, 2000.

Yabuuti, Kiyosi, *Une histoire des mathématiques chinoises*, translated Catherine Jami and Kaoru Baba, Belin, 2000.

Ziggelaar, August, *Le physicien Ignace Gaston Pardies, S.J. (1636–1673)*, Odense Universitetsforlag, 1971.

CHAPTER 1.4

The internationalization of mathematics in a world of nations, 1800–1960

Karen Hunger Parshall

Mathematics has a history with elements of both the contingent and the transcendent. Over the course of the nineteenth century, as the emergence of nation states increasingly defined a new geopolitical reality in Europe, competition among states manifested itself in the self-conscious adoption of the contingent, cultural standards of those states viewed as the 'strongest'. In the case of mathematics, these self-consciously shared cultural standards centred on educational ideals, the desire to build viable and productive professional communities with effective means of communication, and the growing conviction that personal and national reputation was best established on an international stage (Parshall 1995).

In this context, mathematics also increasingly became a 'language spoken' and an endeavor developed internationally, that is, between and among the mathematicians of different nations.[1] For example, in the late nineteenth and early

1. The terminology is important. The word 'international' connotes, as indicated here, something *shared between or among* mathematicians. 'Internationalization', the topic of this chapter, is the process by which a globalized community of mathematicians, which *shares* a set of values or goals, has developed. That process, however, has sometimes involved merely 'transnational' communication, that is, communication *across* national borders, whether or not that communication is understood or appreciated. Transnational communication may ultimately lead to mutually appreciated, *shared* values and goals, but this is not a necessary consequence. The words 'international' and 'transnational' will be used in these respective senses in what follows. For more on the terminology that has developed in the historical literature on the process of the internationalization of science, see Parshall and Rice (2002, 2–4).

twentieth centuries, an Italian style of algebraic geometry with its own very idiosyncratic method of theorem formation and proof—a language of algebraic geometry that essentially only Italians spoke—developed in the context of a newly united Italian nation state seeking to demonstrate its competitiveness in the international mathematical arena and in parallel to the very different German tradition (Brigaglia and Ciliberto 1995). By the mid-twentieth century, however, following the advent of modern algebra with its structural approach to, and organization of, mathematics, algebraic geometers whether in the British Isles, Germany, or Italy, or in the United States, China, or Japan all spoke largely the same, nationally transcendent, mathematical language and tackled important, open problems recognized as such by all (Schappacher 2007).

That mutual recognition had stemmed, among other things, from the internationalization of journals and from the institutionalization of the International Congresses of Mathematicians (ICMs) beginning in 1897 for the direct communication of mathematical results and research agendas. It also manifested itself, at least symbolically, in the awarding of the first Fields Medals in 1936 in recognition of that mathematical work judged 'the best' worldwide. This chapter traces the evolution of mathematics as an international endeavor in the context both of the formation of professional communities in a historically contingent, geopolitical world and of the development of a common sense of research agenda via the evolution of a nationally transcendent mathematical language.

The establishment of national mathematical communities in the nineteenth century

Although scientific communities began to coalesce in the seventeenth century around societies like the Accademia dei Lincei in Rome, the Royal Society in London, and the Académie des Sciences in Paris, the evolution of national mathematical communities, indeed the evolution of national communities regardless of the specialty, was largely a nineteenth-century phenomenon. In mathematics as well as in other academic disciplines, Prussia was in the vanguard in the last half of the nineteenth century, serving as a model for other emergent nation states and ultimately supplanting France as the dominant mathematical nation in western Europe (Grattan-Guinness 2002).

