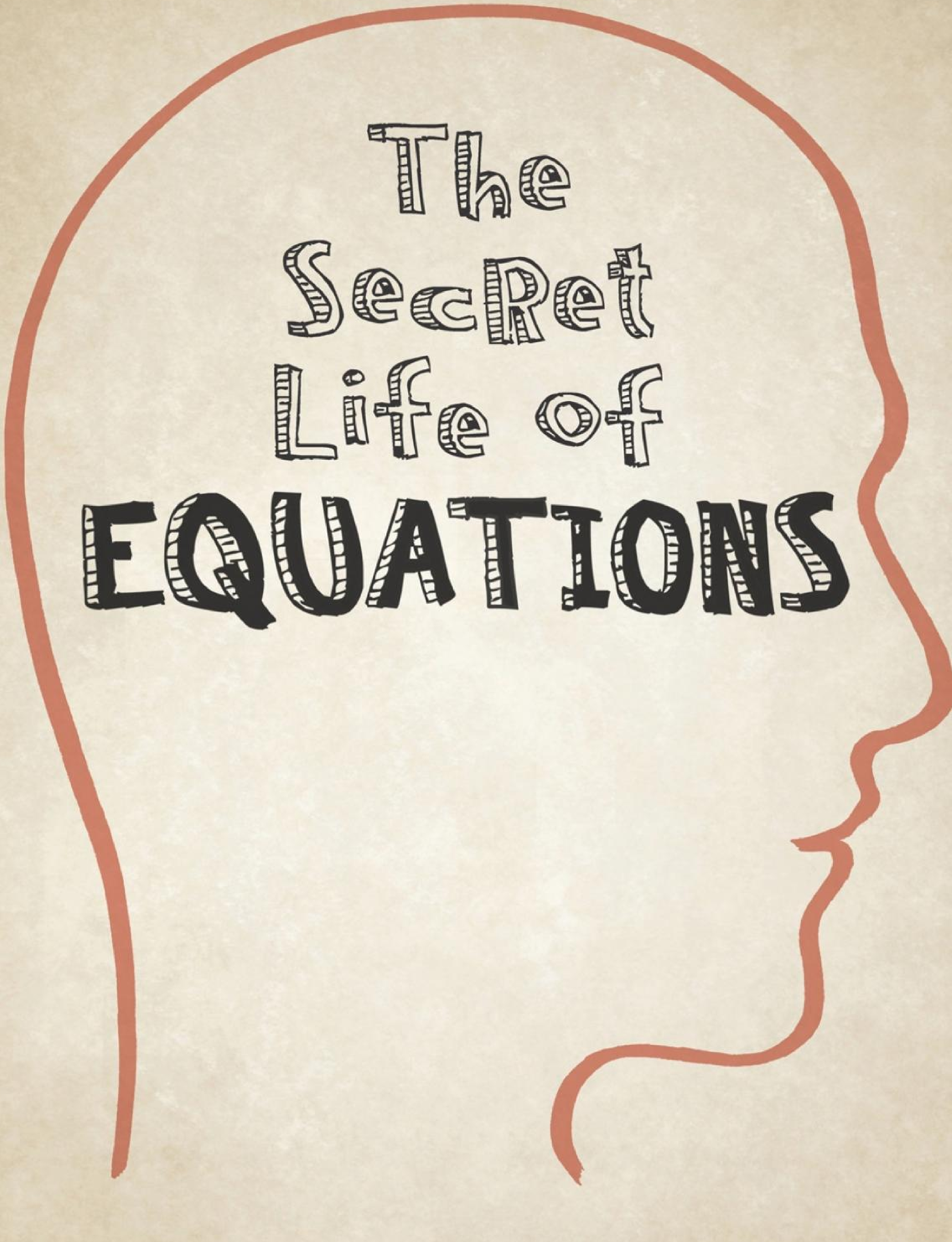
The Secret Life of EQUATIONS



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Introduction

Some time around AD 820 the Persian mathematician Abu 'Abdallah Muhammad ibn Musa al-Khwarizmi wrote his *Compendious Book on Calculation by Completion and Balancing*. There he gave us the word 'algebra' and gathered together some of its basic principles. Fundamental to algebra is the notion of balance, which the equation has come to embody: if we put an apple on one side and an orange on the other, the scales balance when the two weights are equal. And that's what every equation says: *these two things balance*.

How to use this book

This book could be read from cover to cover, like a novel, but most people don't read maths books that way. Maths is a vast, interconnected network of ideas that asks to be explored rather than watching it whizz by in a predefined order. For that reason this book contains many cross-references and you may well find that one section makes more sense when you come back to it after reading another, later one. Don't be dismayed by this; it's how most of us feel most of the time when studying maths. Even great mathematicians sometimes report feeling lost and confused when learning a new area of the subject. They also describe the joy of finding unexpected connections, some of which can be very profound and beautiful.

When meeting any new mathematical idea, most of us need to start with an intuitive picture of what's going on. This book can't get too technical – every one of its sections has been the subject of whole books, sometimes many advanced and complicated ones. What it can do is make the general ideas plain and indicate ways in which those ideas can talk to each other, sometimes across widely different parts of mathematics, science and everyday life. This inevitably involves some quite drastic simplifications, which I hope beginners will appreciate and experts will forgive. For similar reasons most of the graphs don't have numerical scales and whatnot; I suppose this will enrage maths teachers, but the removal of extraneous details keeps our focus on the overall shape of what's going on.

Notation

Still, this is a book about equations. Many popular maths books carefully chart their route to avoid too many scary-looking formulas. This one takes the opposite approach. Mathematicians' notation is designed to make life easier, not more difficult. In this respect it's just like other special forms of notation, such as that used by musicians, editors, choreographers, knitters and chess-players. If you can't parse it, it's completely incomprehensible. But if you can, then, like a picture, this notation can do the work of many cumbersome words.

Our way of writing maths down isn't always logical. It developed over hundreds of years and it can be quirky, weird or downright silly; like most things it shows traces of the historical process that produced it. Perhaps someone could invent a whole new way of writing equations that would be more coherent, but only a foolhardy reformer would dare to try. So don't worry if sometimes it's clear to you what a symbol represents but not why it looks the way it does. At some point you learned to read the words on this page; that was a much more difficult task, involving a system of notation that's almost completely arbitrary. If you managed that, you can surely do this too.

I've assumed you know about positive and negative whole numbers and what a fraction is, along with the following principles from algebra. Letters (or other symbols) can be used to stand for numbers that are unknown or that can vary. Multiplying these unknown quantities can be

Table of Symbols

Here's a list of the most important symbols that crop up in multiple sections, along with the place where they're first introduced.

