The Unimaginable Mathematics of Borges' Library of Babel

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OXFORD
UNIVERSITY PRESS
2008

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Oxford University Press, Inc., publishes works that further
Oxford University's objective of excellence
in research, scholarship, and education.

Oxford New York
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Kuala Lumpur Madrid Melbourne Mexico City Nairobi
New Delhi Shanghai Taipei Toronto

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Published by Oxford University Press, Inc.
198 Madison Avenue, New York, New York 10016

www.oup.com

Library of Congress Cataloging-in-Publication Data

Bloch, William Goldbloom.
The unimaginable mathematics of Borges' Library
of Babel / William Goldbloom Bloch.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-19-533457-9
1. Borges, Jorge Luis, 1899–1986—Knowledge—Mathematics.
2. Mathematics and literature. 3. Mathematics—Philosophy. I. Title.
PQ7797.B635Z63438 2008
868—dc22 2008017271

9 8 7 6 5 4 3 2 1

Printed in the United States of America
on acid-free paper

Contents

We do not content ourselves with the life we have in ourselves and in our own being; we desire to live an imaginary life in the mind of others, and for this purpose we endeavor to shine. We labor unceasingly to adorn and preserve this imaginary existence and neglect the real.

—Blaise Pascal, *Pensées,* no. 147

Acknowledgments

Pigmæos gigãtum humeris impositos plusquam ipsos gigantes videre.
—Didacus Stella (Diego de Estella), *In sacrosanctum Jesu Christi Domini nostri Evangelium secundùm Lucam Enarrationum*

I say with Didacus Stella, a dwarf standing on the shoulders of a giant may see farther than a giant himself.
—Robert Burton, *Anatomy of Melancholy*, "Democritus to the Reader"

IT IS A PLEASURE TO ACKNOWLEDGE THE MANY DEBTS OF gratitude I owe; indeed, so much so that it's difficult to affix a starting point. Rather arbitrarily, I'll begin with Joe Roberts, the professor who introduced me to the concept of elegance in mathematics via the study of combinatorics. Around the same time, I read Rudy Rucker's *Geometry, Relativity and the Fourth Dimension*, which contains a lovely exposition of Riemann's century-old idea that a universe could be both finite and limitless. Another well-deserved "thank you" to the unremembered friend who, many years ago, put a copy of *Labyrinths* into my hand.

Leaping to the present day, I thank Tricia Arnold for endowing the fellowship that enabled me to travel to Buenos Aires. In a similar vein, I thank Susanne Woods and Wheaton College for supporting this project with time, resources, and encouragement. Not surprisingly, two librarians, Martha Mitchell and TJ Sondermann, were extremely helpful in identifying and obtaining old books and journal articles linking mathematics and Borges. Another marvelous staff member at Wheaton, Kathy Rogers, consistently provided vital textual support.

I am grateful to everyone in Buenos Aires who assisted me, most especially Fernando Palacio, cultural mediator and translator *por excelencia*. The Director of the National Library of Argentina, Silvio Maresca, and the Associate Director, Roberto Magliano, were kind enough to meet with me and do whatever was within their power to facilitate this project. Clara Bayá, webmaster and semiofficial translator for the National Library, provided invaluable aid in guiding me first around the building and then around various rules that turned out to be surprisingly pliant. An anonymous guard at the old National Library and an anonymous librarian at the Miguel Cané Municipal Library were both also willing to bend rules and show me parts of their respective buildings that are generally off-limits to the public. The librarian, who was delighted that someone from the United States cared enough about Borges to visit the Miguel Cané Municipal Library, informed me that Argentine civil servants can't bear to read "The Library of Babel." Apparently, they take the Kafkaesque qualities of the tale quite personally, viewing the story as an extended slap against their daily work-life and their organizational systems.

My colleagues from the Humanities, Michael Drout and Hector Medina, acted as a pushmi-pullyu—see Lofting, 77–85—in jump-starting the project, in devoting the time to read and comment on my manuscript, and to talk over many of its points with me. Drout also provided etymologies for me when necessary, encouraged me to create the word "slimber" out of "slim" and "limber," and reassured me whenever I feared that I was using too many infinitives.

