The Philosophy and Physics of Noether's Theorems

A Centenary Volume

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Introduction

JAMES READ AND NICHOLAS J. TEH

Few articles can reasonably be described as epoch-making. Einstein's 'Zur Elektrodynamik bewegter Körper' (1905) is undoubtedly one such; Turing's 'On Computable Numbers, with an Application to the Entscheidungs problem' (1936) is undoubtedly another. But standing equally tall among these ranks should surely be the article to which this volume – and so much else besides – owes its existence: Emmy Noether's 'Invariante Variationsprobleme' (1918). In that one article, Noether proved two theorems (and their converses), forging links between symmetries and conserved quantities which were to go on – whether by her intentions or not – to constitute the bedrock of modern theoretical physics.

But – perhaps surprisingly, perhaps not – the significance of an epoch-maker is not always recognised in the moment. Whether this is so in the case of Einstein is debatable; it is certainly true for Turing – and arguably even more so for Noether. Strikingly, the significance of what Noether proved in her 1918 article was not well appreciated until as late as the 1970s: only at that point were all of the theorems of the 1918 piece widely understood; and only at that point did they begin to be generalised and applied in substantially novel ways. Since then, progress has not stopped, and this volume – born out of an international 2018 centenary conference held at the London campus of the University of Notre Dame – represents the next episode in the same continuation. Bringing together historians, physicists, mathematicians, and philosophers, the volume constitutes the cutting edge of our understanding of (the application of) Noether's seminal work on variational problems, one hundred years on from her original article.

Why do we add 'the application of' in parentheses above? It is now well-known that Noether remarked little on the physical applications of her mathematical results; see, for example, the contributions of Kosmann-Schwarzbach and Rowe in this volume. In light of this, one should distinguish the creativity of Noether's methods – the creativity of her *technique* – from the creativity of their application – the creativity of the physical *representations* effected on the basis of her methods. In the post-1970s literature, the former are at least relatively well understood (albeit still not completely; see, for example, the contributions of Baez and Olver to this volume, which continue to add to such understanding); not so for the latter, in the case of which we are only beginning to explore a rich orchard of fruits.

The contributions to this volume pursue a number of distinct threads on both Noether's techniques and their applications to physics; these we will summarise here as succinctly as possible. We begin with Noether's history: both regarding specifically her work on variational problems, and more generally. In Chapter 1, **Yvette Kosmann-Schwarzbach** reviews this historical context of 'Invariante Variationsprobleme', from its prehistory, to its doldrums in the mid-twentieth century, to (as alluded to above) its revival in the post-1970s literature. Following on from this, in Chapter 2 **David Rowe** focuses on the interactions between Noether and Felix Klein in the years surrounding the appearance of her 'Invariante Variationsprobleme', and specifically on the role of differential invariants in Noether's two theorems. In Chapter 3, **Tomoko Kitagawa** focuses on another specific episode highlighted by Kosmann-Schwarzbach: namely, Noether's deliberations preceding her move to Bryn Mawr College.

Having presented this updated Noether history, we turn to the mathematics of her theorems, both generalisations and applications. In Chapter 4, **John Baez** illuminates the content of Noether's (first) theorem in the Hamiltonian context by pursuing a (Jordan–Lie) algebraic – rather than the traditional geometric – approach. In Chapter 5, **Kasia Rejzner** continues this study of Noetherian themes from an algebraic point of view (this time via homological algebras), by exploring the 'BV formalism' – an extension of the BRST prescription, in which auxiliary fields enjoying rigid symmetries are introduced, and in which the Noether charges associated with those symmetries are then quantised – from the point of view of perturbative algebraic quantum field theory.

Next, we turn to more philosophical questions regarding the explanatory arrow running from symmetries to conservation laws which is often (misleadingly, our authors would have it!) taken to be an important moral drawn from Noether's theorems. In Chapter 6, **Peter Olver** considers the significance of the fact that one can define infinitely many inequivalent Lagrangians invariant under a stipulated set of variational symmetries: should these Lagrangians be understood as encoding 'equivalent' physics – and if so, why? In Chapter 7, **Harvey R. Brown** questions the reasons for which, in light of the converse of Noether's first theorem, symmetries are often indeed considered to have this explanatory priority over conservation laws. In Chapter 8, **Mark Baker**, **Niels Linnemann**, and **Chris Smeenk** deploy the under-appreciated work of Bessel-Hagen in order to demonstrate how the converse of Noether's first theorem can be used to resolve ambiguities over what should be regarded as the 'physical' energy-momentum tensor in field theories.

The next three chapters address the significance of Noether's theorems in the context in which they were originally developed: the foundations of general relativity. In Chapter 9, **Sebastian de Haro** provides both a crystal-clear survey of the role of Noether's theorems in considerations of gravitational energy in general relativity, as well as a substantial novel contribution to recent philosophical discussions regarding the status of gravitational energy in that theory, including quasi-local notions. In Chapter 10, **James Read** continues these discussions, arguing that the pseudotensorial quantities obtained on application of Noether's theorems to general relativity are best interpreted physically from within the framework of 'perspectival realism'. Finally, in Chapter 11, **Laurent Freidel** and **Nicholas J. Teh** apply Noether's theorems in order to shed new light on three infamously vexed notions in the foundations of spacetime theories: (i) general covariance, (ii) the Principle of Relativity, and (iii) the status of conserved charges (including, again, gravitational energy).

For many years, there was practically no recognition of either of these theorems. Then multiple references to 'the Noether theorem' or 'Noether's theorem' – in the singular – began to appear, referring either to her first theorem, in the publications of those mathematicians and mathematical physicists who were writing on mechanics – who ignored her second theorem – or to her second theorem by those writing on general relativity and, later, on gauge theory. I shall outline the curious transmission of her results, the history of the mathematical developments of her theory, and the ultimate recognition of the wide applicability of 'the Noether theorems'. To conclude, in the hope of dispelling various misconceptions, I shall underline what Noether was *not*, and I shall reflect on the fortune of her theorems.

1.2 A Family of Mathematicians

Emmy Noether was born to a Jewish family in Erlangen (Bavaria, Germany) in 1882. Her life was described in Hermann Weyl's obituary (Weyl 1935). In a manuscript *curriculum vitae*, written for official purposes *circa* 1917, she described herself as 'of Bavarian nationality and Israelite confession'.[3] She died in Bryn Mawr (Pennsylvania) in the United States in 1935, after undergoing an operation. Why she had to leave Germany in 1933 to take up residence in America is clear from the chronology of the rise of the Nazi regime in Germany and its access to power and has, of course, been told in the many accounts of her life that have been published,[4] while numerous and sometimes fanciful comments have appeared in print and in the electronic media in recent years.

She was the daughter of the renowned mathematician, Max Noether (1844–1921), professor at the University of Erlangen. He had been a privatdozent, then an 'extraordinary professor' in Heidelberg before moving to Erlangen in 1875, and was eventually named an 'ordinary professor' in 1888. Her brother, Fritz, was born in 1884 and studied mathematics and physics in Erlangen and Munich. He became professor of theoretical mechanics in Karlsruhe in 1902 and submitted his *Habilitation* thesis in 1912. Later, he became professor in Breslau, from where he, too, was forced to leave in 1933. He emigrated to the Soviet Union and was appointed professor at the University of Tomsk. Accused of being a German spy, he was jailed and shot in 1941.

1.3 The Young Emmy Noether

Emmy Noether first studied languages in order to become a teacher of French and English, a suitable profession for a young woman. But from 1900 on, she studied mathematics, first in Erlangen with her father, then audited lectures at the university. For the winter semester in 1903–4, she travelled to Göttingen to audit courses at the university. At that time, new regulations were introduced which enabled women to matriculate and take examinations. She then chose to enroll at the University of Erlangen, where she listed

[3] Declaring one's religion was compulsory in Germany at the time.

[4] The now classical biographies of Noether can be found in the book written by Auguste Dick (1970), translated into English in 1981, and in the volumes of essays edited by James W. Brewer and Martha K. Smith (1981), and by Bhama Srinivasan and Judith D. Sally (1983).

mathematics as her only course of study,[5] and in 1907, she completed her doctorate under the direction of Paul Gordan (1837–1912), a colleague of her father. Here I open a parenthesis: One should not confuse the mathematician Paul Gordan, her 'Doktorvater', with the physicist Walter Gordon (1893–1939). The 'Clebsch–Gordan coefficients' in quantum mechanics bear the name of Noether's thesis adviser together with that of the physicist and mathematician Alfred Clebsch (1833–72). However, the 'Klein–Gordon equation' is named after Walter Gordon and the physicist Oskar Klein (1894–1977) who, in turn, should not be confused with the mathematician Felix Klein, about whom more will be said shortly.

1.4 Noether's 1907 Thesis on Invariant Theory

Noether's thesis at Erlangen University, entitled 'Über die Bildung des Formensystems der ternären biquadratischen Form' ("On the Construction of the System of Forms of a Ternary Biquadratic Form"), dealt with the search for the invariants of those forms (i.e., homogeneous polynomials) which are ternary (i.e., in 3 variables) and biquadratic (i.e., of degree 4). An extract of her thesis appeared in the *Sitzungsberichte der Physikalisch-medizinischen Societät zu Erlangen* in 1907, and the complete text was published the following year in the *Journal für die reine und angewandte Mathematik* (Crelle's Journal). She later distanced herself from her early work as employing a needlessly computational approach to the problem.

After 1911, her work in algebra was influenced by Ernst Fischer (1875–1954), who was appointed professor in Erlangen upon Gordan's retirement in 1910. Noether's expertise in invariant theory revealed itself in the publications in 1910, 1913, and 1915 that followed her thesis, and was later confirmed in the four articles on the invariants of finite groups that she published in 1916 in the *Mathematische Annalen*. She studied in particular the determination of bases of invariants that furnish an expansion with integral or rational coefficients of each invariant of the group, expressed as a linear combination of the invariants in the basis.

At Erlangen University from 1913 on, Noether occasionally substituted for her ageing father, thus beginning to teach at the university level, but not under her own name.

1.5 Noether's Achievements

Her achievement of 1918, whose centenary was duly celebrated in conferences in London and Paris, eventually became a central result in both mechanics and field theory and, more generally, in mathematical physics, though her role was rarely acknowledged before 1950 and, even then, only a truncated part of her article was cited. On the other hand, her articles on the theory of ideals and the representation theory of algebras published in the 1920s made her world famous. Her role in the development of modern algebra was duly recognised by the mathematicians of the twentieth century, while they either considered her work on invariance principles to be an outlying and negligible part of her work or, more often, ignored

[5] On this, as well as on other oft-repeated facts of Noether's biography, see Dick (1970), English translation, p. 14.

it altogether. In fact, the few early biographies of Noether barely mention her work on invariant variational problems, while both past and recent publications treat her fundamental contributions to modern algebra. I shall not deal with them here. They are, and will no doubt continue to be, celebrated by all mathematicians.