$\sqrt{x}$	Square root of x [Pythagoras's Theorem, page 10]
Σ	Sum [Zeno's Dichotomy, page 18]
lim	Limit [Zeno's Dichotomy, page 18]
∞	Infinity [Zeno's Dichotomy, page 18]
π	Pi [Euler's Identity, page 40]
sin, cos, tan	Trigonometric functions [Trigonometry, page 14]
$\int$	Integral [The Fundamental Theorem of Calculus, page 26]
$\frac{dy}{dx}$	Derivative of y with respect to x (other letters sometimes replace y and x) [The Fundamental Theorem of Calculus, page 26]
$\frac{d^2y}{dx^2}$	Second derivative of y with respect to x (other letters sometimes replace y and x) [The Fundamental Theorem of Calculus, page 26]
x', x''	First and second derivatives of x with respect to time (alternative notation) [Curvature, page 30]
log, ln	Logarithms [Logarithms, page 36]
i	The square root of –1 [Euler's Identity, page 40]
∇^2	Laplacian [The Heat Equation, page 80]
div, curl	Derivatives of vector fields [Maxwell's Equations, page 92]
∇	Gradient [The Navier-Stokes Equation, page 96]
$\neg, \wedge, \vee$	Logical not, and, or [De Morgan's Laws, page 126]
$P(x)$	The probability of event x [The Uniform Distribution, page 162]
$P(x\|y)$	The conditional probability of x given y [Bayes's Theorem, page 168]

represented by putting the letters next to each other, so that

$$a \times b = ab$$

and dividing one number by another is conveniently represented by a fraction:

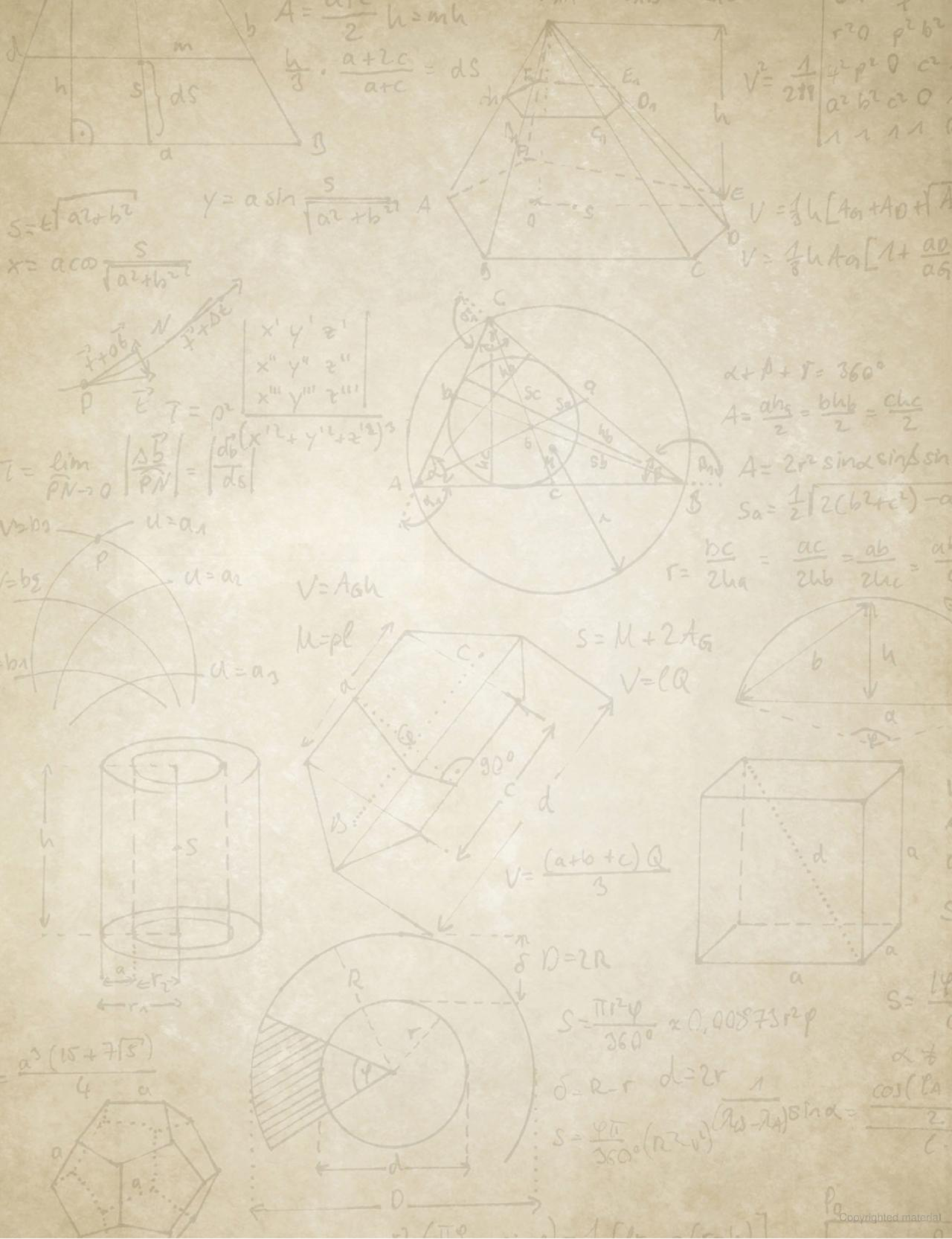
$$a \div b = \frac{a}{b}$$

Finally, the all-important equals sign says that the total of everything on one side of it is exactly the same as on the other. All other notation will be introduced as we go along.

Each equation works like a little machine with moving parts; our main task is to understand what each part does and how it interacts with all the others. Sometimes that means spending time unpacking or decoding notation. Sometimes it means working through a simple example. Sometimes it means getting to the bottom of something obscure or, by contrast, catching a hurried glimpse of it as we zoom past.

In fact, in terms of a traditional sequence of maths education, this book is incredibly uneven: one minute you're dealing with a bit of high-school algebra, then on the next page you hit something you'd meet only late-on in a university degree. I've chosen to ignore that, because mathematical subjects don't come with predefined levels of difficulty. The arithmetic you learned as a child is, it turns out, incredibly deep and mysterious, while many so-called 'advanced' topics are actually pretty easy to grasp once you get past the jargon. Go with the flow, understand what you can and look more deeply into the parts that grab your interest. There isn't a wrong way to do this.

Rich



V=AGh
M=pl
S=M+2AG
V=lQ
α+β+γ=360°
D=2R
δ=R-r
d=2r

The Shape of Space

GEOMETRY AND NUMBER

Pythagoras's Theorem

The sides of a triangle tell us something basic about how space works.

The long side — *The other sides*

$$A^2 = B^2 + C^2$$

What's It About?

Take any three sticks of any lengths. Call the lengths *A*, *B* and *C*, and suppose *A* is the longest one (or joint-longest one, if necessary). You'll find you can make the sticks into a triangle as long as *A* is less than *B* + *C*. If you want to make a triangle with a right angle – a 90° corner, like the kind on a square or rectangle – you have to have a very special set of sticks, though. In fact, if you have *any* two *B* and *C* sticks already fixed in a right angle (making an L-shape), Pythagoras's Theorem tells you how long stick *A* must be to complete the triangle.

At first this might seem less than impressive. First, it only works for a triangle with a right angle in it, which seems a bit of a limitation. Second, when *was* the last time you had to work out the lengths of the sides of a triangle anyway? Well, it turns out that triangles are fantastically important. In a sense a triangle is the simplest two-dimensional shape you can make, so problems involving other 2D shapes can often be turned into problems about triangles. Many 3D problems can, too. What's more, right-angled triangles have a rather special place among all their three-sided siblings (*see* Trigonometry, page 14).

The kind of triangle you get depends on the lengths of its three sides. Some combinations of lengths can't be made into a triangle at all.

Why Does It Matter?

Pythagoras's Theorem is one of the few equations in this book that you might yourself use when, for example, doing a bit of DIY around the house. Still, that doesn't really explain why it's such an important equation. What it captures is something very basic about the way we expect distances to work: in particular, how they relate to the way we typically find our way about.