Defeated during the Napoleonic Wars at the beginning of the nineteenth century, Prussia had responded with a major political, socioeconomic, and educational reorganization aimed at safeguarding against a similar humiliation in the future. One of the masterminds behind the educational reforms, Wilhelm von Humboldt, used the new University of Berlin (founded in 1810) as a platform from which to launch a neohumanist educational agenda aimed at 'provid[ing] a model for scholarship as well as an idealistic framework for galvanizing the German people into action' (Pyenson 1983, 6). In particular, the classical languages and mathematics, but also the physical sciences, were emphasized in an institutional context that was unfettered by political or religious concerns, and that fostered teaching and pure research over what were perceived as the more utilitarian concerns of the French. This evolved into the twin ideals of *Lehr- und Lernfreiheit*, the freedom to teach and to learn in a politically and religiously disinterested university environment characterized by the tripartite mission of teaching and the production of both original research and future researchers. Universities in Berlin, Königsberg, and ultimately Leipzig, Erlangen, Göttingen, and elsewhere produced a generation of mathematicians who matured as researchers not only in professorial lecture halls but also in targeted mathematical seminars. The research they generated, moreover, appeared on the pages of specialized journals like Crelle's *Journal für die reine und angewandte Mathematik* (founded in 1826) and later the *Mathematische Annalen* (founded in 1869).

In the last half of the nineteenth century and up to the outbreak of World War I, educational reformers in general and mathematical aspirants in particular from China (Dauben 2002, 270), Italy (Bottazzini 1981), Japan (Sasaki 2002, 236–238), Spain (Ausejo and Hormigón 2002, 51), the United States (Parshall and Rowe 1994), and other countries took their lead from Prussia in crafting broad reforms as well as more specific mathematics curricula that aimed at transplanting to, and naturalizing in, their respective soils the perceived fruits of the Prussian system. One result of this transplantation and naturalization was the consolidation and growth of mathematical research communities in a number of national settings between the closing decades of the nineteenth century and the opening decades of the twentieth.

After its defeat in the Franco–Prussian War of 1870–1871, France, too, moved toward reforms of its educational system. French scientists, in fact, had long been warning that they were falling behind the Germans (Grattan-Guinness 2002, 24–25; Gispert 2002). In the United States, the Civil War that had divided the nation in the years from 1861 to 1865 was followed by a so called Gilded Age that witnessed not only the development of federally funded institutions of higher education—the land-grant universities—for the promotion especially of the practical sciences of agriculture, mining, and engineering, but also the establishment of new, privately endowed universities. The presidents of both of these new kinds of institutions consciously looked across the Atlantic for exemplars on which to model their new educational experiments. In importing the research ethos of the Prussian universities, two of the privately endowed universities, the Johns Hopkins University (founded in 1876) and the University of Chicago (founded in 1892), set the tone for higher educational reform in the United States. In mathematics, this translated into the formation of at least two programmes that enabled research-level mathematical training competitive with—although not yet equal to—that attainable, for example, at Berlin or Göttingen (Parshall 1988; Parshall and Rowe 1994, 367, note 9). At the University of Chicago, in particular, two of the three original members of the mathematics faculty—Oskar Bolza and Heinrich Maschke—were Göttingen-trained, German mathematicians, and they, together with their American colleague E H Moore, directly imported the ideas of mathematicians like Felix Klein on elliptic and hyperelliptic function theory and David Hilbert on the foundations of mathematics to their American students (Parshall and Rowe 1994, 372–401). Those students—independently and in concert with their mentors—embraced and extended the mathematical ideas to which they were exposed.[2] In so doing, they participated in what was an increasingly transatlantic mathematical dialogue on research questions of common interest,[3] although this kind of direct importation of mathematical ideas did not dissuade American mathematical aspirants, especially in the 1880s, 1890s, and in the first decade of the twentieth century, from travelling abroad for postgraduate training (Parshall and Rowe 1994, 189–259 and 439–445).

By the outbreak of World War I, America's older colleges, notably Harvard, Yale, and Princeton, had made the transition from undergraduate colleges to research-oriented universities. Together, these and other institutions of higher education contributed to the formation of an American mathematical research community that coalesced around the New York Mathematical Society at its founding in 1888 and then around its reincarnation in 1894 as the American Mathematical Society.[4] This national community also sustained specialized journals like the *American Journal of Mathematics* (founded in 1878), the *Annals of Mathematics* (founded in 1884), and the *Transactions of the American Mathematical Society* (first published in 1900) that actively fostered the communication of mathematical results (Parshall and Rowe 1994, 427–453).