1.6 In Göttingen: Klein, Hilbert, Noether, and Einstein

In 1915, the great mathematicians Felix Klein (1849–1925) and David Hilbert (1862–1943) invited Noether to Göttingen in the hope that her expertise in invariant theory would help them understand some of the implications of Einstein's newly formulated general theory of relativity. In Göttingen, Noether took an active part in Klein's seminar. It was in her 1918 article that she solved a problem arising in the general theory of relativity and proved 'the Noether theorems'. In particular, she proved and vastly generalised a conjecture made by Hilbert concerning the nature of the law of conservation of energy. Shortly afterwards, she returned to pure algebra.

At the invitation of Hilbert, Einstein had come to Göttingen in early July 1915 to deliver a series of lectures on the general theory of relativity, which is to say, on the version that preceded his famous paper, 'Die Feldgleichungen der Gravitation' ("The Field Equations of Gravitation"), of November of that year. Noether must have attended these lectures. It is clear from Hilbert's letter to Einstein of 27 May 1916 that she had by then already written some notes on the subject of the problems arising in the general theory of relativity:

> My law [of conservation] of energy is probably linked to yours; I have already given Miss Noether this question to study.

Hilbert adds that, to avoid a long explanation, he has appended to his letter 'the enclosed note of Miss Noether'. On 30 May 1916, Einstein answered him in a brief letter in which he derived a consequence of the equation that Hilbert had proposed 'which deprives the theorem of its sense', and then asks, 'How can this be clarified?' He continues,

> Of course it would be sufficient if you asked Miss Noether to clarify this for me.[6]

Thus, her expertise was conceded by both Hilbert and Einstein as early as her first year in Göttingen, and was later acknowledged more explicitly by Klein when he re-published his articles of 1918 in his collected works (Klein 1921), a few years before his death.

1.7 Noether's Article of 1918

In early 1918, Noether published an article on the problem of the invariants of differential equations in the *Göttinger Nachrichten*, 'Invarianten beliebiger Differentialausdrücke' ("Invariants of Arbitrary Differential Expressions"), which was presented by Klein at the meeting of the *Königliche Gesellschaft der Wissenschaften zu Göttingen* (Royal Göttingen

[6] Einstein, *Collected Papers*, 8A, nos. 222 and 223.

Scientific Society) of 25 January. It was then, in the winter and spring of 1918, that Noether discovered the profound reason for the difficulties that had arisen in the interpretation of the conservation laws in the general theory of relativity. Because she had left Göttingen for a visit to Erlangen to see her widowed and ailing father, her correspondence remains and yields an account of her progress in this search. In her postcard to Klein of 15 February, she already sketched her second theorem, but only in a particular case. It is in her letter to Klein of 12 March that Noether gave a preliminary formulation of an essential consequence of what would be her second theorem, dealing with the invariance of a variational problem under the action of a group which is a subgroup of an infinite-dimensional group. On 23 July, she presented her results to the *Mathematische Gesellschaft zu Göttingen* (Göttingen Mathematical Society). The article which contains her two theorems is 'Invariante Variationsprobleme' ("Invariant Variational Problems"). On 26 July, Klein presented it at the meeting of the more important – because it was not restricted to an audience of pure mathematicians – Göttingen Scientific Society, and it was published in the *Nachrichten* (Proceedings) of the Society of 1918, on pages 235–47. A footnote on the first page of her article indicates that 'The definitive version of the manuscript was prepared only at the end of September.'

1.8 What Variational Problems Was Noether Considering?

We consider variational problems which are invariant under a continuous group (in the sense of Lie). ... What follows thus depends upon a combination of the methods of the formal calculus of variations and of Lie's theory of groups.[7]

Noether considers a general n-dimensional variational problem of order κ for an $\mathbb{R}^\mu$-valued function, where n, κ, and μ are arbitrary integers, defined by an integral,

$$I = \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^\kappa u}{\partial x^\kappa}\right) dx,$$

where $x = (x_1, \ldots, x_n) = (x_\lambda)$ denote the independent variables, and where $u = (u_1, \ldots, u_\mu) = (u_i)$ are the dependent variables. In footnotes, she states her conventions and explains her abbreviated notations: 'I omit the indices here, and in the summations as well whenever it is possible, and I write $\frac{\partial^2 u}{\partial x^2}$ for $\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\gamma}$, etc', and 'I write dx for $dx_1 \ldots dx_n$ for short'.

Noether then states her two theorems:

In what follows we shall examine the following two theorems:
I. If the integral I is invariant under a [group] $\mathfrak{G}_\rho$, then there are ρ linearly independent combinations among the Lagrangian expressions which become divergences – and conversely, that implies the invariance of I under a [group] $\mathfrak{G}_\rho$. The theorem remains valid in the limiting case of an infinite number of parameters.

[7] I cite the English translation of Noether's article that appeared in *The Noether Theorems* (2010).

II. If the integral I is invariant under a [group] $\mathfrak{G}_{\infty\rho}$ depending upon arbitrary functions and their derivatives up to order σ, then there are ρ identities among the Lagrangian expressions and their derivatives up to order σ. Here as well the converse is valid.[8]

Noether proves the direct part of both theorems in section 2, then the converse of theorem I in section 3 and that of theorem II in section 4. In section 2, she assumes that the action integral $I = \int f dx$ is invariant. Actually, she assumes a more restrictive hypothesis, the invariance of the integrand, $f dx$, which is to say, $\delta(f dx) = 0$. This hypothesis is expressed by the relation

$$\bar{\delta} f + \mathrm{Div}(f \cdot \Delta x) = 0.$$

Here Div is the divergence of vector fields and $\bar{\delta} f$ is the variation of f induced by the variation

$$\bar{\delta} u_i = \Delta u_i - \sum \frac{\partial u_i}{\partial x_\lambda} \Delta x_\lambda.$$

Thus, Noether introduced the evolutionary representative, $\bar{\delta}$, of the vector field δ, and $\bar{\delta} f$ is the Lie derivative of f in the direction of the vector field $\bar{\delta}$. What she introduced, with the notation $\bar{\delta}$, is a generalised vector field, which is not a vector field in the usual sense, on the trivial vector bundle $\mathbb{R}^n \times \mathbb{R}^\mu \to \mathbb{R}^n$. In fact, if

$$\delta = \sum_{\lambda=1}^{n} X^\lambda(x) \frac{\partial}{\partial x^\lambda} + \sum_{i=1}^{\mu} Y^i(x,u) \frac{\partial}{\partial u^i},$$

then $\bar{\delta}$ is the vertical generalised vector field

$$\bar{\delta} = \sum_{i=1}^{\mu} \left(Y^i(x,u) - X^\lambda(x) u^i_\lambda \right) \frac{\partial}{\partial u^i},$$

where $u^i_\lambda = \frac{\partial u^i}{\partial x^\lambda}$. It is said to be 'generalised' because its components depend on the derivatives of the $u^i(x)$. It is said to be 'vertical' because it contains no terms in $\frac{\partial}{\partial x^\lambda}$.[9]

By integrating by parts, Noether obtains the identity

$$\sum \psi_i \, \bar{\delta} u_i = \bar{\delta} f + \mathrm{Div}\, A,$$

where the ψ_i's are the 'Lagrangian expressions', i.e., the components of the Euler–Lagrange derivative of f, and A is linear in $\bar{\delta} u$ and its derivatives. In view of the invariance hypothesis which is expressed by $\bar{\delta} f + \mathrm{Div}(f \cdot \Delta x) = 0$, this identity can be written as

$$\sum \psi_i \, \bar{\delta} u_i = \mathrm{Div}\, B, \quad \text{with} \quad B = A - f \cdot \Delta X.$$

[8] In a footnote, Noether announces that she will comment on 'some trivial exceptions' in the next section of her article.

[9] The evolutionary representative of an ordinary vector field has also been called the vertical representative. Both terms are modern. Noether does not give $\bar{\delta}$ a name. An arbitrary vertical generalised vector field is written locally, $Z = \sum_{i=1}^{\mu} Z^i \left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots \right) \frac{\partial}{\partial u^i}$.

of the most important points of this history. Lagrange, in his *Méchanique analitique* (1788), claimed that his method for deriving 'a general formula for the motion of bodies' yields 'the general equations that contain the principles, or theorems known under the names of the *conservation of kinetic energy*, of the *conservation of the motion of the center of mass*, of the *conservation of the momentum of rotational motion*, of the *principle of areas*, and of the *principle of least action*'.[12] In the second edition of his *Mécanique analytique*, in 1811, as a preliminary to his treament of dynamics, he presented a detailed history of the diverse 'principes ou théorèmes' (principles or theorems) formulated before his *Mécanique*, thus recognising the contributions of his predecessors in the discovery of these principles – Galileo, Huyghens, Newton, Daniel Bernoulli, Maupertuis, Euler, the Chevalier Patrick d'Arcy and d'Alembert – and in this second edition, he explicitly observed a correlation between these principles of conservation and invariance properties. After Lagrange, the correlation between invariances and conserved quantities was surveyed by Jacobi in several chapters of his *Vorlesungen über Dynamik*, lectures delivered in 1842–3 but published only posthumously in 1866. The great advances of Sophus Lie (1842–99) – his theory of continuous groups of transformations that was published in articles and books that appeared between 1874 and 1896 – became the basis of all later developments, such as the work of Georg Hamel (1877–1954) on the calculus of variations and mechanics in 1904, and the publication of Gustav Herglotz (1881–1953) on the 10-parameter invariance group of the [special] theory of relativity in 1911. In her 1918 article, Noether cited Lie very prominently, as his name appears three times in the eight lines of the introductory paragraph, but with no precise reference to his published work. Both Hamel and Herglotz were cited by her. In her introduction, she also referred to publications, all of them very recent, by '[Hendrik] Lorentz and his students (for example, [Adriaan Daniel] Fokker), [Hermann] Weyl, and Klein for certain infinite groups' and, in a footnote, she wrote, 'In a paper by [Adolf] Kneser that has just appeared (Math. Zeitschrift, vol. 2), the determination of invariants is dealt with by a similar method.' In fact, while Noether was completing the definitive version of her manuscript, in August 1918, Kneser had submitted an article, 'Least Action and Galilean Relativity', in which he used Lie's infinitesimal transformations and, as Noether would do, emphasised the relevance of Klein's Erlangen program, but he did not treat questions of invariance. Noether stressed the relation of her work to 'Klein's second note, *Göttinger Nachrichten*, 19 July 1918', stating that her work and Klein's were 'mutually influential' and referring to it for a more complete bibliography. In section 5 of her paper, she cited an article, 'On the Ten General Invariants of Classical Mechanics' by Friedrich Engel (1861–1941), that had appeared two years earlier. Indeed, scattered results in classical and relativistic mechanics, tying together properties of invariance and conserved quantities, had already appeared in the publications of Noether's predecessors which she acknowledged. However, none of them had discovered the general principle contained in her Theorem I and its converse. Her Theorem II and its converse were completely new. In the expert opinion of

[12] Lagrange (1788, p. 182), italics in the original.

the theoretical physicist Thibaut Damour,[13] the second theorem should be considered the most important part of her article. It is certainly the most original.