Eric Denton coordinated the first group reading and offered salient suggestions and collegial encouragement duly leavened with cynicism. ("Bill, your book is neither fish nor fowl.")

Anni Baker, Bernard Bloch, Tom Brooks, Michael Chesla, Bev Clark, Betsey Dyer, Lisa Lebduska, Shelly Leibowitz, Shannon Miller, Laura Muller, Rolf Nelson, John Partridge, Joel Relihan, Dorothea Rockburne, Pamela Stafford, David Wulff, and Paul Zeitz read this book in manuscript form and provided worthy and meaningful feedback. Any errors or infelicities remaining are, of course, solely my own.

Domingo Ledezma helped me out by translating some thorny passages in the story and Doug Jungreis confirmed my intuitions about Hopf fibrations. Julio Ortega encouraged me and introduced me to Borges' widow, the remarkable Maria Kodama.

John Wronoski of Lame Duck Books in Cambridge, Massachusetts first let me hold Borges' autograph manuscript of "La biblioteca de Babel" in my shaking hands, and then kindly let me use images of it in this volume. (By the way, the manuscript is for sale for approximately $650,000. Prior to learning this, I never actually ached to be a multimillionaire, but now I hereby publicly promise that if this book sells over three million copies, I will cheerfully call Mr. Wronoski to negotiate a price.)

Throughout the process, my editor, Michael Penn, combined abiding wisdom, keen grammatical insight, calming patience, and sly humor. Working with him was a continuous pleasure. Stefano Imbert, the illustrator, did a marvelous job capturing the ambience of the Library. Other people associated with Oxford University Press who helped shape the final result are Ned Sears, Stephen Dodson, and Keith Faivre.

On a number of occasions, my mother-in-law and my parents gave generously of their time and energy by watching my young children, allowing me to devote myself to this work. Speaking of my children, Dylan always loved the "pokey things" in the illustrations and Levi was always willing to cheer me up with a cartwheel performance. Finally, my wife Ingrid tolerated my obsessions, disjunctions, and corporeal absences as I wrangled with various parts of this book. Her multiform support was, and continues to be, vital and cherished.

Thank you, one and all.

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Preface

One feels right away that this is the kingdom of books. People working at the library commune with books, with the life reflected in them, and so become almost reflections of real-life human beings.

—Isaac Babel, "The Public Library"

"WHO IS THE INTENDED AUDIENCE FOR THIS WORK IN progress?" This question, asked almost apologetically by a friend, stumped me for only a fraction of a second. With the clarity and explosiveness usually reserved for a rare mathematical insight, the answer burst from me: *Umberto Eco!* Polymath, brilliant semiotician, editor of the journal *Variaciones Borges*, interpreter of "The Library of Babel," and a favorite author for many years—Eco struck me as the ideal reader of this writing. (And Umberto, I hope you do read and enjoy this, someday.)

Of the more than six billion people who are *not* Umberto Eco, I imagine that those who'd find this work appealing would share, to varying degrees, the following traits: a familiarity with and affinity for Borges' works, especially "The Library of Babel"; a nodding, perhaps cautious, acquaintance with the thought that mathematics might not be the root of all evil; and the habit of rereading sentences, paragraphs, and stories for sheer delight, as well for playing with the superpositions of layers of available meanings.

While it's possible to set up a straw man and use it to wonder which way of presenting information is "better," A Multi-Claused Sentence vs. A Picture of Overlapping Sets

I take the view that the approaches are complementary; they aren't two opponents locked into a zero-sum game for which one side must prevail. So, since part of my not-so-hidden agenda is to persuade those of a literary temperament that mathematics can be more than the "problem/solution" model of much rudimentary education, I present a Venn diagram that visually encapsulates the speculations of the previous paragraph (figure 1).