If the United States provides an illustration of a national mathematical community that formed in the nineteenth century in fairly direct emulation of the

2. Students from Italy—notably, Luigi Bianchi, Gregorio Ricci-Curbastro, and Gino Fano—also went to Germany expressly to work with Felix Klein first at the Technische Hochschule in Munich from 1875 to 1880 and then at Göttingen after 1886.

3. See, for example, Fenster (2007) for an account of the transnational development between A Adrian Albert in the United States and Richard Brauer, Emmy Noether, and Helmut Hasse in Germany of the theory of finite-dimensional algebras over the rationals.

4. The American Mathematical Society (AMS) modeled itself on the London Mathematical Society (LMS), which had formed in 1865 (and which, despite its name, was a national society). The LMS was the first such society but other national societies soon followed; for example, the Société mathématique de France began in 1872 and the Tokyo Mathematical (later Mathematico-Physical) Society started in 1877. The more localized Moscow Mathematical Society actually predated them all; it was founded in 1864. By the early decades of the twentieth century even more countries—like Spain (Ausejo and Hormigón 2002, 53–57), Italy, Japan, and China (see below)—had followed suit. The specialized national mathematical society—like the specialized mathematical journal—came to define national mathematical communities internationally.

Prussian model, England represents a country in which a national mathematical community developed with only occasional glances across the Channel, and those perhaps more at France than at Germany. In 1830, Charles Babbage famously caricatured English science in his *Reflections on the decline of science in England*. For Babbage, that decline had resulted from many factors, not the least of which were the ineffectiveness of the Royal Society and the absence of true cultural and professional inducements for science in England.

As with all caricatures, Babbage's contains elements of truth. His rhetorical salvos—as well as those of others like John F W Herschel and Augustus Bozzi Granville—came just as the new British Association for the Advancement of Science was being founded and the Royal Society of London was entering into a period of reorganization and renewal. If English science had been in decline before 1830, its trajectory had a strongly positive slope by the middle of the nineteenth century as exemplified by John Couch Adams's mathematical prediction—independent of that of the French astronomer, Urbain Leverrier—of the existence of the planet Neptune in 1845–6. As the case of Adams also suggests, if, as Herschel famously averred in 1830, 'in mathematics we have long since drawn the rein, and given over a hopeless race', things were improving on that score as well (Babbage 1830, ix).

Although mathematics had long been published in the British Isles in the context of the journals of general science societies, the decades immediately following mid-century witnessed there as in Germany, France, Italy, Russia, and elsewhere the development of specialized, research oriented journals that helped to distinguish a community of mathematical researchers (Despeaux 2002).[5] Of particular importance in the British context was the *Quarterly Journal of Pure and Applied Mathematics* which began under that title in 1855 but which had resulted from an evolutionary process that had transformed the highly localized, undergraduate-oriented *Cambridge Mathematical Journal* (founded in 1837) into the more self-consciously research-oriented and trans-Britannic *Cambridge and Dublin Mathematical Journal* (in 1845) (Crilly 2004).

In 1855 and under the editorial leadership of James Joseph Sylvester and Norman Ferrers, the *Quarterly Journal* not only followed France's *Journal de mathématiques pures et appliquées* in emulating in name Crelle's *Journal für die reine und angewandte Mathematik* but also specifically articulated an internationalist view (albeit with nationalistic overtones) of the mathematical endeavor. As the editors put it in their 'address to the reader' in the journal's first number, their aim was

5. Crelle's *Journal für die reine und angewandte Mathematik* and the *Mathematische Annalen* have already been mentioned. In France, among others, were Liouville's *Journal de mathématiques pures et appliquées* (begun in 1836) and later the *Bulletin de la Société mathématique de France* (started in 1873), while Italy supported the publication of, for example, the *Annali di matematica pura ed applicata* (first published in 1858), and mathematicians in Moscow launched *Matematicheskii Sbornik* in 1866.