1.11 How Modern Were Noether's Two Theorems?

What Noether simply called 'infinitesimal transformations' are, in fact, vast generalisations of the ordinary vector fields, and they are now called generalised vector fields. They would eventually be re-discovered, independently, in 1964 by Harold H. Johnson, then at the University of Washington, who called them 'a new type of vector fields', and in 1965 by Robert Hermann (1931–2020). They appeared again in 1972 as Robert L. Anderson, Sukeyuki Kumei, and Carl Wulfman published their 'Generalization of the Concept of Invariance of Differential Equations. Results of Applications to Some Schrödinger Equations' in *Physical Review Letters*. In 1979, R. L. Anderson, working at the University of Georgia in the United States, and Nail Ibragimov (1938–2018), then a member of the Institute of Hydrodynamics at the Siberian branch of the USSR Academy of Sciences in Novosibirsk – such east-west collaboration was rare at the time – in their monograph, *Lie-Bäcklund Transformations in Applications*, duly citing Klein and Noether while claiming to generalise 'Noether's classical theorem', called them 'Lie-Bäcklund transformations', a misleading term because Albert V. Bäcklund (1845–1922) did not introduce this vast generalisation of the concept of vector fields, only infinitesimal contact transformations. The concept of a generalised vector field is essential in the theory of integrable systems which became the subject of intense research after 1970. On this topic, Noether's work is modern, half a century in advance of these re-discoveries. Peter Olver's book, *Applications of Lie Groups to Differential Equations* (1986a), is both a comprehensive handbook of the theory of generalised symmetries of differential and partial differential equations, and the reference for their history, while his article of the same year on 'Noether's theorems and systems of Cauchy–Kovalevskaya type' is an in-depth study of the mathematics of Noether's second theorem. His article (Olver 2018), written for the centenary of Noether's article, stresses the importance of her invention of the generalised vector fields.

In Göttingen, Noether had only one immediate follower, Erich Bessel-Hagen (1898–1946), who was Klein's student. In 1921, he published an article in the *Mathematische Annalen*, entitled 'Über die Erhaltungssätze der Elektrodynamik' ("On the Conservation Laws of Electrodynamics"), in which he determined in particular those conservation laws that are the result of the conformal invariance of Maxwell's equations. There, Bessel-Hagen recalls that it was Klein who had posed the problem of 'the application to Maxwell's equations of the theorems stated by Miss Emmy Noether about two years ago regarding the invariant variational problems' and he writes that, in the present paper, he formulates the two Noether theorems 'slightly more generally' than they had been formulated in her article. How did he achieve this more general result? By introducing the concept of 'divergence symmetries' which are infinitesimal transformations which leave the Lagrangian invariant

[13] Damour is a professor at the Institut des Hautes Études Scientifiques and a member of the Académie des Sciences de l'Institut de France.

up to a divergence term, or 'symmetries up to divergence'. They correspond not to the invariance of the Lagrangian $f dx$, but to the invariance of the action integral $\int f dx$, i.e., instead of satisfying the condition $\delta(f dx) = 0$, they satisfy the weaker condition $\delta(f dx) = \mathrm{Div}\, C$, where C is a vectorial expression. Noether's fundamental relation remains valid under this weaker assumption, provided that $B = A - f \cdot \Delta x$ is replaced by $B = A + C - f \cdot \Delta x$. Immediately after he stated that he had proved the theorems in a slightly more general form than Noether had, Bessel-Hagen added: 'I owe these [generalised theorems] to an oral communication by Miss Emmy Noether herself'. We infer that, in fact, this more general type of symmetry was also Noether's invention. Bessel-Hagen's acknowledgment is evidence that, to the question, 'Who invented divergence symmetries?', the answer is: Noether.

1.12 How Influential Were Noether's Two Theorems?

The history of the reception of Noether's article in the years 1918–70 is surprising. She submitted the 'Invariante Variationsprobleme' for her *Habilitation*, finally obtained in 1919, but she never referred to her article in any of her subsequent publications. I know of only one mention of her work of 1918 in her own writings, in a letter she sent eight years later to Einstein, who was then an editor of the journal *Mathematische Annalen*. In this letter, which is an informal referee report, she rejects a submission 'which unfortunately is by no means suitable' for the journal, on the grounds that 'it is first of all a restatement that is not at all clear of the principal theorems of my "Invariante Variationsprobleme" (Gött[inger] Nachr[ichten], 1918 or 1919), with a slight generalization – the invariance of the integral up to a divergence term – which can actually already be found in Bessel-Hagen (Math[ematische] Ann[alen], around 1922)'.[14]

I found very few early occurrences of Noether's title in books and articles. While Hermann Weyl, in *Raum, Zeit, Materie*, first published in 1918, performed computations very similar to hers, he referred to Noether only once, in a footnote in the third (1919) and subsequent editions. It is clear that Richard Courant must have been aware of her work because a brief summary of a limited form of both theorems appears in all German, and later English editions of 'Courant–Hilbert', the widely read treatise on methods of mathematical physics first published in 1924. It is remarkable that we found so few explicit mentions of Noether's results in searching the literature of the 1930s. In 1936, the little-known physicist Moisei A. Markow (1908–94), who was a member of the Physics Institute of the USSR Academy of Sciences in Moscow, published an article in the *Physikalische Zeitschrift der Sowjetunion* in which he refers to 'the well-known theorems of Noether'. Markow was a former student of Georg B. Rumer (1901–85), who had been an assistant of Max Born in Göttingen from 1929 to 1932. Rumer, in 1931, had proved the Lorentz invariance of the Dirac operator but did not allude to any associated conservation laws, while in his articles on the general theory of relativity published in the *Göttinger Nachrichten* in 1929 and 1931, he

14 For a facsimile, a transcription, and a translation of Noether's letter, see Kosmann-Schwarzbach (2010, pp. 161–5), and see comments on this letter, Kosmann-Schwarzbach (2010), pp. 51–2.

cited Weyl but never Noether. Similarly, it seems that V. A. Fock (1898–1974) never referred to Noether's work in any of his papers to which it was clearly relevant, such as his celebrated 'Zur Theorie des Wasserstoffatoms' (On the Theory of the Hydrogen Atom) of 1935. Was it because, at the time, papers carried few or no citations? Or because Noether's results were considered to be 'classical'? The answers to both questions are probably positive, this paucity of citations being due to several factors.

An early, explicit reference to Noether's publication is found in the article of Ryoyu Utiyama (Utiyama 1916–90), then in the department of physics of Osaka Imperial University, 'On the Interaction of Mesons with the Gravitational Field. I', which appeared in *Progress of Theoretical Physics* (Utiyama 1947), four years before he was awarded the PhD. His paragraph I begins with the 'Theory of invariant variation' for which he cites both Noether's 1918 article and page 617 of Pauli's 'Relativitätstheorie' (1921). Following Noether closely, he proves the first theorem, introducing 'the substantial variation of any field quantity', which he denotes by δ^* – i.e., what Noether had denoted by $\bar{\delta}$ – and also treats the case where the dependent variables 'are not completely determined by [the] field equations but contain r undetermined functions'. This text dates, in fact, to 1941, as the author reveals in a footnote on the first page: 'This paper was published at the meeting[s] of [the] Physico-mathematical Society of Japan in April 1941 and October 1942, but because of the war the printing was delayed'. Such a long delay in the publication of this scientific paper is one example – among many – of the influence of world affairs on science. It appears that this publication is a link in the chain leading from Noether's theorems to the development, by the physicists, of the gauge theories, where the variations of the field variables depend on arbitrary functions. Episodes in this history, told by Utiyama himself, were published in Lochlainn O'Raifeartaich's book (1997), from which we learn that, although Utiyama published his important paper 'Invariant Theoretical Interpretation of Interaction' in the *Physical Review* only in 1956, two years after the famous article of Yang and Mills, he had worked independently and had treated more general cases, showing that gauge potentials are in fact affine connections. In this paper, Utiyama gave only six references: one is (necessarily) to the publication of Yang and Mills, another is to his own 1947 paper, clearly establishing the link from his previous work to the present one, and another reference is to page 621 of Pauli (1921). This time, however, a reference to Pauli serves as a reference to Noether, so that her name does not appear.

In later developments, in the Soviet Union in 1959, Lev S. Polak published a translation of Noether's 1918 article into Russian and, in 1972, Vladimir Vizgin published a historical monograph whose title, in English translation, is *The Development of the Interconnection between Invariance Principles and Conservation Laws in Classical Physics*, in which he analysed both of Noether's theorems. At that time, new formulations of Noether's first theorem had started to appear with the textbook of Israel M. Gel'fand and Sergeĭ V. Fomin on the calculus of variations, published in Moscow in 1961, which contains a modern presentation of Noether's first theorem – although not yet using the formalism of jets as would soon be the case – followed by a few lines about her second theorem. This book appeared in an English translation two years later. In the 1970s, Gel'fand published several articles with Mikhael Shubin, Leonid Dikiĭ (Dickey), Irene Dorfman, and Yuri Manin on the 'formal calculus of variations', not mentioning Noether because they dealt mainly with the Hamiltonian formulation of the problems, while Manin's 'Algebraic Theory of Nonlinear Differential Equations' (1978) as well as Boris Kupershmidt's 'Geometry of Jet Bundles and the Structure of Lagrangian and Hamiltonian Formalisms' (1980) both contain a 'formal Noether theorem', which is a modern, generalised version of her first theorem. A few years earlier already, in the article 'Lagrangian Formalism in the Calculus of Variations' (1976), Kupershmidt had presented an invariant approach to the calculus of variations in differentiable fibre bundles, and Noether's first theorem was formulated for the Lagrangians of arbitrary finite order.

Further research in geometry in Russia yielded new genuine generalisations of the concepts introduced by Noether and of her results. Alexandre Vinogradov (1938–2019), who had been a member of Gel'fand's seminar in Moscow, left the Soviet Union for Italy in 1990 and the second part of his career was at the University of Salerno. Beginning in 1975, Vinogradov, together with Joseph Krasil'shchik – who worked in Moscow, then for several years in the Netherlands, and again in Moscow at the Independent University – published extensively on symmetries, at a very general and abstract level, greatly generalising Noether's formalism and results, and on their applications, a theory fully expounded in their book (Krasil'shchik and Vinogradov 1997).

Searching for other lines of transmission of Noether's results, one finds that in the early 1960s Enzo Tonti (later professor at the University of Trieste) translated Noether's article into Italian, but his translation has remained in manuscript. It was transmitted to Franco Magri in Milan who, in 1978, wrote an article in Italian where he clearly set out the relation between symmetries and conservation laws for non-variational equations, a significant development, but he did not treat the case of operators defined on manifolds. In France, Jean-Marie Souriau (1922–2012) was well aware of 'les méthodes d'Emmy Noether' which he cited as early as 1964, on page 328 of his book, *Géométrie et relativité*. In 1970, independently of Bertram Kostant (1928–2017), Souriau introduced the concept of a momentum map. The conservation of the momentum of a Hamiltonian action is the Hamiltonian version of Noether's first theorem. Souriau called that result 'le théorème de Noether symplectique', although there is nothing Hamiltonian or symplectic in Noether's article! Souriau's fundamental work on symplectic geometry and mechanics was based on Lagrange, as he himself claimed, but it was also a continuation of Noether's theory.