The intended audience is the intersection of the three different sets of character traits. Judging mainly from the steady sales of Borges' fiction, I have managed to convince myself that besides you (presumably), there are at least several hundred thousand people who fit this description.

If, however, an unimaginative education or a particularly unpleasant teacher left a lingering distaste for all things mathematical, I hope this book acts as a corrective. Mathematics can be creative, whimsical, and revelatory all at once. More to the point, as embodied in the different meanings of the word "analysis," it is simultaneously a process and an intellectual structure. Borges, a great imbiber of mathematics, seems to have understood this idea and instantiated it in many of his stories—most especially "The Library of Babel." His imagination works in, through, out, about, and all around logical strictures.

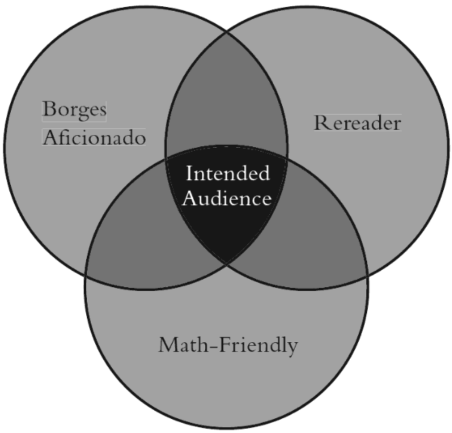

FIGURE 1. Who is the intended audience?

Conversely, for those of a mathematical bent who've not read Borges, I hope this volume inspires two things: a desire to explore more of Borges' work—there are many riches to be found—and, equally, a desire to learn more about the math tools I employ. We, as a society, are gifted these days; many books introducing math to the casual reader are readily available.

The chapters that are mathematical in nature will generally begin with the introduction of a mathematical idea. Some exposition, and perhaps a few examples, are given to help concretize the concept. Finally, the ideas will be applied to some aspects of "The Library of Babel" towards the desired end of producing an unimaginable (or unimagined) result.

Andrew Wiles, who proved Fermat's last theorem, memorably analogized the process of doing mathematics as follows:

> You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. (Singh, pp. 236–37)

Reading the math chapters of this work might be likened to stumbling around in a dark room, bumping into furniture, and finally, after finding the light switch, learning that you're not in a mansion after all, but rather facing away from the screen in a movie theater, and that the switch is really a fire alarm.

After the suite of introductory material comes the touchstone for this work: Andrew Hurley's superb translation of "The Library of Babel." After the story, and unlike most math books, the chapters are logically independent and can be dipped and skimmed as fancy dictates. (Of course, some intratextual references are unavoidable.) Although I've endeavored to structure the book so that it may be enjoyed from start to finish, based on predilections, nonlinear routes may be better suited for different kinds of readers.

In fact, it's safe to say that there are three main themes woven into this book. The first one digs into the Library, peels back layers uncovering nifty ideas, and then runs with them for a while. The second thread is found mostly in the "Math Aftermath" sections appended to the chapters: in them, I develop the mathematics behind the ideas to a greater degree and, in some cases, give step-by-step derivations for formulas used in the main body of the chapter. (Allow me to emphasize that the Math Aftermaths are—I hope—clear and engaging, but they certainly aren't required in order to understand and enjoy any other parts of the book.) The third focus is on literary aspects of the story and Borges; the chapters playing with these motifs come after those concerned with the math.

In the first chapter, "Combinatorics: Contemplating Variations of the 23 Letters," I use millennia-old ideas, alluded to in the story itself, to calculate the number of books in the Library. Once the basic concept of exponential notation is absorbed, the number is unexpectedly easy to find; it is understanding the magnitude of that number that occupies the bulk of the chapter. A number of previous critics also calculate this number, and several have provided similar means of understanding its size. By contrast, I fully explain the underlying mathematics and, moreover, add a new twist to the calculation. Expanding on some of the ideas raised, the Math Aftermath shows how to use a property of the logarithm function to recast the number of distinct books of the Library in terms more familiar, more amenable to our understanding. The chapter ends with the derivation of an ancient counting formula.