Imagine a big field with a solitary wooden post somewhere around the middle of it. Suppose I've hidden some treasure in a secret location in this field and I need to direct you to the right spot to dig it up by passing you a message that's as concise as possible. As long as I know you'll have a compass with you (or you know how to find north by looking at the sky), I can give you the necessary information using just two numbers: I can tell you to stand at the post and go so many metres (or yards) north, and then so many metres (or yards) east.

What if I've hidden the treasure somewhere south-east of the post? No problem – I can give you a negative number for the north distance, and you'll be able to interpret –10m (–10yd) north as meaning 10m (10yd) south. In this way I can identify any point in the field, however large it is, with just those two numbers. In fact this is a standard way to find our way around flat, two-dimensional spaces, and it was formalized by the French mathematician René Descartes in the early 1600s. Instead of north and east we often use *x* and *y*, which you might remember from school. Sometimes physicists use i and j to mean more or less the same thing.

It's not even important where the post is; in fact, if the post moves, I can always adjust the numbers I've given you to take that into account. In a sense, then, this allows us to get from any one point (the post) to any other (the treasure). Here's what Pythagoras does for us: in this setup we know the distances north and east, and these form two sides of a right-angled triangle (because east is at right angles to north). So Pythagoras's Theorem tells us what the direct distance is between the post and the treasure. That makes it a fundamental fact about distances in space.

Perhaps you can see how to extend it to three dimensions, too: simply add another number indicating 'height above ground' (*see* illustration, page 12). If the number is negative, it tells you how deep to dig downwards! Pythagoras's Theorem still works in three dimensions, and even in higher dimensions, too. We call these setups 'rectangular coordinate systems', and Pythagoras's Theorem gives us a way to calculate lengths and distances. This is some of the most basic information we need in maths, physics and engineering where these systems – and this equation – are used every day.

In More Detail

We don't know much about Pythagoras; he lived in the Greek world in the 5th century BC and became the leader of a religious cult whose beliefs were steeped in numerology. Many weird stories have been told about his life and teachings, but if he himself ever wrote any of it down, none of it survives. The fact known as Pythagoras's Theorem probably wasn't discovered or proved by him alone, but it certainly seems to have been in circulation among his followers. In the book *The Ascent of Man*, the 20th-century mathematician and author Jacob Bronowski called it 'the most important single theorem in all of mathematics'; that might be pushing it a bit, but it's surely one of the ancient mathematicians' great achievements.

The first thing to notice is that on paper this looks like an equation about areas rather than lengths. After all, if A is a length, say 10cm (or 10in or 10 anything else), then A^2 is the area of a 10cm × 10cm square, that is, $100cm^2$ (one

To find the point *X* from the origin *O*, we walk a certain distance up and then to the right. Here, that distance to the right is negative!

hundred square centimetres). That, in fact, is the ancient view of it, summed up in a slogan that schoolchildren down the centuries were made to recite: 'The square on the longest side is equal to the sum of the squares on the other two sides.' This, though, hardly makes it clear why anyone should care about it, since we very rarely come across three squares arranged so neatly in real life.

The power of the theorem comes from our ability to take square roots. The square root of a number is just the number that, when you multiply it by itself, takes you back to where you started. So the square root of 9 is 3, because $3 \times 3 = 9$. In other words, if you want to lay out a square room whose area is $9m^2$ you should make each side of the room 3m long. In modern notation we write

$$\sqrt{9} = 3$$

with that odd tick symbol meaning 'square root'.

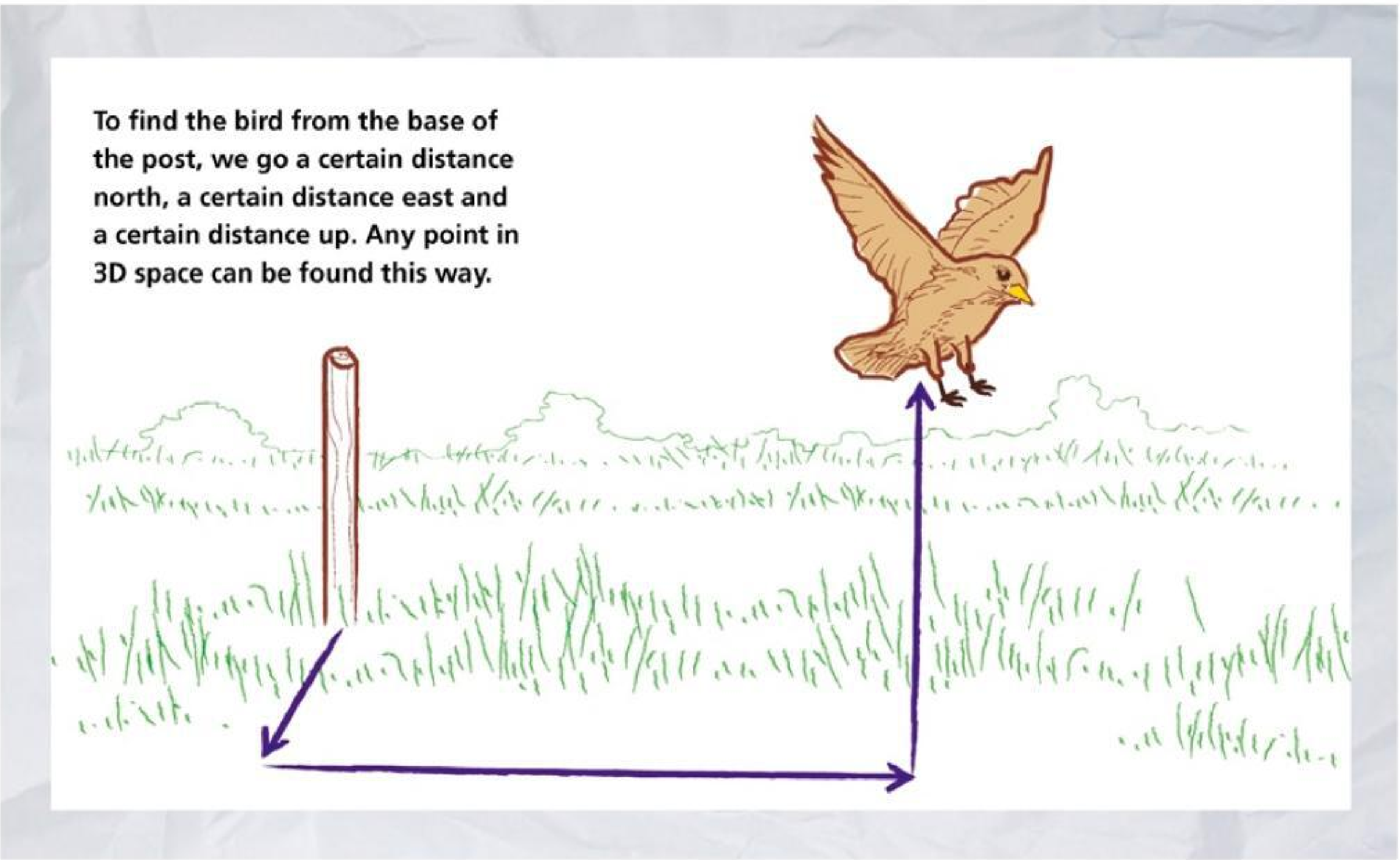
We're now ready to use Pythagoras's Theorem to find the length of stick we need to finish off a triangle or, more excitingly, how far the treasure is from the post. For example, suppose stick *B* is 3cm long and stick *C* is 4cm; they're already fixed in an L-shape; we want to find the length of stick *A* to complete the triangle:

$$\begin{aligned} A^2 &= B^2 + C^2 \\ &= 3^2 + 4^2 \\ &= 9 + 16 \\ &= 25\text{cm}^2 \end{aligned}$$