1.13 From General Relativity to Cohomological Physics

The history of the second theorem – the improper conservation laws – belongs to the history of general relativity. In the literature on the general theory, the improper conservation laws which are 'trivial of the second kind' are called 'strong laws', while the conservation laws obtained from the first theorem are called 'weak laws'. The strong laws play an important role in basic papers of Peter G. Bergmann in 1958, of Andrzej Trautman in 1962, and of Joshua N. Goldberg in 1980. While the second theorem, which explained in which cases such improper conservation laws would exist, had been known among relativists since the early 1950s, it became an essential tool in the non-Abelian gauge theories that were

In the discrete versions of the Noether theorems, the differentiation operation is replaced by a shift operator. The independent variables are now integers, and the integral is replaced by a sum, $\mathcal{L}[u] = \sum_n L(n, [u])$, where $[u]$ denotes $u(n)$ and finitely many of its shifts. The variational derivative is expressed in terms of the inverse shift. A pioneer was John David Logan, who published 'First Integrals in the Discrete Variational Calculus' in 1973. Much more recent advances on the discrete analogues of the Noether theorems, an active and important field of research, may be found in a series of papers by Peter Hydon and Elizabeth Mansfield (2001), published since 2001, including a discrete version of the second theorem.

1.15 Were the Noether Theorems Ever Famous?

Whereas both theorems were analyzed by Vizgin in his 1972 monograph on invariance principles and conservation laws in classical physics, it appears that the existence of the first and second theorems in one and the same publication was not expressed in written form in any language other than Russian before the first edition of Olver's book in 1986 and his contemporaneous article where 'Noether's theorems' appear in the title. At roughly the same time, one can find 'theorems', in the plural, in a few other publications: in Hans A. Kastrup's contribution to *Symmetries in Physics (1600–1980)*, the text of a 1983 communication finally printed in 1987 in this extremely rich collection of essays, and in my mathematical paper, 'Sur les théorèmes de Noether', presented in Marseille-Luminy in 1985 at the 'Journées relativistes' organised by André Lichnerowicz, which also appeared in 1987. Then came David Rowe's survey (Rowe 1999).

Fame came eventually. I quote from Gregg Zuckerman's 'Action Principles and Global Geometry' (1987):

> E. Noether's famous 1918 paper, 'Invariant variational problems' crystallized essential mathematical relationships among symmetries, conservation laws, and identities for the variational or 'action' principles of physics. … Thus, Noether's abstract analysis continues to be relevant to contemporary physics, as well as to applied mathematics.[15]

Therefore, approximately 70 years after her article had appeared in the *Göttingen Nachrichten*, fame came to Noether for this (very small) part of her mathematical *œuvre*. In the 20 page contribution of Pierre Deligne and Daniel Freed to the monumental treatise, *Quantum Fields and Strings: A Course for Mathematicians* (1999), she was credited not only with 'the Noether theorems' but also with 'Noether charges' and 'Noether currents'. For as long as gauge theories had been developing, these terms had, in fact, been in the vocabulary of the physicists, such as Utiyama, Yuval Ne'eman (1999), or Stanley Deser whose discussion of 'the conflicting roles of Noether's two great theorems' and 'the physical impact of Noether's theorems' continues to this day in articles (Deser 2019) and preprints. At the end of the twentieth century, the importance of the concepts she had introduced was

[15] Here Zuckerman cites Olver's book.

finally recognised, and her name was attached to them by mathematicians and mathematical physicists alike.

1.16 In Lieu of Conclusion

One can read in a text published as late as 2003 by a well-known philosopher of science that 'Noether's theorems can be generalised to handle transformations that depend on the $u^{(n)}$ as well.' Any author who had only glanced at Noether's paper, or read parts of Olver's book, would have been aware that Noether had already proved her theorems under that generalised assumption. This, in fact, is one of the striking and important features of Noether's 1918 article. Therefore, *caveat lector*! It is better to read the original than to rely on second-hand accounts. For my part, I shall not attempt to draw any philosophical conclusions from what I have sketched here of Noether's 'Invariante Variationsprobleme', its genesis, its consequences, and its influence, because I want to avoid the mistakes of a non-philosopher, of the kind that amateurs make in all fields.[16]

It is clear that Noether was not a proto-feminist. She was not a practicing Jew. Together with her father, she converted to Protestantism in 1920, which did not protect her from eventual dismissal from the University of Göttingen by the Nazis. She was not an admirer of American democracy, and her sympathies were with the Soviet Union. Even though her 1918 work was clearly inspired by a problem in physics, she was never herself a physicist and did not return to physics in any of her subsequent publications. She never explored the philosophical underpinnings or outcomes of her work – in a word, she was not a philosopher. She was a generous woman admired by her colleagues and students, and a great mathematician.

While the Noether theorems derive from the algebraic theory of invariants developed in the nineteenth century – a chapter in the history of pure mathematics – it is clear from the testimony of Noether herself that the immediate motivation for her research was a question that arose in physics, at the time when the new general theory of relativity was emerging – a fact that she stated explicitly in her 1918 article. The results of this article have indeed become – in increasingly diverse ways which deserve to be much more fully investigated than time and space permitted – a fundamental instrument for mathematical physicists. On the one hand, these results are essential parts of the theories of mechanics and field theory and many other domains of physical science, and on the other, in a series of mainly separate developments, her results have been generalised by pure mathematicians to highly abstract levels, but that was not accomplished in her lifetime. Had she lived longer, she would have witnessed this evolution and the separate, then re-unified, paths of mathematics and physics, and we are free to imagine that she would have taken part in the mathematical discoveries that issued from her 23 page article.

[16] For a philosophical outlook, see, e.g., Brading and Brown (2003).

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yields a stationary value. Formulated in the classical δ symbolism, this reads:

$$\delta \int_a^b F\big(y, y', x\big) dx = 0. \tag{2.2}$$

To solve this, one calculates $\delta F\big(y, y', x\big) = \epsilon \left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi' \right)$, and finds

$$\delta \int_a^b F\big(y, y', x\big) dx = \int_a^b \delta F\big(y, y', x\big) dx = \epsilon \int_a^b \left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi' \right) dx. \tag{2.3}$$

The standard trick at this point is to rewrite the second term on the right by using partial integration:

$$\int_a^b \frac{\partial F}{\partial y'}\phi' dx = \left[\frac{\partial F}{\partial y'}\phi \right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \phi dx. \tag{2.4}$$

Since ϕ vanishes on the boundary, only the second term remains, and substituting in (2.2) yields

$$\delta \int_a^b F\big(y, y', x\big) dx = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \phi dx = 0. \tag{2.5}$$

Since this equation holds for arbitrary $\phi(x)$, the integrand itself must vanish, which yields the classical Euler-Lagrange equation:

$$\psi(x) = \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0. \tag{2.6}$$

This well-known argument lies in the background of the formal methods employed in ‘Invariante Variationsprobleme’ (Noether 1918b). In Noether’s setting, however, the function F can be much more complicated, and instead of a single variable x, one integrates over an n-dimensional region. The corresponding solution (2.27) then involves n Lagrangian expressions ψ_i, which arise as differential invariants from the corresponding variational problem. These mathematical underpinnings are familiar from analytical mechanics, where one takes $F = L = T - V$. The Lagrangian $L(q_1, \ldots, q_n, \dot{q_1}, \ldots, \dot{q_n}, t)$ uses generalized coordinates $q_i(t)$ and their derivatives $\dot{q}_i(t)$ to describe the position and velocity of a physical system over a time interval $t \in [a, b]$. The equations of motion are then given by the n equations

$$\psi_i(t) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i(t)} - \frac{\partial L}{\partial q_i(t)} = 0. \tag{2.7}$$

As Noether remarked at the beginning of ‘Invariante Variationsprobleme’, her results combined methods from the formal calculus of variations with those from Lie’s theory of continuous groups. The latter take hold when one requires that the integral remain invariant under the infinitesimal action of the generators of a Lie group, whereas the former hold for *any* variational problem of the type Noether considered. These variational methods happen to be intimately connected with the generation of differential invariants, which will be the primary focal point of interest in this paper. Indeed, this connection was a central theme in

Felix Klein's lectures, beginning with his modernized interpretation of the derivation of the Lagrangian equations of motion (see equation (2.31) below).

Emmy Noether was a true virtuoso in wielding formal variational methods as a tool for systematically generating differential invariants. That this part of her mathematical legacy has been almost entirely overlooked can largely be explained by the fact that, until fairly recently, the formal calculus of variations had fallen out of fashion. This point was made by E. J. McShane, who wrote that her works 'constitute a major contribution to a highly formalistic aspect of the calculus of variations that received much attention in the nineteenth century, but was already becoming less interesting to analysts when her theorems appeared' (McShane 1981, p. 130).[4] As was pointed out in Kosmann-Schwarzbach (2006/2011, pp. 84, 138–9), however, by the mid-1970s there was a resurgence of interest in formal calculus of variations. It should also be added that Noether was well aware of the need to supplement and sharpen formal methods when solving concrete problems in the calculus of variations; see her remarks in Noether (1923, p. 442).

In his lecture courses on relativity, Klein took a retrospective approach. He presented various topics genetically rather than systematically, pulling together threads from earlier publications in order to reveal their underlying connections. Provisional texts from these lectures circulated for a time, either in handwritten or mimeographed form, but these differed markedly from the version prepared by Stephan Cohn-Vossen and published posthumously (Klein 1927). Klein's earlier lectures on the development of mathematics in the nineteenth century, by contrast, were not altered greatly in Klein (1926).[5] Both sets of lectures had ambitious goals that were never completely realized.[6] In spirit, Klein's lectures on relativity were akin to those he delivered earlier on the development of mathematics in the nineteenth century, except that he now had a more specific agenda in mind that required closer attention to technical aspects. Two central topics were invariant theory and variational methods, both major research interests of Emmy Noether at this time. Her famous paper 'Invariante Variationsprobleme' was preceded by a related study on differential invariants (Noether 1918a), which exploited what she called formal variational methods. The problem she attacked there amounted to a far-reaching generalization of E.B. Christoffel's reduction theorem (Christoffel 1869), a key result taken up in Klein's lectures (Klein 1927, pp. 195–9).

An important publication for present purposes is Noether's (1922) rarely mentioned note that appeared in the German *Encyclopedia of the Mathematical Sciences*. Although brief and somewhat opaque, this text will prove highly suggestive for contextualizing her earlier work. One year later, Noether (1923) elaborated on some of the same ideas by sketching recent progress in algebraic and differential invariant theory. Both of these notes refer to Noether's (1918a) generalization of Christoffel's reduction theorem for homogeneous quadratic differential forms. The immediate motivation for Noether's (1918a) work came from Klein's lectures on the mathematical foundations of general relativity, held during the

4 The pitfalls and shortcomings of formal calculus of variations are discussed in Ewing (1985, chap. 4), where the author emphasized that these methods do not appear in the other chapters of his book.