to 'communicate a general idea of *all* that is passing in mathematical circles, *both at home and abroad*, that can be of interest to Mathematicians as such' (Parshall 2007, 139; my emphasis). To that end, they actively fostered contributions from other countries, and especially from France, thanks both to the presence of Charles Hermite on the editorial board and to the ongoing efforts particularly of Sylvester (Despeaux 2002, 243–271).[6] In this way, they brought some of the latest foreign mathematical research directly to their fellow countrymen in an effort to keep them abreast of what was being done abroad. It was not, however, just a matter of keeping current; it also involved becoming actively competitive on what was recognized as an increasingly international mathematical stage. The editors held 'that it would be little creditable to English Mathematicians that they should stand aloof from the general movement, or else remain indebted to the courtesy of the editors of foreign Journals, *for the means of taking part in a rapid circulation and interchange of ideas by which the present era is characterised*' (Parshall 2007, 139; my emphasis). No longer would the British Isles be mathematically insular.[7] It was a national participant in what was increasingly viewed as a trans-European, if not yet fully international, mathematical endeavor.[8]

Transnational and international impulses in the closing decades of the nineteenth century

Mathematics, as the views expressed by Sylvester and his editorial team illustrate, came to be seen during the last half of the nineteenth century as a body of knowledge that develops effectively through the communication of ideas across national political borders. Sometimes that communication produces—as in the case of Liouville and various of his contributions to, for example, mechanics, potential theory, and differential geometry—new results inspired by and built on the work of mathematicians in other countries (Lützen 2002, 95–100). Or it serves, as in the case of Cesare Arzelà during the 1886–7 academic year, to provide a rich literature—the works of Eugen Netto, Peter Lejeune Dirichlet, Joseph Serret, Camille Jordan—from which to craft the first course of lectures on Galois theory ever to be given in Italy (Martini 1999). As these examples illustrate, *transnational* communication could lead to an *internationally* shared set of research

6. Other 'national' journals also accepted and encouraged contributions from abroad in an effort at international communication, for example, Liouville's *Journal* (Lützen 2002, 91–93).

7. Although some Russian mathematicians like Pafnuti Chebyshev traveled to western Europe to make scientific contacts, and some mathematicians like J J Sylvester journeyed to Russia, the Russian mathematical community experienced first a kind of linguistic isolation and then also a political isolation relative to the rest of Europe in the nineteenth and well into the twentieth century. This did not, however, prevent the formation there of strong mathematical traditions in number theory at St Petersburg University and in function theory at Moscow University.

8. On the development of mathematical Europe, see Goldstein *et al.* (1996).

goals. Communication could, however, be complicated by the growing spirit of active competition not only between individual, emerging national communities but also between individuals within those nations to establish their reputations. A striking example of this phenomenon was the development in the British Isles and in Germany of two distinct approaches to, and languages for, the theory of invariants.

Although examples of what would come to be known as invariants may be found, like the germs of so much other modern mathematics, in Gauss's *Disquisitiones arithmeticæ* of 1801, invariant theory developed in a largely algebraic context in the British Isles and in a primarily number-theoretic and geometric context in Germany beginning in the 1840s and continuing strongly through the 1880s (Parshall 1989). In both settings, the basic question was the same: given a homogeneous polynomial in n (although in practice usually just two or three) variables with real coefficients, find all expressions in the coefficients (invariants) or in the coefficients and the variables (covariants) that remain unchanged under the action of a linear transformation.

As the simplest example, and this example appeared in the *Disquisitiones*, consider $Q = ax^2 + 2bxy + cy^2$ and a nonsingular linear transformation of the variables x and y which takes x to $mx + ny$ and y to $m'x + n'y$, for m, n, m', and n' real numbers and for $mn' - m'n \neq 0$. Applying this transformation to Q gives $Ax^2 + 2Bxy + Cy^2$, where A, B, and C are obviously expressions in a, b, c, m, n, m', and n'. It is easy to see that the following equation holds: $B^2 - AC = (mn' - m'n)^2 (b^2 - ac)$, that is, the expression $b^2 - ac$ in the coefficients of Q, the discriminant, remains invariant up to a power of the determinant of the linear transformation.