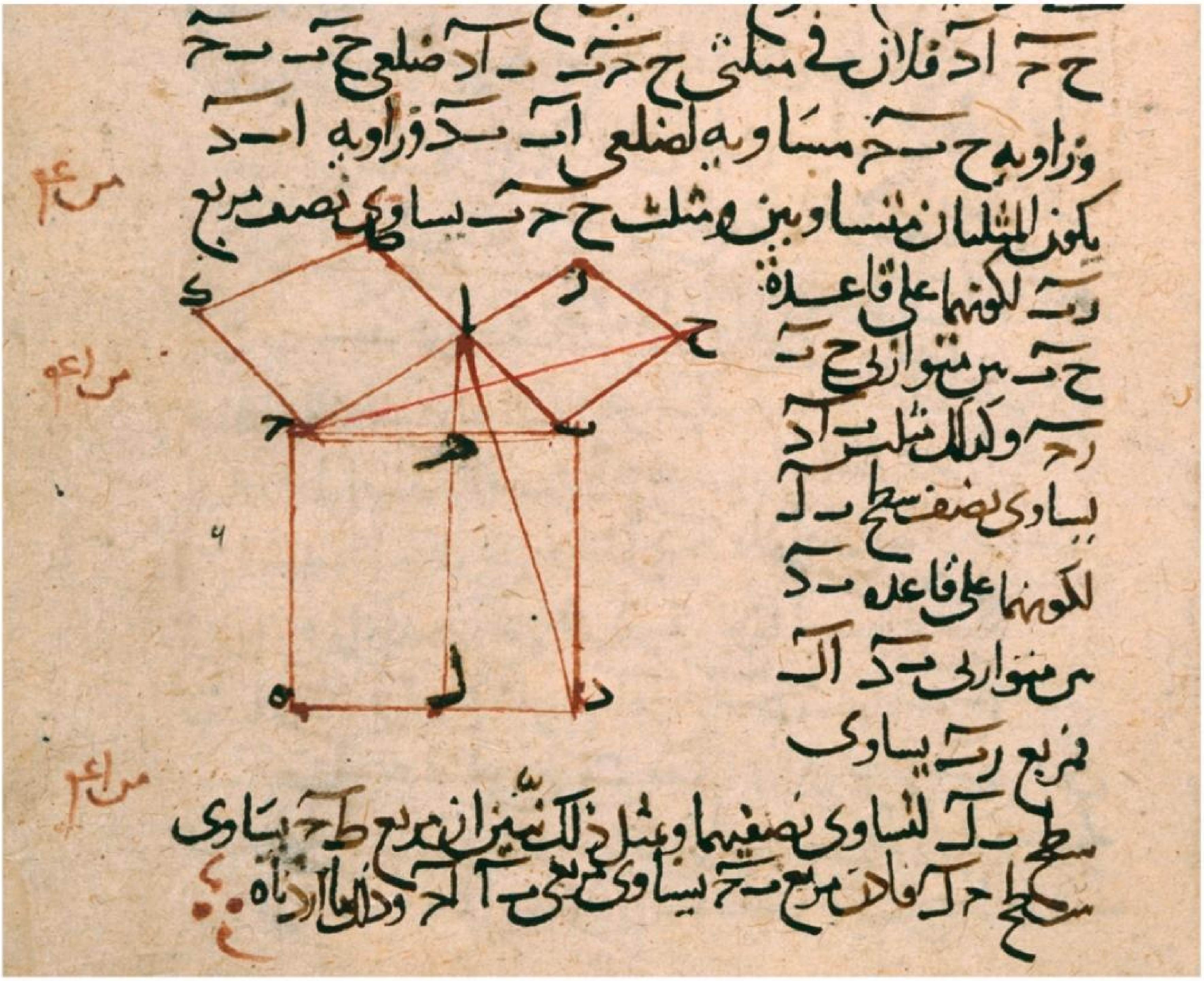
So we know A^2, but we want to find A; that is, we know the area of the square and want the lengths of its sides, which is exactly what the square root gives us:

$$\begin{aligned} A &= \sqrt{25} \\ &= 5\text{cm} \end{aligned}$$

As in the example about areas given above, this sum works equally well with sides of 3in, 4in and 5in, or of any other unit. I didn't pick the numbers 3, 4 and 5 by accident: when *A*, *B* and *C* in Pythagoras's Theorem are all nice whole numbers they're called a 'Pythagorean Triple'. These aren't



To find the bird from the base of the post, we go a certain distance north, a certain distance east and a certain distance up. Any point in 3D space can be found this way.



The renowned Persian scholar Nasir al-Din al-Tusi published his version of Euclid's Proof of the Pythagorean theorem in Arabic in 1258.

so easy to come up with by trial and error, but the ancient Greek geometer Euclid figured out a clever way to find them. Pick any two different whole numbers – call them p and q – and suppose p is the bigger one. Then make

$$A = p^2 + q^2$$
$$B = 2pq$$
$$C = p^2 - q^2$$

and you have a Pythagorean Triple. If you know a little bit of algebra, try proving for yourself that this works: $B^2 + C^2$ will always equal A^2 if Euclid's recipe is followed.

Pythagoras's Theorem looks like a relationship between the areas of three squares, but actually it tells us how to work out distances between points in space.

Trigonometry

Circles make the world go round; triangles give us a handle on them.

The angle

The opposite side

$$\sin(a) = \frac{O}{H}$$

The long side

The adjacent side

$$\cos(a) = \frac{A}{H}$$

$$\tan(a) = \frac{O}{A}$$

What's It About?

The word 'trigonometry' means something like 'the art of measuring triangles'. Triangles are some of the most basic shapes in geometry – they come up everywhere in areas such as surveying, building and astronomy, so it's not surprising that this is a very old art indeed. In fact, in some ways trigonometry is older than anything we would recognize as geometry, or even mathematics: we can find its beginnings in practical techniques used in ancient Egypt and Babylon a good 4,000 years ago.