5 For details concerning Klein's lecture courses during this period, see Tobies (2019, pp. 455–63).

6 Richard Courant nevertheless decided to publish edited versions of both posthumously; whether or not Klein ever assented to this plan remains unclear.

summer semester of 1917.[7] In these, Klein highlighted the methodological advantages of the approach taken earlier by Bernhard Riemann and Rudolf Lipschitz for attacking the problem of equivalence for differential forms, a problem first posed and solved by Christoffel (see Klein (1927, pp. 187–8, 195–9)). Noether also worked closely with Hermann Vermeil, who also attended Klein's lectures and contributed to the latter's research program, above all in a paper in which he cited her assistance (Vermeil 1919). Vermeil became Klein's last assistant in 1919 and played a major role in preparing all three volumes of his collected works (Klein 1921–3).

Although none of these investigations can be discussed in any detail here, they clearly point to Felix Klein's strong interest in older mathematical ideas as well as his desire to develop these further. Apart from this, his collaboration with Noether reveals how quickly she mastered the methodological tools needed to pursue the program he had in mind. It was Klein who submitted Noether's 'Invariante Variationsprobleme' (Noether 1918b) as well as her earlier note (Noether 1918a) for publication in the *Nachrichten* of the Göttingen Scientific Society. Both of these papers employed invariant theory and variational principles, and both arose from their mutual research interests at that time. These joint efforts of Klein and Noether, however, were fairly quickly forgotten, in part because they were superseded by subsequent developments and results.[8] The contributions of Göttingen mathematicians to the new physics, on the other hand, were to some extent canonized in Richard Courant's classic textbook *Methoden der mathematischen Physik* (Courant and Hilbert 1924), which went through several editions and was later translated into English.[9] As noted by Kosmann-Schwarzbach, this was one of the few early texts that provided an account of both Noether theorems (Kosmann-Schwarzbach 2006/2011, 2011, p. 96).

2.2 On Klein's Research Agenda, 1916–1918

Most discussions of general relativity in Göttingen have focused somewhat narrowly on David Hilbert's efforts to wed Einstein's theory of gravitation with Gustav Mie's electromagnetic theory of matter. Seen in retrospect, this was the first in a series of abortive attempts to establish a unified field theory. Einstein, who later explored numerous approaches for constructing a viable UFT, dismissed Hilbert's initial program out of hand as naive. Hilbert's own initial enthusiasm for this project seems to have waned as well (see Renn and Stachel 2007). Very few sources exist that might shed some light on Emmy Noether's work with him, but allusions in Hilbert's lecture course from 1916 to 1917 suggest that he hoped to exploit insights from invariant theory to advance his physical program

7 During the previous two semesters, Klein offered lectures on 'Invariant Theory of Linear transformations' (SS 1916) and 'Invariant Foundations of Special Relativity' (WS 1916/17). His lectures during the SS 1917 were entitled 'Invariant Theory of General Point Transformations.' He began the WS 1917/18 with lectures on 'Invariant Foundations of General Relativity,' but broke these off after the Christmas vacation. These final lectures were never published, but the others appeared ten years later in a posthumously published edition prepared by Stephen Cohn-Vossen (Klein 1927). Some of Klein's contemporaries read earlier *Ausarbeitungen*, which already circulated during the war years. Arnold Sommerfeld, who worked closely with Klein and who edited Band 5 of the *Encyklopädie*, helped to distribute these texts from Klein's lectures.

8 One of the puzzles surrounding the differential form for energy-momentum conservation in general relativity disappeared once it was realized that this was an immediate consequence of the contracted Bianchi identities; see Rowe (2018, pp. 263–72).

9 This was Volume I, but the second volume did not appear until 1937 when Courant was in the United States.

(Sauer and Majer 2009, pp. 287–9). If so, these hopes came to naught, nor does it seem likely that Noether's expertise could have been of any help in this venture, given her limited knowledge of physics. In fact, she was already pursuing her own ideas in algebra and invariant theory, inspired by her reading and still-ongoing collaboration with Ernst Fischer.[10] By the final year of the Great War, Hilbert's research shifted away from mathematical physics to the foundations of mathematics. Nevertheless, his work opened an array of mathematical problems that served as a catalyst for Klein's initial publications on general relativity. The latter then prompted Noether to analyze the underpinnings of Hilbert's Theorem 1, which he had stated without proof (Hilbert 1915).

Notwithstanding the central importance of these matters, they represent only one particular aspect within the larger complex of problems Klein pursued during the years 1917–19. One can gain a clear idea of Klein's interests from that time by reading Klein (1927) while comparing it with Part II of Wolfgang Pauli's *Theory of Relativity*, the part on 'Mathematical Tools' in his report for the *Encyklopädie* (Pauli 1958, pp. 21–70). Doing so also reveals that Pauli had carefully studied the *Ausarbeitungen* from Klein's lectures as well as the latter's published works, which he highlighted in the final section of Part II on variational principles (Pauli 1958, Sec. 23). Klein read the proofs of Pauli's report and sent him several letters in response, some published (Pauli 1979), though for some mysterious reason Pauli failed to mention Noether's "Invariante Variationsprobleme", which was written in order to clarify the status of conservation laws and energy principles in general relativity. Instead, he cited Klein's papers from the final section of Volume 1 of his collected works (Klein 1921–3), entitled 'Zum Erlanger Programm.' The final three papers under that heading reflect Klein's research agenda during the period he worked closely with Noether on topics in general relativity.

In general terms, Klein aimed to link spacetime physics with the approach to geometry he outlined in his 'Erlanger Programm' (Klein 1872). Originally, he conceived of this as a method for unifying geometrical research by studying transformation groups and their associated invariants. Already decades earlier, global transformations had begun to assume a central place in geometrical investigations, though it was only around 1870 that geometers associated these infinite families of transformations with the finite groups studied by algebraists. Klein's 'Erlanger Programm' marked the culmination of his collaboration with Sophus Lie during the period 1869–72. As Thomas Hawkins pointed out (Hawkins 1984), its ideas at first met with a slow reception, though this changed by the the 1890s, the decade during which Lie's novel theory of continuous groups drew widespread international attention.[11]

The original conception outlined by Klein (1872) amounts to the following idea. Klein identified a geometry with a coordinatized space (or a 'manifold') M and a group G of coordinate transformations that acted on it. Given a geometry (M, G), one could then induce an equivalent geometry on a manifold M' by means of a bijective mapping $f\colon M \longrightarrow M'$, since the group G acting on M will by means of f also act on M'. Furthermore, the properties

[10] For an idea of the scope of her research program in invariant theory, see Noether (1923).
[11] On the history of Lie groups and Lie algebras, see Hawkins (2000).

of a geometry (M, G) could be studied systematically by developing the invariant theory for G. Historically, this had been done only for projective groups, once projective geometry emerged as a major field of research in the nineteenth century. Within the context of Klein's 'Erlanger Programm,' projective and Euclidean geometry were the two paradigmatic cases, although he described several other geometries as well.

In his old age, Felix Klein tried to picture this scheme as marking the beginning of a grand narrative that culminated with Einstein's special and general theories of relativity, both of which inspired a great deal of interest among mathematicians. While working closely with Noether, he published three papers on general relativity in the *Göttinger Nachrichten* (Klein 1918a, 1918b, 1919). He then incorporated these in Volume I of his collected works, along with commentaries in a special section entitled 'Zum Erlanger Programm' (Klein 1921–3, I, pp. 411–612). Emmy Noether supported this undertaking from beginning to end, when she read the page proofs for this section of the volume.[12]

Klein had no difficulty interpreting special relativity within the context of his 'Erlanger Programm,' since the Poincaré group acts globally on Minkowski space. General relativity, on the other hand, was based on the principle of general covariance. In this setting, one takes the coordinate transformations to be defined by bijective analytic functions with a non-vanishing functional determinant; this ensures the existence of local bijective inverses, and thus a local transformation group. Sophus Lie had begun to develop such a theory in the 1880s by studying local differential invariants (Lie 1884), whereas Klein's research after 1872 dealt with applications of finite and infinite discrete groups. Eduard Study, a leading expert on algebraic invariant theory, later exposed several weaknesses in Lie's approach, which differs in many respects from modern Lie theory (Study 1908).

During the mid-1870s, Klein continued working closely with his former Erlangen colleague, Paul Gordan, a leading expert on formal methods in invariant theory. Gordan was joined in Erlangen by Max Noether in 1875, the year Klein departed for Munich, and it was Gordan who later guided the doctoral research of his colleague's daughter.[13] Emmy Noether completed her dissertation in 1907, after which time she assisted her father and other Erlangen mathematicians. After 1911, she struck up an intense collaboration with Ernst Fischer, an Austrian mathematician who came to Erlangen as Gordan's successor. Soon thereafter, she began churning out papers closely related to Hilbert's famous invariant-theoretic works from the late 1880s and early 1890s. These eventually caught the latter's attention, and so it happened that both Hilbert and Klein became well aware of Emmy Noether's expertise in invariant theory. As the war dragged on and the usual reservoir of mathematical talent in Göttingen began to dissipate, both were keen to gain her assistance for their ongoing work. This turning point was the one and only time in her career when she could actually profit from not being a man.[14]

[12] Klein (1921–3, 1, p. v). Hubert Goenner has pointed out that modern spacetime theories involve fields attached to the geometry. Physicists typically study the symmetries of Lie groups or Lie algebras associated with such fields rather than groups that act on the spacetime itself; see Goenner (2015).

[13] On mathematics in Erlangen during this period, see Rowe (2021, pp. 1–37).

[14] Her petition to habilitate in Göttingen initially failed, however, and she only gained the right to teach courses in 1919; see Rowe (2021, pp. 39–61).

2.4 On Klein's Göttingen Lectures, 1916–1917

As is well known, Riemannian geometry and the theory of differential forms took on new life in the wake of Einstein's general theory of relativity, which highlighted the importance of Gregorio Ricci's absolute differential calculus, dubbed tensor analysis by Einstein. In the 1880s, Ricci was the first to systematize the theory of differential invariants in the form of a calculus. During the last years of the war, Klein turned to these topics in his historical lectures, delivered in part to celebrate the legacy of the Göttingen mathematical tradition. He described an analytical tradition – associated with the names of Christoffel, Bianchi, and Ricci – setting this against a second tradition, which he identified with the names of Riemann and Rudolf Lipschitz; the latter happened to have been his formal doctoral supervisor in Bonn, following the sudden death in 1868 of his actual mentor, Julius Plücker.