Developing a theory of how to find all such expressions occupied Arthur Cayley, J J Sylvester, George Salmon, and others in the British Isles as well as Otto Hesse, Siegfried Aronhold, Alfred Clebsch, Paul Gordan, and others in Germany. The British employed very concrete calculational techniques to seek explicit Cartesian expressions of the invariants, as in the form $b^2 - ac$ above; the Germans developed a more abstract notation and approach, although they, too, aimed at finding complete systems of covariants for homogeneous polynomials of successive degrees. Each group also worked largely in isolation from the other, with the British publishing primarily in their own journals and the Germans in theirs, until 1868 when Gordan proved the finite basis theorem—namely, for any homogeneous form in two variables, a finite (minimum generating) set of covariants generates them all—and explicitly called attention to a major flaw in the British invariant-theoretic superstructure. The British, and especially Sylvester, then went to work to correct the error and to vindicate their techniques. Nothing less than national mathematical pride was at stake, yet neither side could really understand the work of the other. They had literally been speaking different mathematical languages that had been created in their respective national contexts, yet their confrontation over the finite basis theorem also evidenced the increasingly transnational—if perhaps not yet fully international—nature of mathematics by the last quarter of the nineteenth century (Parshall 1989).

Coincidentally, but symptomatic of the kind of situation that had presented itself in invariant theory, a new type of mathematical publication, the reviewing journal, was launched in Germany in 1868 expressly 'to provide for those, who are not in a position to follow independently every new publication in the extensive field of mathematics', to give them moreover 'a means to gain at least a general overview of the development of the science', and 'to ease the efforts of the scholar in his search for established knowledge'.[9] The *Jahrbuch über die Fortschritte der Mathematik* represented a collaborative effort among mathematicians to survey the international mathematical landscape and to report, in German, on the research findings of mathematicians throughout Europe and eventually in the United States and elsewhere. By the end of the century, the *Jahrbuch* had been joined by two additional reviewing journals—the French *Bulletin des sciences mathématiques et astronomiques* (begun in 1895) and the Dutch *Revue semestrielle des publications mathématiques* (started in 1897)—in the ongoing quest effectively to disseminate mathematical results transnationally (Siegmund-Schultze 1993, 14–20).[10]

These reviewing efforts, moreover, were supplemented by great synthetic undertakings like the *Enzyclopädie der mathematishen Wissenschaften*, begun in 1894 under the direction of Felix Klein, and the French translation and update, the *Encyclopédie des sciences mathématiques*, started in 1904 with Jules Molk as editor. Both of these works aimed, in some sense, to go beyond the reviewing journals by surveying contemporary mathematics and indicating promising lines for future research. In so doing, they had the potential to create shared research agendas across national boundaries.[11]

Transnational impulses also manifested themselves at this time in the form of new, expressly international research journals, although these ventures also had nationalistic or regionalistic overtones. As one case in point, the Norwegian mathematician Sophus Lie encouraged his Swedish friend and fellow mathematician Gösta Mittag-Leffler to found a new journal, *Acta mathematica* (first

9. For the quote, see the 'Vorrede' of the *Jahrbuch* as translated in Despeaux (2002, 297–298).

10. In the twentieth century, the *Zentralblatt für Mathematik und ihre Grenzgebiete* (begun in 1931 by the German publishing house of Julius Springer) and the *Mathematical Reviews* (started in 1940 by the American Mathematical Society) represented two rival, national, international reviewing journals. The *Mathematical Reviews* was founded largely in response to the dismissal of the Italian Jewish mathematician, Tullio Levi-Civita, as editor of the *Zentralblatt* and to the *Zentralblatt*'s National Socialist policy of debarring Jewish mathematicians from reviewing the work of German mathematicians (Siegmund-Schultze 2002, 340–341). As the case of these two journals makes clear, even the ostensibly international—or at least transnational—reviewing journal was not immune to broader geopolitical currents.