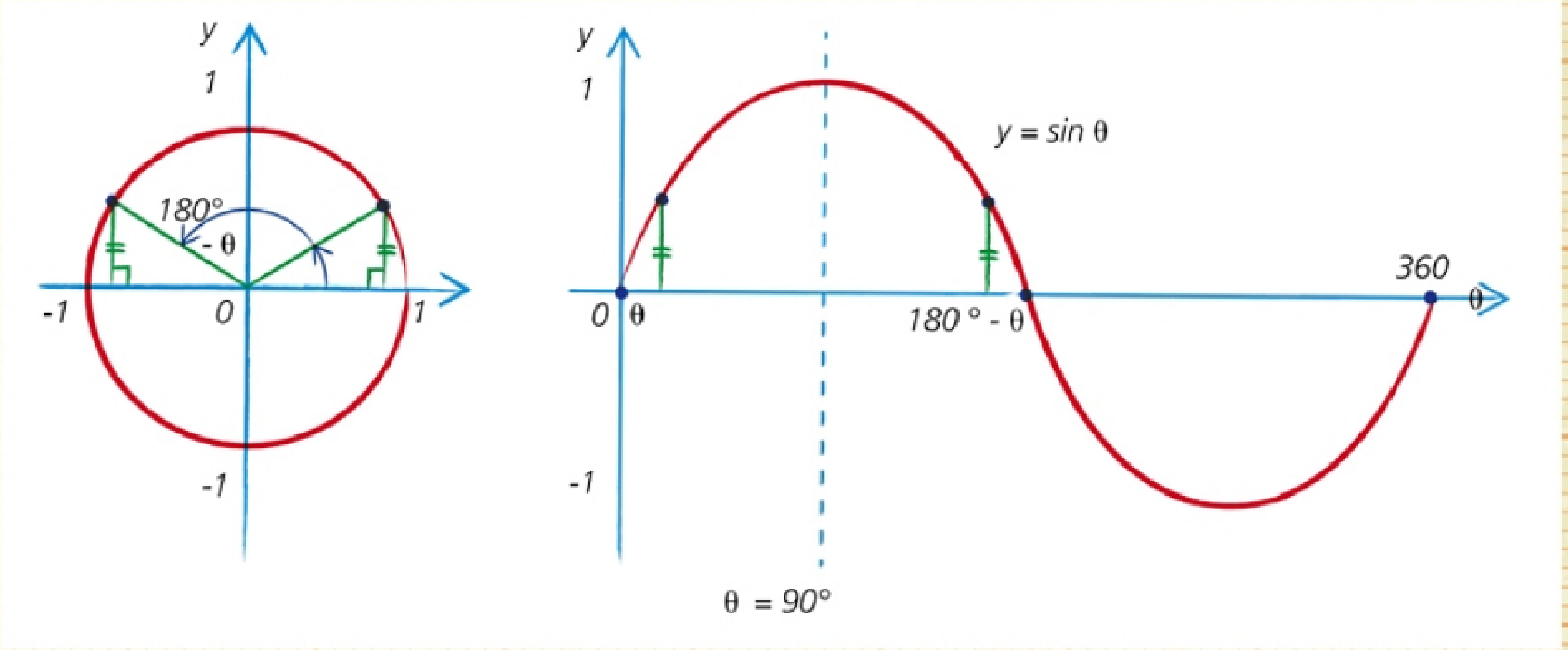
It turns out that trigonometry has intimate connections with circles, even though circles don't look much like triangles. This, too, was known intuitively from very early on: a point moving in a circle can be described by the trigonometric functions, and they appear in many mathematical models that involve circular or smooth back-and-forth motions. As a result, they pop up in several equations in this book.

In More Detail

When it comes to measuring triangles, two things spring to mind: the lengths of the three sides and the sizes of the three angles. These are obviously connected: to see this, take any three sticks and you'll find there's only one triangle you can make with them, so the lengths seem to determine the angles in advance.

This relationship is evidently more about proportions than about actual lengths, though, since two triangles can have the same angles but different-length sides. In other words, they are the same shape but different sizes – the jargon from geometry class is that they're 'similar triangles'. So it is the ratios of the lengths of the sides that determine the angles in the triangle, not the actual lengths themselves.

Around AD600, Indian scholars created the main trigonometric ratios as we know them today, though they went under different names: we call



The sine function (on the right) is the height of the green triangle (on the left) as its longest side sweeps around the red circle.

them sine ('sin'), cosine ('cos') and tangent ('tan'). There have been quite a few others, some of which are still in regular use because they make certain formulas or operations more convenient, but these are the best-known ones. For a long time they were laboriously calculated by measuring different triangles. But why would anyone feel the need to do this? The answer, at the time, was simple: trigonometry helps us deal with common, real-life problems that are tough to solve without it.

Suppose you want to measure the height of a tall tree that's too difficult to climb. If you lie down on the ground you can measure the angle you have to look up at to see the top of the tree. This can be done quite accurately with some simple equipment. You can also easily measure the distance along the ground from where you're looking to the bottom of the tree. From this information, trigonometry enables us to find the height of the tree.

We know an angle, *x*, and the length of side adjacent to it, *A*; we'd like to find the length of the opposite side, *O*. The formulas tell us that

$$\tan(x) = \frac{O}{A}$$

Suppose we measure the angle to be 40°. We look up tan(40) in a table – or use a calculator – and find the value is about 0.839. Suppose we also measured the distance *A* to be 10m (or, again, 10 yards or 10 of any other unit). Then we have

$$0.839 = \frac{O}{10}$$

which means that *O*, the height of the tree, must be 8.39m. As you can imagine, this was a very useful technique for ancient surveyors and builders to know about, and their successors still use it today.

Circles, angles and distances are basic building blocks of some of the maths we use most often – trigonometry brings them together in a strange kind of harmony.

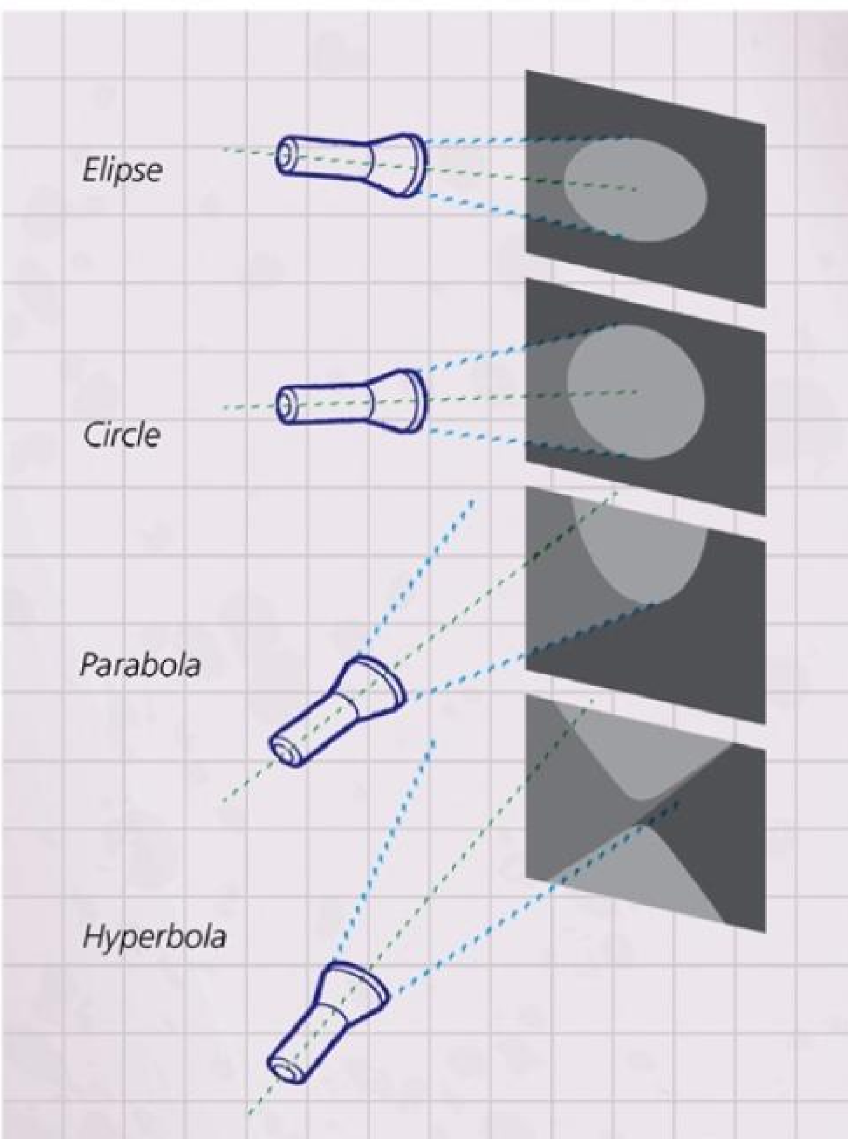
Conic Sections

The circle, ellipse, parabola and hyperbola are found everywhere in nature and have a simple geometric description.