As Abraham Pais pointed out in his scientific biography of Einstein (Pais 1982/2005, p. 217), Felix Klein situated Einstein's work in the analytic tradition of Christoffel and Ricci. Pais warmly recommended Klein's posthumously published lectures (Klein 1927),[20] but he apparently spent little time reading these himself. Had he done so, he surely would have noticed that Klein sharply contrasted between Riemann's methodological approach and the methods Einstein and Grossmann adopted from Ricci and Levi-Civita. To put the matter plainly, Klein's interpretation placed Einstein's work outside the mainstream of the Riemannian tradition. This latter direction drew on variational methods and employed so-called normal coordinates, methods developed further by Lipschitz, who explicitly cited Riemann's work (Lipschitz 1869, 1870). Christoffel's methods for classifying differential forms, on the other hand, were entirely formal and algebraic. Ricci, in fact, saw these as major virtues, and he went out of his way to criticize the use of the calculus of variations as a method for finding differential invariants, a technique employed in earlier works of Jacobi and Beltrami (Ricci and Levi-Civita 1901, p. 127). Klein mentioned Ricci's dismissive attitude toward Beltrami's work, before he went on to say:

> The Christoffel-Ricci representation has found widespread transmission. It takes the place of honor in Edmund Wright's monograph *Invariants of Quadratic Differential Forms* (Cambridge Tracts 1908), whereas Riemann is only dealt with in passing and Lipschitz not at all. Einstein, too, grew up in this tradition. (Klein 1927, p. 189)[21]

In short, Klein's principal goal in his lectures on differential invariants was to resurrect this Riemannian approach. His presentation culminated with a fairly detailed account of its methods and results, tools that provided the background and motivation for Noether (1918a). As will be described below, these ideas formed a major role in Klein's overall agenda, and Noether pursued them with remarkable success. She also attended some of his earlier and better-known courses on the development of mathematics in the nineteenth century. He taught these over a span of three semesters, starting in the winter of 1914–15. Although she was then still in Erlangen, Noether evidently received a copy of the manuscript from

[20] The third chapter stems from his SS 1917 lectures, which were supplemented by the editor, Stephan Cohn-Vossen. This chapter is entitled, 'Gruppen analytischer Punkttransformationen bei Zugrundelegung einer quadratischen Differentialform.'

[21] Wright's text (Wright 1908), published roughly a decade before Klein's lectures, summarized the state of the art prior to the advent of general relativity.

those initial lectures that Klein's daughter, Elisabeth Staiger, wrote up for him. Klein made a note of this in preparation for a meeting with Noether on 28 April 1915, when he began his second series of lectures.[22]

Klein's lectures on the mathematics of the past century were to have ended with a fourth and final series, held during the summer semester of 1916, in which he originally planned to discuss the works of Lie and Henri Poincaré. Spurred by Einstein's and Hilbert's recent publications, however, he opted instead to lecture on mathematical developments related to the theory of relativity. His assistant, Walter Baade, wrote up these lectures in a 100-page manuscript to which Klein appended a provisional table of contents.[23] Nevertheless, he conceived of these as a direct continuation of his earlier courses; he thus labeled these lectures Teil IV, Kap. 9, so as to follow directly on the eight chapters in the earlier *Ausarbeitungen*.[24] Judging by the many topics touched upon, these lectures represent Klein's first attempt to gain an overview of the literature. Some of the material he covered was probably reworked into subsequent *Ausarbeitungen*, which he circulated through Sommerfeld and Einstein.[25]

Klein's interest in special relativity largely focused on the invariant theory of the Lorentz group, starting with Maxwell equations, which he wrote in a manifestly covariant form. Some years before this, Klein (1910) gave a projective interpretation of Minkowski space, which placed it within the larger context of metric geometries of constant curvature. Like Euclidean geometry, it corresponds to a flat space, but one in which the metric is indefinite rather than positive definite. Klein had been the first to exploit the possibility of attaching different types of quadrics to a projective space in order to introduce a metric, an idea inspired by Arthur Cayley's realization of Euclidean geometry by means of the so-called Cayley metric.[26] In this special case, the quadric is a degenerate imaginary figure, whereas the Minkowski metric corresponds to a real degenerate quadric. These possibilities for deriving different metrical geometries served as a major inspiration for Klein's 'Erlanger Programm,' especially since the same general approach could be applied in many different settings. After Minkowski geometrized what later came to be called 'special' relativity, Klein proposed calling 'invariant theory relative to a group of transformations the relativity theory of a group' (Klein 1910, p. 539). Emmy Noether underscored this viewpoint at the very end of "Invariante Variationsprobleme" (Noether 1918b, p. 257) in order to emphasize how her various results fully accorded with this position.

[22] Nachlass Klein 21J, SUB, Göttingen. Noether was enrolled as one of the 28 auditors for these lectures, which Klein held in Carathódory's seminar. She was also one of the 20 who attended his third series of lectures during the winter semester of 1915–16 (Nachlass Klein 21K).

[23] Nachlass Klein 21N, SUB, Göttingen.

[24] These lectures were entitled 'Die Infinitesimalgeometrie bei Gauß und Riemann und ihre Bedeutung für die neueste mathematische Physik'.

[25] Einstein reacted in a letter to Klein from 15 December 1917, in which he criticized the latter's tendency to overestimate the importance of formal methods (Einstein 1998a, pp. 569–70). Pais later commented about the irony of this pronouncement in light of Einstein's later views about the guiding role of mathematical ideas for physical theories (Pais 1982/2005, p. 325).

[26] In a metric geometry, the distance between two points is an elementary invariant under congruence transformations (isometries); whereas four collinear points determine an invariant, the cross ratio, in projective geometry. By attaching a quadric to the space as an 'absolute' figure, one can introduce a metric, owing to the fact that the line joining any two points will meet the quadric in two more. If the transformation group is then restricted to automorphisms that leave the 'absolute' figure fixed, the invariance of the cross ratio ensures that the metric thus defined will remain invariant under these transformations.

A prime goal of Klein's lectures on the mathematical foundations of relativity was to describe how the relevant ideas first arose. Invariant theory and the calculus of variations both had a long history; so did group theory, including the theory of transformation groups first launched by Klein and Lie in the early 1870s. In 1904, Henri Poincaré noted that the Lorentz transformations form a group, and four years later Minkowski geometrized special relativity by inaugurating spacetime physics.[27] One of the major strands in Klein's lectures dealt with the posthumous reception of Bernhard Riemann's famous ideas on what came to be called Riemannian geometry. In his younger years, Klein had been strongly influenced by Riemann's geometric theory of complex functions, in particular the notion of so-called Riemann surfaces. He also read the habilitation lecture Riemann delivered in 1854 on a topic chosen by Carl Friedrich Gauss (Riemann 1868). Hermann von Helmholtz first became aware of Riemann's reflections after reading a lecture delivered by Ernst Schering in December 1866, the year of Riemann's death (Schering 1867). As he emphasized soon thereafter in his celebrated essay (Helmholtz 1868), his own reflections on the empirical roots of human space perception ran along somewhat parallel lines as those set out by Riemann. Klein recalled how deep and mysterious Riemann's ideas seemed to him and his generation at the time they first appeared (Klein 1927, p. 165).

One reason for this had to do with the circumstance that Riemann was speaking to the entire philosophical faculty. His allusions to mathematical concepts were thus exceedingly condensed, though sufficient for Gauss to follow and appreciate, especially since Riemann indicated clearly how his concept of curvature for n-dimensional manifolds was related to Gaussian curvature for surfaces. Richard Dedekind reported on Gauss's praise for this accomplishment in the biographical essay he wrote for the first edition of Riemann's *Werke* (Riemann 1876, pp. 507–26). This volume also contained Riemann's 'Commentatio Mathematica' (Riemann 1876a), a paper he composed in 1861 in response to a prize problem set by the Paris Academy. Since his submission failed to win a prize, the manuscript languished among Riemann's papers until it was rescued by Heinrich Weber, who wrote a fairly lengthy commentary on it (Riemann 1876, pp. 384–99, and 1892, pp. 405–23). About half of his commentary was devoted to a single page of the text, which contained the analytic formulas that Riemann suppressed in his lecture from 1854.[28] Klein's 'Erlanger Programm' was written long before the advent of the modern notion of differential manifolds. In fact, it appeared even before Heinrich Weber published Riemann's 'Commentatio' (Riemann 1876a) in the first edition of his *Werke*. Klein's longstanding interest in Riemann's work eventually led to the discovery of new material from his lectures and manuscripts, which were then edited by Max Noether and Wilhelm Wirtinger for publication in 1902.[29] Despite such efforts,

[27] Einstein initially considered this a largely superfluous formalism, but after Arnold Sommerfeld and Max von Laue developed it further, he soon came to appreciate the virtues of 4-vectors (Walter 2007). In the mathematical section of Einstein and Grossmann (1913, p. 328), Grossmann cited works by these authors for readers who wished to study the tensor calculus used in special relativity.

[28] Weber elucidated these in Riemann (1876, pp. 384–91), but to meet criticisms of that commentary he gave a revised version in the second edition (Riemann 1892, pp. 405–15). Olivier Darrigol recently offered a somewhat different interpretation based on a study of fragments found in Riemann's literary estate; see Darrigol (2015).

[29] This undertaking coincided with another, much larger project, namely the Gauss edition, which Klein took over after Schering's death in 1897.

the thread of sources tracing back to Riemann that might shed light on his approach to Riemannian geometry is exceedingly thin.

Riemann's habilitation lecture presents only the idea behind normal coordinates in qualitative terms (Riemann 1892, pp. 276–8), whereas Weber was the first to work out the analytic form (Riemann 1892, pp. 405–11). Starting with an arbitrary point O as origin and the bundle of vectors v_O emanating from it, one coordinatizes the points P at a distance d from O in the direction of v_O by the lengths along the geodesic curve determined by the given vector. Since these geodesic curves are invariants of the metric attached to a differential manifold, Riemannian normal coordinates are part of its intrinsic structure.[30]

Klein traced these ideas through Gaussian surface theory in his lectures, introducing the Lagrangian differential equations for geodesics on a surface. His starting point was the variational principle

$$\delta \int ds = \delta \int \sqrt{Edu^2 + 2Fdudv + Gdv^2} = 0, \tag{2.8}$$

which leads to two differential equations when u and v are varied independently. In order to connect this with classical physics, Klein introduced a time variable t so that the geodesic curves correspond to the paths of inertial motion of test particles on the surface once launched with a given initial velocity. This well-known fact evidently provided Einstein with an important clue in 1912, the year he began searching for a non-scalar theory of gravitation. As he later recalled in his Kyoto address from 1922:

> I suddenly realized that there was good reason to believe that the Gaussian theory of surfaces might be the key to unlock the mystery. I realized at that point the great importance of Gaussian surface coordinates. However, I was still unaware of the fact that Riemann had given an even more profound discussion of the foundations of geometry. I happened to remember that Gauss's theory had been covered in a course I had taken during my student days with a professor of mathematics named Geiser.[31] From this I developed my ideas, and I arrived at the notion that geometry must have physical significance. (Einstein 2012, p. 638, my translation)

Following Lipschitz (1870), Klein rewrote (2.8) for a general quadratic differential form as

$$\delta \int \frac{\sum g_{ik} dx^i dx^k}{dt^2} dt = 0,$$

which implies that for arbitrary dx^r the invariant

$$2d\left(\sum g_{ik} dx^i \delta x^k\right) - \delta\left(\sum g_{ik} dx^i dx^k\right) = 2\sum \Psi_r(d,d)\delta x^r = 0. \tag{2.9}$$

This implies that $\Psi_r(d,d)$ is a covariant vector, and the equation describes the geodesics for an n-dimensional differential manifold R^n. When written out, it amounts to the usual form written with Christoffel symbols. Lipschitz obtained $3n$ differential equations in all:

$$\Psi_r(d,d) = 0, \qquad \Psi_r(d,\delta) = 0, \qquad \Psi_r(\delta,\delta) = 0, \tag{2.10}$$

30 For an overview of the uses of normal coordinates, see Pauli (1958, pp. 44–52).
31 The geometer Carl Geiser was a professor at the ETH Zürich from 1869 to 1913.

where the first and third are special cases of the second, which can be written as

$$\Psi_r(d,\delta) = \sum_i g_{ir} d\delta x^i + \sum_{i,k} \frac{1}{2}\left(\frac{\partial g_{ir}}{\partial x_k} + \frac{\partial g_{kr}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_r}\right) dx^i dx^k = 0. \tag{2.11}$$

Lipschitz applied this formalism to derive Riemann's curvature form $[\Omega]$ using methods adapted by Emmy Noether (1918a). Starting from the differential form $f = \sum g_{ik} dx^i dx^k$, Riemann first introduced

$$\Omega = \delta\delta \sum g_{ik} dx^i dx^k - 2d\delta \sum g_{ik} dx^i \delta x^k + dd \sum g_{ik} \delta x^i \delta x^k. \tag{2.12}$$

The operators d and δ commute and have the same formal properties as in ordinary calculus. Thus,

$$\delta\delta \sum g_{ik} dx^i dx^k = \delta\left(\sum \delta g_{ik} dx^i dx^k + \sum g_{ik}\left(\delta dx^i dx^k + dx^i \delta dx^k\right)\right),$$

where $\delta g_{ik} = \sum_r \frac{\partial g_{ik}}{\partial x_r} \delta x_r$. In connection with Riemann and Lipschitz, Noether referred to (2.12) as the 'normal form of the second variation.' She showed how similar methods could be used to derive corresponding normal forms of higher degree (Noether 1918a).