11. Translations were yet another manifestation of efforts at transnational communication. On, for example, a sustained nineteenth-century French translation effort, see Grattan-Guinness (2002, 39–44).

published in 1882), that was to be international in outlook while highlighting the best of Scandinavian mathematical research (Barrow-Green 2002, 140–148). Similarly, the Italian mathematician Giovan Battista Guccia was instrumental not only in founding the Circolo Matematico di Palermo in 1884, a society that despite its local name soon became Italy's *de facto* national mathematical organization, but also the Circolo's *Rendiconti* (first published in 1887). By the outbreak of World War I, both the Circolo and its *Rendiconti* had succeeded in the agenda Guccia had explicitly articulated, namely, 'to internationalize, to diffuse, and to expand mathematical production of the whole world, making full use of the progress made by modern civilization in international relations' (Brigaglia 2002, 187–188).

The International Congresses of Mathematicians and the impact of World War I

Guccia's efforts in Italy, especially in the 1890s and up to the outbreak of World War I, reflected a widely spreading sense among mathematicians that the time was ripe for fostering greater international contact and cooperation. The German mathematician Georg Cantor was one of the first actively to advocate the idea of mounting an actual international congress of mathematicians. Frustrated by the hostile reception that his work on transfinite set theory had received within the hierarchical and paternalistic German university system, Cantor sought as early as 1890 to create a venue for the presentation of new mathematical ideas that would be free of internal mathematical politics and prejudices. In Cantor's view, an international arena would provide the openness and diversity of perspective that he found so lacking in his parochial national context (Dauben 1979, 162–165). By 1895, he had succeeded through what was effectively an international letter-writing campaign in enlisting the support for his efforts of mathematicians like Charles Hermite, Camille Jordan, Charles Laisant, Émile Lemoine, and Henri Poincaré in France, Felix Klein and Walther von Dyck in Germany, and Alexander Vassiliev in Russia, among others (Lehto 1998, 3).

After much discussion and negotiation, the first International Congress of Mathematicians was held in 1897 in Zürich, in politically neutral Switzerland. In all just over two hundred mathematicians from sixteen countries—among them, Austria-Hungary, Finland, France, Germany, Great Britain, Italy, Russia, Switzerland, and the United States—took part in the congress. In addition to hearing a full and rich program of mathematical lectures, the participants succeeded in formulating a set of objectives for future congresses. These events would aim 'to promote personal relations among mathematicians of different countries', to survey 'the present state of the various parts of mathematics and its applications and to provide an occasion to treat questions of particular importance', 'to advise the organizers of future Congresses', and 'to deal with questions related to bibliography, terminology, etc. requiring international cooperation' (Lehto 1998, 7–11, quotes on 9–10). In light of the emphasis on treating 'questions of particular importance' and on issues like terminology that might require 'international cooperation', those present at the Zürich ICM clearly foresaw a mathematical world in which researchers, regardless of their nationalities, communicated in ever more common mathematical terms in their pursuit of answers to questions commonly viewed as 'important'. At the second ICM, held in Paris in 1900, David Hilbert did much to shape this new, international, mathematical world order.

In the address he gave on 'Mathematical problems', Hilbert famously charted the courses of a number of mathematical fields by isolating in them what he viewed as key unsolved problems. As he explained in his introductory remarks, he aimed 'tentatively as it were, to mention particular definite problems, drawn from the various branches of mathematics, from the discussion of which an advancement of science may be expected' (Hilbert 1900, 7). Among these, the first six problems highlighted what became, owing in no small part both to Hilbert's Paris lecture and to the publication in 1899 of his *Grundlagen der Geometrie*, an emphasis in twentieth-century mathematics on an axiomatic, foundational, and ultimately structural approach (Mehrtens 1990, 108–165; Corry 1996, 137–183). In some sense, this not only provided a vernacular in which mathematicians, regardless of their nationality, could communicate, but also delineated specific structures—groups, rings, fields, algebras, topological spaces, vector spaces, probability spaces, Hilbert spaces, and so on—for further mathematical development.