Fixed numbers

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Coordinates



The light from a torch forms a cone, which is 'cut off' when it hits a wall. The resulting shape on the wall is a conic section.

What's It About?

Shine a torch directly at a wall: it should make a circle of light. Now slowly tilt the torch upwards a little bit and watch the circle stretch out into a new shape. If you keep going, at some point the shape will suddenly seem to open out indefinitely as it goes upwards. You can even carry on, and for a while at least you'll see the shape moving upwards and changing shape more subtly. Those shapes, though they look quite different, are all 'conic sections'. Each one is literally a cross-section of the cone of light coming from the end of the torch.

As well as cropping up in many natural settings that seem to have nothing to do with walls and torches, conic sections share a surprising geometric unity. This comes from the fact that your torch is really producing a constant, three-dimensional cone of light, as you can see if the room is smoky or very dusty. The two-dimensional shape you see changes simply because of the changing angle at which the wall 'chops off' the cone.

In More Detail

As you tilt the torch the shapes you see are, in sequence, a circle, a series of ellipses, a parabola and then a series of hyperbolas (*see* illustration).

Like many power stations, this one in Didcot, England, uses cooling towers whose curved outline is a parabola.

Jets of water often form parabolas, like these at the University of Adelaide, Australia.

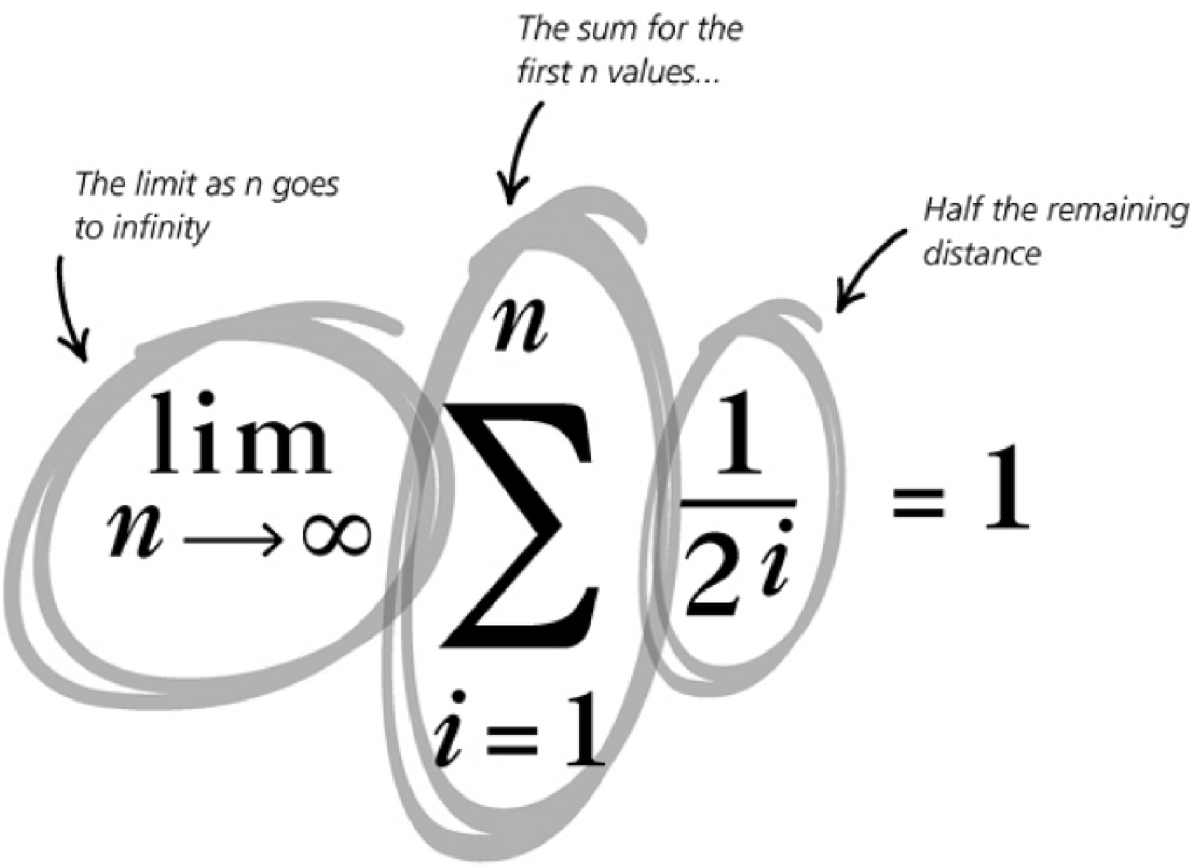
These are some of the most important curves in all of mathematics. When you throw a ball, its path is a parabola (*see* Newton's Second Law, page 56); the same shape is used to make mirrors, microphones and other objects that use reflection to concentrate a signal onto a point, and Archimedes is even said to have used parabolic mirrors to set fire to ships during the Siege of Syracuse in the third century BC. The planets move in ellipses around the sun [*see* Kepler's First Law, page 52] and the ellipse has its own reflective properties, exploited in the 'whispering galleries' of St Paul's Cathedral in London and in the treatment of gallstones by sound waves. Hyperbolas can be found in soap films and electrical fields and are frequently used in architecture and design. The image of the torch's beam on the wall changes more subtly from parabolic to hyperbolic when the torch is parallel to the wall – for example, when pointing directly upwards – so lamps close to walls usually create hyperbolic shapes.

To draw a curve using the equation given above, first fix values for A, B, C, D, E and F. The other letters (x and y) define points in a two-dimensional space, so that every point gives a unique pair of values for x and y [*see* Pythagoras's Theorem, page 10]. Now we try each point to see if the equation is true there – if so, it belongs to the curve, and if not, it doesn't. Most of the points we try won't work; when we calculate everything on the left-hand side of the equals sign, we'll get something other than zero, so that point isn't on the curve. We only choose the points where we get zero, and perhaps we imagine marking them with a dot. What we'll find is that our dots always join up to make one of the shapes the torch made on the wall: a circle, ellipse, parabola or hyperbola, depending on the values we choose for our fixed numbers. Actually there are two other possibilities, for if we choose the numbers really carefully we can either get two straight lines that cross each other or just one single point.

The curves known as conic sections fascinated the ancient Greeks and have found a surprising range of applications in the modern world, from lens making to architecture.

Zeno's Dichotomy

A 'proof' that motion is impossible comes close to inventing calculus two millennia early.



What's It About?

Suppose, says the ancient philosopher Zeno of Elea, that you're in the middle of a room and want to get out. The door is open and there's nothing blocking your path. Go ahead and walk to the door – except there's a tiny problem. To get there you must first walk halfway to the door. Then, you must walk halfway from where you are to the door. You still won't have got there, so you must repeat this again and again…how many times? Zeno thinks the answer is an *infinite* number of times. After all, with each motion you get closer to the door but the next step only covers half of the remaining distance, so you never quite close the gap. Well, he concludes, nobody can do an infinite number of things in a finite amount of time, so getting out of the room is impossible!

Zeno's argument isn't quite as silly as it sounds – as far as we can tell, it was one of a set of four arguments that work together to criticize some specific ancient ideas about space, time and motion – but for us the interest is more mathematical than philosophical. What Zeno has noticed is that a given distance seems to be equal to the sum of all those halves: we halve it, halve it again, halve it again and so on. In modern language he's discovered the idea of a limit, which in the 18th century became a basic tool in maths and physics.