Riemann derived $[\Omega]$ by using the geodesic differential equations (2.10) to find $d^2, d\delta, \delta\delta,$ whereas Lipschitz eliminated the higher-order differentials directly and found:

$$[\Omega] = \Omega - 2\Big\{\sum g^{rs}\Psi_r(d,\delta)\Psi_s(d,\delta) - \sum g^{rs}\Psi_r(d,d)\Psi_s(\delta,\delta)\Big\}. \tag{2.13}$$

Klein used Grassmannian coordinates to express the numerator and denominator of the Riemannian curvature tensor K_R. Starting with $f(d,d) = \sum g_{ik} dx^i dx^k$, he formed its polar $f(d,\delta) = \sum g_{ik} dx^i \delta x^k$, and then the determinant

$$F = \begin{vmatrix} f(d,d) & f(d,\delta) \\ f(\delta,d) & f(\delta,\delta) \end{vmatrix},$$

writing this in Grassmannian notation as

$$F = \sum (g_{ir}g_{ks} - g_{kr}g_{is}) p_{ik} p_{rs}. \tag{2.14}$$

He did the same with the numerator, writing $[\Omega] = \sum (ik,rs) p_{ik} p_{rs},$ and then

$$K_R = -\frac{[\Omega]}{2F}. \tag{2.15}$$

Klein had long been familiar with Grassmannian coordinates, which in the case $n = 4$ provide an elegant method for representing the lines in projective 3-space. Taking two points with coordinates (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4), the six determinants $p_{ik} = x_i y_k - x_k y_i$ yield homogeneous coordinates for the line joining them. These line coordinates satisfy a quadratic identity, namely $P = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$. The same relations hold for arbitrary n, where for any two vectors d, δ, the $p_{ik} = dx_i \delta x_k - dx_k \delta x_i$ form an anti-symmetric tensor. Klein was later astonished to learn that these were connected with Riemannian differential geometry, an insight Lipschitz never conveyed to him in the late 1860s when Klein was immersed in his dissertation topic, a systematic investigation of quadratic line complexes in line geometry. Thus, in his 'Erlanger Programm,' he noted the significance

2.5 On Klein's Collaboration with Noether

The collaboration between Klein and Noether began in 1917 when she was attending his private lectures on the mathematical foundations of relativity theory. The fact that they never co-authored a single paper should come as no surprise. Co-authorship was still relatively uncommon during this period, and German professors were famous for exploiting work done by their assistants. In the case of Klein and Noether, she was obviously in a subservient role; not until 1919 was she allowed to join the faculty as an unpaid lecturer (*Privatdozent*). Nevertheless, Klein's papers contain many references to her work as well as allusions to her vital assistance, whereas she included similar comments in her papers, indicating how her work was closely tied with his. Emmy Noether was, in fact, a key member in a small team of researchers who were engaged in Klein's large-scale research program aimed at elaborating mathematical methods in modern physics. Others who assisted him included Hermann Vermeil, Vsevolod Frederiks, and Walter Baade.

It appears that Noether was no longer working closely with Hilbert by the summer of 1917, since he had ended his lectures on relativity theory and was now teaching set theory. Her work with Klein, on the other hand, began to build momentum around this time. On 22 August 1917, she wrote Fischer to announce that she had finally solved a problem that had occupied her attention since spring, namely the extension of a theorem proved by E. B. Christoffel and G. Ricci for quadratic differential forms to arbitrary forms of any degree (Dick 1981, p. 33). Noether presented a lecture on her general 'Reduction Theorem' on 15 January 1918 at a meeting of the Göttingen Mathematical Society, and 10 days later, Klein submitted her paper (Noether 1918a) for publication. Drawing on methods in the calculus of variations going back to Lagrange, she showed how the problem of finding a complete system of differential invariants could be reduced to classical invariant theory, i.e. finding all algebraic invariants of a corresponding projective group. One begins with one or more differential expressions $f(x, dx) = f(x_1, \ldots, x_n; dx_1, \ldots, dx_n)$ and the group of all analytic transformations $x_i = x_i(y_1, \ldots, y_n)$, where $f(x, dx)$ is mapped to $g(y, dy)$. On the level of the differentials, this general group corresponds to the group of all linear transformations:

$$dx_i = \sum \frac{\partial x_i}{\partial y_k} dy_k; \qquad \delta x_i = \sum \frac{\partial x_i}{\partial y_k} \delta y_k.$$

A general differential invariant is an analytic function J, defined for any f and any number of its partial derivatives with respect to either the variables, the differentials, or combinations of both, that remains unchanged under the general transformation group. If, however, none of the derivatives involves the variables x_i, and only first-order differentials appear in J, then these are called projective invariants, as they are restricted to the general linear group. Noether's argument thus aimed to show how to generate a system of differential invariants that can then be reduced to projective invariants by systematically removing higher-order differentials.

Noether (1922, p. 407) briefly described how her theorem was related to classical variational problems (2.2), thus leading to the introduction of Riemannian normal coordinates, which transform the geodesic curves in a differential manifold into lines. The corresponding coordinate transformations will consequently be linear mappings, and the system of differential invariants can then be transformed to another equivalent system whose underlying transformation group consists of linear transformations. How all this can be carried out is only sketched by Noether (1918a, pp. 245–6), as she originally planned to publish a detailed analysis in *Mathematische Annalen*, a plan that never materialized. She noted, however, that the case for quadratic forms was worked out in detail by Vermeil (1919), who closely followed her general ideas. Above all, Noether emphasized that her note demonstrated the scope and power of formal variational methods as opposed to the far less transparent methods of Christoffel using elimination theory (Noether 1922: 407). Noether (1922) refers explicitly to Klein's seminar lectures from the summer semester of 1917 on mathematical methods in relativity, which clearly served to launch her work on variational methods in the theory of differential invariants. Still, it seems very doubtful that Klein imagined one could extend Christoffel's 'Reduction Theorem' to all possible differential forms. Only Noether had the full grasp of technical resources to take on such a difficult problem.

Einstein also read her paper and wrote to Hilbert on 24 May 1918:

> Yesterday I received from Frl. Noether her very interesting paper on the construction of invariants. It impresses me that one can view these things from such a general standpoint. It would have done no harm to the troops returning to Göttingen from the field if they had been sent to school under Frl. Noether. She appears to know her métier well! (Einstein 1998b, p. 774)

Klein had submitted Noether's (1918a) paper in January, and in the meantime his collaboration with Noether had intensified. It was during this time that their attention shifted to understanding the status of the various formulations of energy-momentum conservation in general relativity. Unlike Einstein, Klein distinguished sharply between these new findings and traditional conservation laws in classical mechanics. This soon became part of his program for promoting the 'Erlangen Programm' (Klein 1872) as a framework for the new physics. In Klein's view, relativity theory was best conceived as a broad approach linking mathematics and physics. He thus saw it not in terms of two groups – the Lorentz group and the group of general point transformations – but rather as the invariant theory *relative to some given group* bearing on a particular physical theory. He thus had this general context in mind when he emphasized the distinction between conservation laws in classical mechanics, special relativity, and the general theory of relativity.

These issues surfaced when Klein and Hilbert exchanged their views regarding the status of the various conservation laws in Klein (1918a). Hilbert not only agreed with Klein's general position, he went further by claiming there was no analogue for classical energy conservation in general relativity. He even asserted that *one could prove a theorem* effectively ruling out conservation laws for general transformations analogous to those that hold for the transformations of the orthogonal group. That remark caught Klein's attention, and he replied: 'It would interest me very much to see the mathematical proof carried out that you alluded to in your answer' (Klein 1918a, p. 565). Emmy Noether was clearly very interested, too.

She wrote Klein from Erlangen on 29 February 1918: 'I thank you very much for sending me your note Klein (1918a) and today's letter, and I'm very excited about your second note Klein (1918b); the notes will certainly contribute much to the understanding of the Einstein–Hilbert theory.'[33] Noether then proceeded to explain her progress on the problem of distinguishing between classical and relativistic conservation laws. Twelve days later, she wrote Klein again about some key ideas that she would later publish in "Invariante Variationsprobleme" (see Rowe 2021, p. 81). The next day, 13 March, her father, the eminent algebraic geometer Max Noether, was pleased to report on her progress in a letter to Klein. Alluding to her collaboration with Klein, he wrote, 'it has been a great source of satisfaction for me that our mutual relations have been revived again through the activities of my daughter; I see every day how her creative powers grow and hope that these will lead to many new results' (Nachlass Klein 12, SUB Göttingen). Four months later, on 23 July, Emmy Noether presented her main results to the Göttingen Mathematical Society. Klein then submitted a preliminary version of her findings to the Göttingen Scientific Society three days later. After receiving proofs, she made the final revisions in late September. Noether (1918b) gave a precise answer to Hilbert's conjecture, though he apparently never acknowledged this. In "Grundlagen der Physik" (Hilbert 1924, p. 5), he cited her paper in a footnote, though only because her second theorem provided a proof for Hilbert's Theorem I in "Die Grundlagen der Physik I" (Hilbert 1915).

As emphasized above, Noether's methods for constructing differential invariants were closely related to those of Riemann and Lipschitz. Moreover, as can be seen from her brief survey article (Noether 1922), the same invariant-theoretic methods she used in Noether (1918a) were central for setting forth the results in her far more famous paper (Noether 1918b). In the latter paper, Noether began by noting that one can derive identities involving the Lagrangian expressions ψ_i for any general variational problem, defined for a function f that depends on n independent variables $x_1, \ldots, x_n$, a set of functions of these variables, $u_1(x), \ldots, u_\mu(x)$, and their derivatives. One varies the $u(x)$ so that δu and all derivatives vanish on the boundary ∂D of the domain D, from which follows:

$$\delta I = \delta \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}\right) dx = \int \cdots \int \delta f \, dx. \tag{2.26}$$

This leads to

$$\int \cdots \int \sum \left(\psi_i \left(x, u, \frac{\partial u}{\partial x}\right) \delta u_i\right) dx, \tag{2.27}$$

where ψ_i are the usual Lagrangian expressions that arise in solving $\delta I = 0$.[34] Since the δu are varied arbitrarily, the integrands must be equal, and applying integration by parts, Noether obtains the general identity:

$$\sum \psi_i \delta u_i = \delta f + Div\, A = \delta f + \frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n}, \tag{2.28}$$

[33] E. Noether to F. Klein, 29 February 1918, Nachlass Klein 22B, SUB, Göttingen.