The import of Hilbert's address at the Paris ICM was sensed immediately. In addition to its publication in French translation in the Congress proceedings, the address was published in German in the *Nachrichten von der königlichen Gesellschaft der Wissenschaften zu Göttingen* and in the *Archiv der Mathematik und Physik* as well as in English translation in the *Bulletin of the American Mathematical Society*. German, French, and English speakers could all participate in the agenda that Hilbert had laid out.[12]

The next three ICMs took place in Heidelberg, Rome, and Cambridge, at four-year intervals from 1904 to 1912. The number of attendees steadily increased as did non-European participation. At the Cambridge ICM, in particular, of the five hundred and seventy-four participants, eighty-two were non-European with two from Africa, six from Asia, sixty-seven from North America, and seven from South America (Lehto 1998, 14). It was decided on that occasion that, following

12. To date, at least sixteen of Hilbert's twenty-three problems can be considered to have been solved in whole or in part by mathematicians from the Baltic States, France, Germany, Japan, the former Soviet Union, and the United States (Yandell 2002).

Mittag-Leffler's invitation, the next congress would be held in Stockholm in 1916. Those plans, however, were scuttled owing to the outbreak of World War I in 1914.

The politics of internationalization in the West during the interwar period

At the war's close in 1918, Mittag-Leffler immediately renewed the invitation to Stockholm; he sensed an urgency to resume the ICMs and to get mathematics back on its international track. The new political realities that prevailed in postwar Europe worked counter to his efforts, however. The French, and especially the noted complex analyst and algebraic geometer Émile Picard, actively opposed any relations with the former Central Powers. Picard's answer to the question '*veut-on, oui ou non, reprendre des relations personnelles avec nos ennemis?*', 'do we want, yes or no, to resume personal relations with our enemies?' was a resounding 'no' (Lehto 1998, 16). While some in the British mathematical community agreed, others like G H Hardy strongly supported the resumption of normal scientific relations. Hardy, a well known pacifist, had done his best even during the war to maintain working relations with his mathematical colleagues despite the political agendas of nations. In 1915, for example, the book *General theory of Dirichlet series* that he co-authored with the Hungarian Marcel Riesz appeared as volume twenty-six in the series of Cambridge Mathematical Tracts and bore the avowal '*auctores hostes idemque amici*', 'the authors, enemies, and all the same friends' (Segal 2002, 363).

As these differing opinions make clear, there was little agreement in the immediate aftermath of the war on how best—or even whether—to proceed with the international initiatives that had begun with such promise some two decades earlier. Still, two initiatives did go forward: plans for an ICM to be held not in Stockholm but in Strasbourg in 1920 and plans for an International Mathematical Union (IMU) to be founded officially at the Strasbourg ICM and to oversee, among other things, the planning of future ICMs. Both of these efforts—international only in name in 1920—were fraught with political difficulties from the start.

First, the former Central Powers were barred from attending the Strasbourg ICM and were ineligible both for membership in the IMU and for participation in future ICMs. In the view of the majority, the Central Powers had 'broken the ordinances of civilization, disregarding all conventions and unbridling the worst passions that the ferocity of war engenders'; in order for them to be readmitted into the international confraternity of mathematicians, moreover, they 'would have to renounce the political methods that had led to the atrocities that had shocked the civilized world' (Lehto 1998, 18). As a result, Germany, in particular, the mathematical trendsetter since the mid-nineteenth century, would not be able to participate.

Second, the selection of Strasbourg as the locale for the ICM had blatantly political overtones, given that Alsace-Lorraine in general and Strasbourg in particular had been returned to French control as a result of the Germans' defeat in the war. As Mittag-Leffler bitterly put it, '*ce congrès est une affaire française qui ne peut nullement annuler le congrès international à Stockholm*', 'this congress is a French affair which can in no way annul the international congress in Stockholm' that he had originally proposed (Lehto 1998, 24).

When mathematicians finally convened in Strasbourg in September 1920, it was indeed, as Mittag-Leffler had predicted, 'a French affair'. The unwaveringly anti-German Picard was elected one of the first Honorary Presidents of the Executive Committee of the IMU as well as the President of the Strasbourg ICM, and he took the occasion of his opening ICM address publicly to uphold the decision to debar mathematicians from the former Central Powers. In his words, '*pardonner à certains crimes, c'est s'en faire le complice*', 'to pardon certain crimes is to become an accomplice in them' (Lehto 1998, 29).