Why Does It Matter?

Infinities bother people, and not just in philosophy seminars. The idea that you can add up an infinite number of things and get something perfectly ordinary and finite seems dodgy from the outset; after all, nobody actually could add them all up, since they'd never finish the process. In response to problems like this Aristotle made an important distinction between actual infinities and merely potential ones, which can go on as long as you like without any definite end-point. The easiest

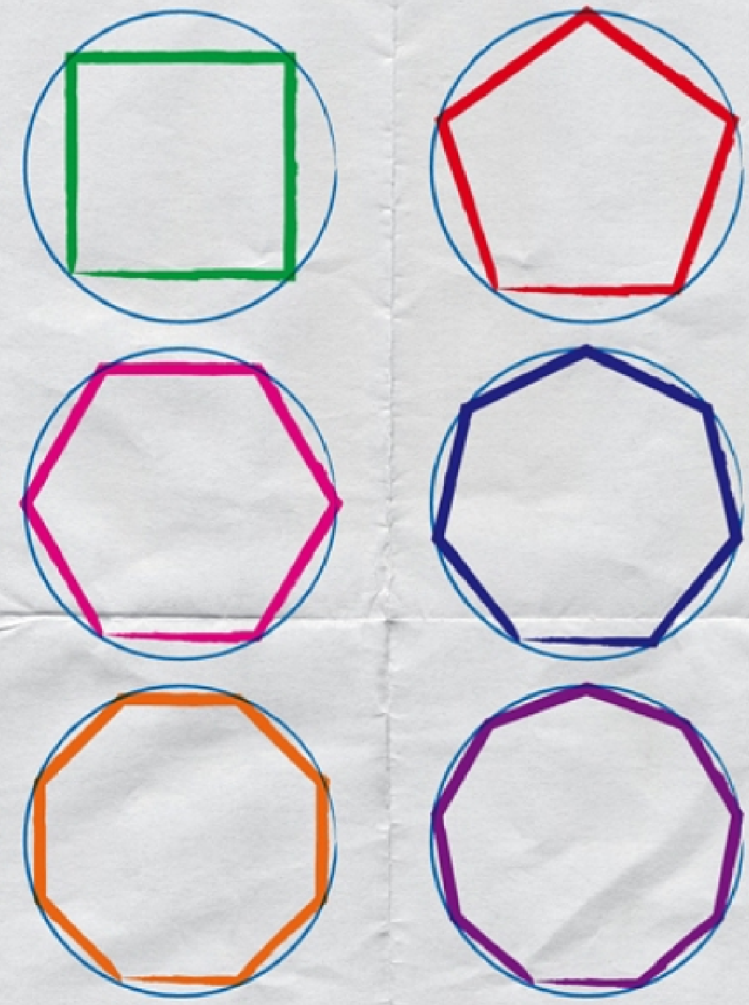
As n gets bigger and bigger, $^{1}\tilde{o}_n$ gets smaller and smaller. In fact, it gets awfully close to 0 when n is very big. What's more, if you give me any 'margin of error', however small, I can find a value of n so that $^{1}\tilde{o}_n$ is closer to 0 than your margin of error is and, from that point onwards as n increases $^{1}\tilde{o}_n$ always stays within that margin. In English we say 'the limit of $^{1}\tilde{o}_n$ as n goes to infinity is 0'. We don't mean that n ever becomes infinity, just that it's allowed to grow larger and larger. That, in a nutshell, is what a limit is.

Let's now look at the claim made by our equation:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2^i} = 1$$

In English: 'the limit, as n goes off towards infinity, of the sum of $^{1}\tilde{o}_{2^i}$ for every i from 1 to n, is equal to 1'. Quite a mouthful, admittedly: but it exactly expresses the geometric intuition that as you take each step covering half the remaining distance you get closer to the door (that is, to having covered a total distance of 1 unit) and that you can get as close as you like to the door if you're allowed to take a lot of steps (though you can never actually get to it).

In fact this sophisticated, 18th-century idea ƒwas already almost-formed when Archimedes used his 'method of exhaustion' to find the circumference of a circle. He noticed that if you fit regular polygons into a circle, letting the number of sides increase without bounds, they get closer and closer to being circles. In modern terms, Archimedes realized that 'the limit of the circumference of a regular polygon, as the number of sides increases, is the circumference of a circle', which gave him an approximate value of pi [see Euler's Identity, page 40].



Archimedes' 'method of exhaustion' produces a sequence of polygons that get closer and closer to the blue circle.

Approaching a limit by an infinite number of smaller and smaller steps sounds like philosophical wordplay but it lies at the heart of calculus, one of the most useful of all mathematical inventions.

Fibonacci Numbers

What number links pentagons, ancient mysticism and rabbit-breeding?

The next Fibonacci number — *The one before it* — *The one before that*

$$F_n = F_{n-1} + F_{n-2}$$

What's It About?

In 1202 Leonardo of Pisa, known as Fibonacci, published and solved the following problem. Imagine you're a farmer breeding a special kind of rabbit that reaches sexual maturity at one month old and has a very long lifespan. Each mature female can produce one male and one female each month. You take a new-born male and female and put them in a big field with plenty of food and no predators. Now suppose you let nature take its course and return after a certain number of months, *n*. How many mating pairs of rabbits will you have? The answer is F_n, the *n*th Fibonacci number, and our equation tells us how to calculate it.

The problem may seem a bit trivialized: after all, it does not describe a very realistic situation. Yet the Fibonacci numbers are an extraordinary discovery. They have a close relationship with an ancient number known as the Golden Ratio, which many have believed to be sacred or mystical and which itself has many surprising connections to other mathematical puzzles. There are a great many alleged sightings of these numbers in nature, especially in biology, where a simple rule like Fibonacci's might explain how organic growth that produces complicated-looking forms can be encoded by relatively little DNA.

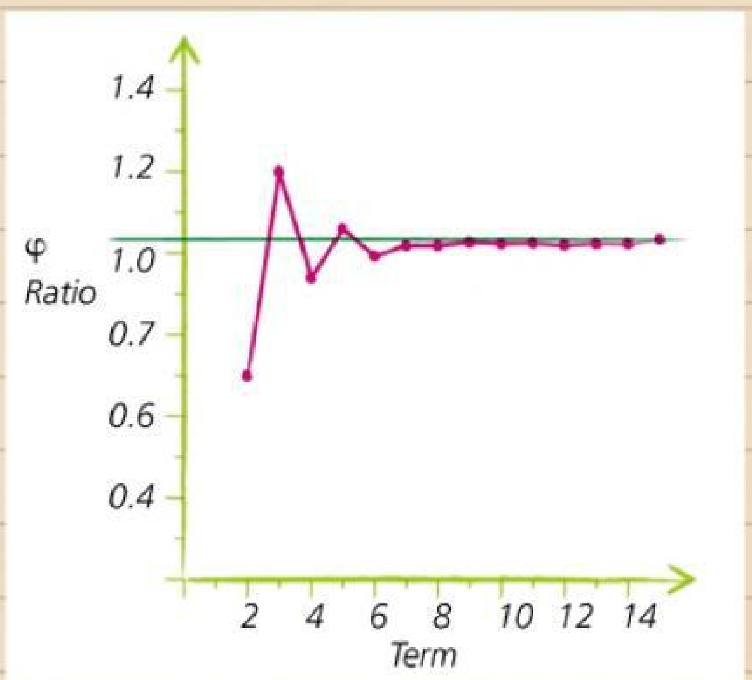
Why Does It Matter?