[34] To compute such an n-fold integral, one takes D to be a Cartesian product of closed intervals in R^n and then iterates the calculation of the n single integrals.

where *Div A* arises from the boundary terms. She noted that for the simplest case, a single integral and a function f that depends on no derivatives of the u higher than the first, the corresponding identity takes the form:

$$\sum \psi_i \delta u_i = \delta f - \frac{d}{dx}\left(\sum \frac{\partial f}{\partial u_i'} \delta u_i\right), \tag{2.29}$$

where $u_i' = \frac{du_i}{dx}$.

Noether calls this Heun's central Lagrangian equation, but without giving any explanation or reference. It appears that this offhand remark has never received much notice, but, in fact, this was one of the most important threads connecting Klein's lectures with Noether's publications from this time. To recognize what this says in a more familiar setting, let us look again at the Lagrangian equations (2.7), where $f = L, u = q_i$, and $u' = \dot{q}_i$. Then (2.29) reads[35]

$$\begin{aligned} \sum \psi_i \delta q_i &= \sum \left[\frac{\partial L}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i(t)}\right] \delta q_i \\ &= \sum \frac{\partial L}{\partial q_i(t)} \delta q_i - \frac{d}{dt}\left(\sum \frac{\partial L}{\partial \dot{q}_i(t)}\right) \delta q_i \\ &= \delta L - \frac{d}{dt}\left(\sum \frac{\partial L}{\partial \dot{q}_i(t)} \delta q_i\right). \end{aligned} \tag{2.30}$$

Equation (2.29) also occupies a central place in Noether's note on 'Formal Variational Calculus and Differential Invariants' (Noether 1922), which appeared at the end of the report on algebraic and differential invariants in the *Encyklopädie der mathematischen Wissenschaften*. The report itself was written by Roland Weitzenböck, a leading expert on invariant theory who was then working in Graz. He submitted the final manuscript in 1921, around the time that Felix Klein was giving Wolfgang Pauli fairly detailed feedback on the proofs for his far better-known report of relativity (Pauli 1921), which contains numerous references to related works by Klein and his young collaborators. Extant correspondence probably no longer survives that would confirm how Noether's note came to be attached to Weitzenböck's report, but the circumstances alone strongly suggest that Klein asked her to write about a special topic that had been at the center of their collaboration during the last two years of the war. Noether references a page in Heun's lengthy report on 'Ansätze und allgemeine Methoden der Systemmechanik' in Band IV of the *Encyklopädie*, which was edited by Klein. Since her brother Fritz was Karl Heun's assistant in Karlsruhe when this article appeared in 1913, she could have learned about the so-called central Lagrangian equation from him. Far more likely, though, she picked up on the mathematical importance of this equation from Klein's lectures.

In those lectures, which Klein delivered in the summer of 1917, his very first topic was Lagrangian mechanics interpreted by way of the invariant theory of a general group of point transformations (or, as one would say today, a local diffeomorphism group). Introducing Lagrangian coordinates q_α, Klein derived the equations of motion in the usual way by

[35] The final step follows from $\frac{d\delta q_i}{dt} = \delta \frac{dq_i}{dt} = \delta \dot{q}_i$.

varying the q_α over the time interval while leaving them fixed on the boundary. Writing T for the kinetic energy and $\sum P_\alpha \delta q_\alpha$ for the work associated with a virtual displacement, Klein inquires as to why the equations of motion remain invariant under general coordinate transformations. His simple answer was that the integrand in the variational formulation

$$\delta \int \left[\sum \left(\frac{\partial T}{\partial q_\alpha} - \frac{d\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)}{dt} + P_\alpha \right) \delta q_\alpha \right] dt = 0 \tag{2.31}$$

is an invariant. He then noted that Lagrange had avoided using variational analysis and instead employed the principle of virtual displacement to derive the same result using what Heun called the central Lagrangian equation (Klein 1927, pp. 141–2).

In Noether (1922) as well as in Noether (1918b), the physics behind Heun's equation has been stripped away. Instead it becomes a tool for showing how the Lagrangian expressions ψ_i enter as differential invariants, which can be used to elucidate several important relations with other differential invariants. For quadratic forms, Noether writes the equation in the form:

$$\delta \sum g_{ik} dx_i dx_k - 2d \sum g_{ik} dx_i \delta x_k = -2 \sum \psi_\mu(d, d) \delta x_\mu. \tag{2.32}$$

The notation on the right is suggestive for exploiting other identities. Indeed, one can derive a more general identity using d and δ that contains the above as a special case:

$$\begin{aligned} D \sum g_{ik} dx_i \delta x_k - \delta \sum g_{ik} dx_i Dx_k - \\ d \sum g_{ik} \delta x_i Dx_k = -2 \sum \psi_\mu(d, \delta) Dx_\mu. \end{aligned} \tag{2.33}$$

Setting $\delta = d$, we recover the identity above, where D now appears in place of δ. Noether then observes that $\psi_\mu(d, \delta)$ is a covariant vector, which leads to a contravariant vector $p(d, \delta)$, where $p(d, d) = 0$ are equations for the geodesic curves. By means of $p(d, \delta)$, one can carry out covariant differentiation, etc. The condition $p(d, \delta) = 0$ was used by Levi-Civita to interpret curvature in terms of parallel displacement of vectors.

Emmy Noether hardly considered her work on differential invariants the last word on the subject, though she herself wrote about it only retrospectively. In a report on 'Algebraic and Differential Invariants' (Noether 1923) that she delivered in September 1922 in Leipzig at the annual meeting of the German Mathematical Society, she summarized recent progress but also alluded to a number of still-outstanding problems. In her opening remarks, she recalled Hilbert's earlier synopsis of developments in algebraic invariants as having unfolded in three phases: a naive period, followed by a formal, and then a critical stage (his own work). Noether remarked that she would only speak with reference to this final phase, noting that Hilbert's work had reformed algebraic invariant theory by exploiting arithmetical methods in algebra. For differential invariants, she identified the critical developments with Riemann's name, as he was the first to treat this branch of research with formal variational methods. Her report ended with some brief remarks about the role these methods play in her reduction theorem (proved in Noether 1918a). Pointing to recent work by Weyl and J. A. Schouten, which was no longer based on variational principles, Noether threw open the question whether higher-order differentials could nevertheless be eliminated in analogy

As might be expected, these two lectures confirm that Klein and Noether were very different types of mathematicians whose respective styles practically defy comparison. Weyl noted that Noether shared with several other major figures – Dedekind, Kronecker, and even Weierstrass – a strong inclination to algebraicize the broad terrain of analysis, whereas Klein's general tendency was to invoke continuity arguments, or as Weyl expressed it, topological reasoning, which gave his analytical and algebraic works a strong geometric flavor. Since Weyl's own mathematical tastes lay closer to Klein's, he waxed more than a little enthusiastically when speaking of the latter's 'Erlanger Programm' from 1872, which he clearly saw as the centerpiece of Klein's mathematical legacy. Weyl made the particularly striking claim – one that would have greatly pleased Klein – that the understanding of geometry in the latter's 'Erlanger Programm' was 'nothing other than *relativity theory* in its general, mathematically formulated form' (Weyl 1930, p. 299).

Weyl's lecture in honor of Emmy Noether has been cited far more often and for good reason. This was a dramatic occasion, the capstone of truly harrowing times that both of them had gone through together (Rowe 2021, pp. 199–265). In the fall of 1933, Hermann Weyl abandoned his chair in Göttingen for a research professorship at Princeton's Institute for Advanced Study (IAS). Beginning in February 1934, he again saw Emmy Noether on a regular basis, as Oswald Veblen had arranged for her to give weekly lectures at the IAS. Most commentators who have subsequently surveyed Noether's career have followed Weyl in viewing it as composed of three periods: (1) 1907–19, period of relative dependence; (2) 1920–26, ideal theory; (3) 1927–35, non-commutative algebras with applications to commutative number fields (Weyl 1935, p. 439). As a general scheme, this tripartite division is certainly useful, even though the boundaries between these periods should not be drawn too sharply. More problematic is the label 'relative dependence' for (1), when Emmy Noether emerged as a leading authority in the field of *invariant theory*, producing work that carried over into the second period.

Weyl treated the first period rather dismissively, though he was well aware of her two papers (Noether 1918a, 1918b) written in collaboration with Klein. He recalled those times, when Hilbert was 'over head and ears in the general theory of relativity' and Klein saw 'its connection with his old ideas of the Erlangen program [that] brought the last flareup of his . . . mathematical production' (Weyl 1935, pp. 430–1). He noted how the posthumously published lectures (Klein 1927) bore witness to this, but also how Emmy Noether's expertise in invariant theory had supported Hilbert's research as well as Klein's. Commenting on her two papers, he wrote:

> For two of the most significant sides of the theory of general relativity she gave at that time the genuine and universal mathematical formulation: First, the reduction of the problem of differential invariants to a purely algebraic one by the use of 'normal coordinates'; second, the identities between the left sides of Euler's equation of a problem of variation which occur when the (multiple) integral is invariant with respect to a group of transformations involving arbitrary functions (identities that contain the conservation theorem of energy and momentum in the case of invariance with respect to arbitrary transformations of the four world coordinates). (Weyl 1935, p. 431)

It is worth noting that Weyl's brief allusion to 'Invariante Variationsprobleme' addresses only Noether's second theorem, not the first. When physicists write about her paper today, they often refer to *the* Noether theorem, meaning the first of her two main results, thereby overlooking that she proved two theorems. Seen in historical context, her second theorem was the central finding, since it clarified the status of identities arising from invariant variational problems; these were first noticed by Hilbert and played a major role in his 1915 publication. Hermann Weyl had, of course, an excellent vantage point from which to judge the motivation behind these two studies, but by 1935, the year of her death, only a small number of experts knew what was in these papers.

Around the centenary of her birth, Emmy Noether's career was celebrated in several new publications and with the publication of her *Collected Papers* (Noether 1983), edited by Nathan Jacobson. Noether's reputation at that time rested squarely on her contributions to modern algebra, an area of research that had expanded greatly during the half-century since Noether's death. By this time, some mathematical physicists had come to appreciate the importance of her "Invariante Variationsprobleme", but few seem to have appreciated the larger context of interests that motivated her work, despite the fact that she cited several other related papers in it. In "Formale Variationsrechnung und Differentialinvarianten" (Noether 1922, p. 408), she repeated the citations of these six earlier works, noting that they were motivated by physical problems.[38] Noether characterized her own paper as motivated by questions of principle.

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