These overtly political sentiments clouded not only the Strasbourg ICM but also efforts to mount the next ICM scheduled for 1924. Mathematicians from the United States and British Isles had begun to push for an end to the exclusionary rules imposed by the IMU, and only efforts by the Canadian mathematician John C Fields to host the 1924 ICM in Toronto ultimately rescued it from complete political entanglement. In some sense, matters were no better in 1928 when the Congress met in Bologna. While some in the IMU continued to insist on exclusion, Salvatore Pincherle (IMU President from 1924 to 1928 and President of the 1928 Congress) and his Italian co-organizers, implemented an open door policy at the Bologna ICM. Although some German mathematicians like Ludwig Bieberbach vociferously opposed German participation on political grounds, David Hilbert rallied his countrymen, who ultimately formed the largest non-Italian national contingent at the ICM (Lehto 1998, 33–46).

This ongoing politicization soon took its toll. By the time the next ICM concluded in Zürich in 1932, the IMU had essentially ceased to exist. The prevailing sentiment among the almost seven hundred mathematicians in attendance in Zürich was that the unabashedly political agenda of the IMU had been detrimental to the international health of the community, and that national politics should thenceforth remain separate from mathematics.

One corrective that followed was the establishment in 1932 of the Fields Medal, the equivalent in mathematics to the Nobel Prize, to be awarded on the occasion of the ICMs to acknowledge outstanding achievements made by mathematicians regardless of nationality. The first of these were given at the Oslo Congress in 1936 to the Finnish mathematician Lars Ahlfors for his work on the theory of Riemann surfaces and to the American Jesse Douglas for his solution of Plateau's problem on minimal surfaces (Monastyrsky 1997, 11). Another corrective that had, in fact, already been at work during the troubled postwar years of the ICMs was the Rockefeller Foundation and its International Education Board (IEB), which expressly sought to encourage international scientific and mathematical development in the interwar period. The Foundation, through the IEB, had, for example, funded the building of both the new Mathematics Institute in Göttingen and the Institut Henri Poincaré in Paris in the late 1920s for the international encouragement and exchange of mathematical research. Unfortunately, the activities of the Göttingen Institute were curtailed from 1933 with the rise of National Socialism in Germany and the subsequent ousting of Jews, not least the Institute's director Richard Courant (see Siegmund-Schulze, Chapter 9.4 in this volume); a little later the Institut Henri Poincaré was fundamentally affected by the outbreak of World War II (Siegmund-Schultze 2001).

Internationalization: West and East

The confused political situation in the interwar period in the West did not prevent international mathematical relations more globally, and especially between West and East.[13] Prior to the nineteenth century, Japan and China were largely closed to Western scientific influences, the most notable exception being the introduction of some Western science by Jesuit missionaries in China in the seventeenth century (Jami, Chapter 1.3 in this volume). Following the Meiji Restoration in 1868, however, Japan looked increasingly to the West for educational, scientific, and cultural models that would help them to compete more effectively in the modern world. The same became true of China after its defeat in the first and second Opium Wars (1839–42 and 1856–60) and in the first Sino-Japanese War in 1895.

In the case of Japan, Westernization was officially mandated, and it was swift. Although the infiltration of Western science—notably mathematics and aspects of naval and military science—had begun after Japan opened some of its ports to

13. The interwar period also witnessed international mathematical relations between the northern and southern hemispheres. In particular, soon after he took office in 1933, US President Franklin Delano Roosevelt announced what came to be known as the 'Good Neighbor Policy' between the United States and the countries of Central and South America. In the sciences and mathematics, this translated into support from private foundations like the Rockefeller Foundation and the John Simon Guggenheim Foundation for intellectual exchanges beginning in the 1930s, carrying on through the war and afterward. In 1942, for example, Harvard mathematician George David Birkhoff went on a mathematical 'good neighbor' lecture tour of Latin America (Ortiz 2003) to be followed in 1943 by his former student and then Harvard colleague Marshall Stone (Parshall 2007). These trips resulted in North American study tours for a number of talented Latin American students and in the establishment of ties between mathematical communities in the Americas.