The truth is, the Fibonacci numbers have mostly fascinated mathematicians, not scientists or technologists. We'll discuss some mathematical reasons to be interested in them shortly. However, in inventing them, Fibonacci gave birth to the really important general idea of a 'recurrence relation'. This is, crudely put, any sequence of numbers whose next term depends only on one or more of the terms before it, according to a rule that never changes.

The way recurrence relations evolve the next value from the ones that have gone before makes them very useful for describing processes that develop over time. Extremely simple recurrence relations govern the amount of money in your savings account (assuming you put by the same amount every month), for example, and the amount you owe on your mortgage [*see* Logarithms, page 36]. Economists often use much more complex recurrence relations than these, as do biologists and engineers. What are known as Markov Chains – essentially, recurrence relations that only rely on the immediately preceding value, usually including an element of chance – appear in a perplexing array of applications, from the physics of heat diffusion [*see* The Heat Equation, page 80] to financial forecasting [*see* Brownian Motion, page 70].

Some recurrence relations continue to entertain the pure mathematicians, too, and the most notorious is the following. The first item in the sequence is any whole number you choose. The rule is: if the last number was even, halve it; otherwise treble it and add 1. So if we begin with 7 the sequence starts off like this:

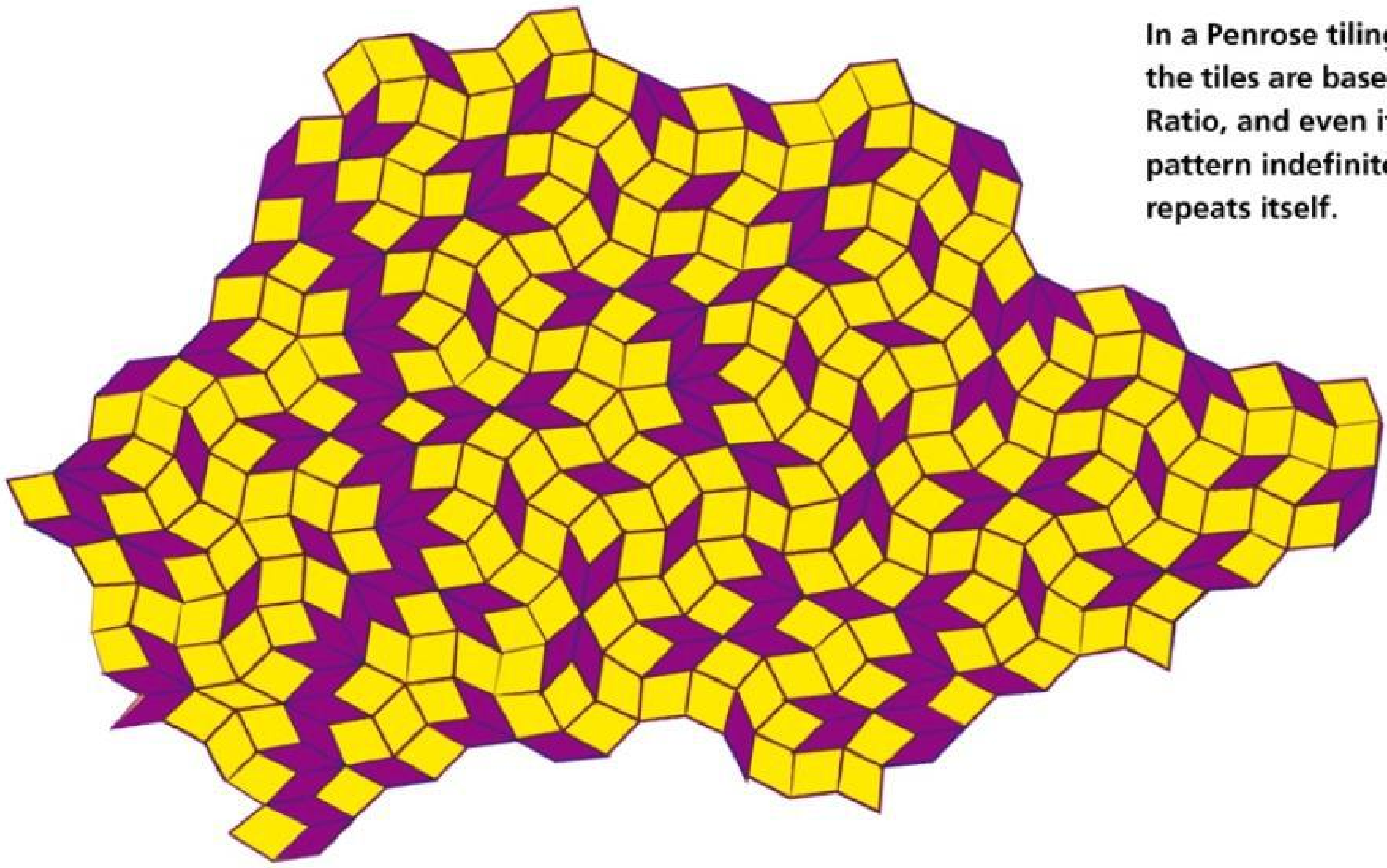
7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2,...



The ratio of adjacent Fibonacci numbers quickly settles down to being close to the Golden Ratio, approaching it as a limit.

Notice that, after a bit of jumping around, this sequence reaches the number 1, and after that it settles down into a simple cycle of three numbers. Will this always happen, no matter which number you start with? That is, do these sequences always, eventually, hit 1? The so-called Collatz Conjecture says they do; though easy to understand, nobody knows whether it's true or not. A solution, if it's found, will almost certainly involve the development of brand new ideas, perhaps with wide applications.


The family tree of Fibonacci's rabbits should follow his recurrence relation – at least in theory. Real life is somewhat more complicated.



In a Penrose tiling, the angles in the tiles are based on the Golden Ratio, and even if you continue the pattern indefinitely it never exactly repeats itself.

draw a regular pentagon (five-sided shape) using only a ruler and compass; this was an important practical technique for artists and craftsmen and led to the ancient Greeks discovering the ratio, long before Fibonacci and his imaginary rabbits.

The Golden Ratio has an extremely odd history and even today there are people who believe it has something approaching magical powers. They claim it governs many natural phenomena and has been used by architects and artists to create work with intrinsically pleasing proportions. I'm sorry to say that many of these claims turn out to be false. The Golden Ratio does, though, appear in the natural structures known as 'quasicrystals', which chemists are still actively researching.

The wilder claims often turn up in relation to the so-called Golden Rectangle. If you cut a square off this type of rectangle, the leftover part has exactly the same proportions as the one you started with. This means you can carry on cutting off smaller and smaller squares forever (or until you get bored). If you cut up the Golden Rectangle repeatedly, in just the right way, you can draw a quarter-circle in each square and produce a Golden Spiral, which is very pretty.This only works when the rectangle's longest side is φ times its shortest. Let's briefly see why.
If it's going to work, the main rectangle will have sides of length, say, 1 and r such that once you've cut a 1×1 square off it the remaining rectangle will have short side equal to $r-1$ and long side equal to 1. So r must satisfy the equation

$$\frac{1}{r} = \frac{r-1}{1}$$

A little rearranging turns this into the equation $r^2 - r - 1 = 0$, and a little high-school algebra yields φ as one of the two possible solutions (the other also works; you just end up with the rectangle flipped over on its side).

Recurrence relations create complex results by repetition, just as many natural processes do, and their long-term behaviours are often full of surprises.