After that, in "Information Theory: Cataloging the Collection," I consider the meaning of a catalogue for the Library and the forms that it might take. The Math Aftermath takes some basic results in number theory and applies them to aspects of the Library and the unknowability of certain pieces of compressed information. Then, in "Real Analysis: The Book of Sand," I apply elegant ideas from the seventeenth century and counterintuitive ideas of the twentieth century to the "Book of Sand" described in the final footnote of the story. Three variations of the Book, springing from three different interpretations of the phrase "infinitely thin," are outlined.

Next, in "Topology and Cosmology: The Universe (Which Others Call the Library)," I employ late nineteenth- and early twentieth-century mathematics to explore possible shapes of the Library. Ultimately, I propose a rapprochement between the apparently conflicting views outlined by the narrator of the story. In the Math Aftermath section of the chapter, the discussion moves into somewhat more sophisticated domains by introducing two possible variations of the Library, each of which possesses noteworthy traits, one example being *nonorientability*.

Following this, in "Geometry and Graph Theory: Ambiguity and Access," I use Borges' descriptions of the Library to abstract the architecture of each floor of the Library and use it to unfold a surprising consequence. Interested readers can continue the tale of the chapter by following along in the Math Aftermath as I unpack an even stronger mathematical result stemming from the story.

The next chapter, "More Combinatorics: Disorderings into Order," is a kind of a fantasia on the possibilities inherent in ordering and disordering the distribution of books in the Library, and it concludes the mathematical section of the book.

After this, despite a desire to resist interpretation of the story, by drawing on metaphors from Alan Turing and information theory, I propose a new reading in "A Homomorphism: Structure into Meaning." Following that, in "Critical Points," prior work on "The Library of Babel" serves as a springboard to some compelling ruminations about life in the Library and other topics. Finally, in "Openings," a "What did he know and when did he know it? How did he know it?" attitude is adopted vis-à-vis Borges and mathematics. Was he a mathematician? A philosopher? A visionary writer blithely unaware of the depth of his insights?

The literary chapters are followed by a cortege of back matter, beginning with an appendix, "Dissecting the 3-Sphere," for those who want a refresher on how equations capture the characteristics and properties of multidimensional spheres. The appendix may sound scarier than it really is; I don't use much beyond the Pythagorean theorem, and I even provide a review of that.

In general, I avoid mathematical notation beyond that encountered in middle school or perhaps the early years of high school. However, in case it is unfamiliar, following the appendix is a short list of notations with definitions. Speaking of definitions, there's a lot to say on the matter. Mathematics is an intellectual discipline built on definitions; indeed, the *axioms* of mathematics are exactly definitions that have been accepted as plausible and true by the concerted critical faculty of millions of thinkers around the world aggregated over the past several millennia. Moreover, these days great theoretical breakthroughs occur when brilliant mathematicians see new interrelations and make definitions that enable a cascade of untold consequences to be discovered by other workers in the field. For us, definitions will be considerably more prosaic; I italicize words that strike me as being of a technical nature, outside the usual range of quotidian use, and provide definitions in a glossary following the notations and the endnotes.

As a reader, when I encounter an endnote, I'm compelled almost against my will to flip to the back of the book to learn what the endnote says.[1] As I writer, I find that despite my best efforts to incorporate them into the body of the book, my work includes diverting digressions, fine points of mathematics that might interest only specialists, and citations to other works. All of these are consigned to the endnotes.

After the glossary, an annotated list of suggested readings is provided for those with curiosity primed to learn more of the mathematics used in the book. A bibliography of references cited or consulted rounds out the end matter.

Introduction

We adore chaos because we love to produce order.

—M. C. Escher

It's an ironic joke that Borges would have appreciated: I am a mathematician who, lacking Spanish, perforce reads "The Library of Babel" in translation. Furthermore, although I bring several thousand years of theory to bear on the story, none of it is literary theory.

Having issued these caveats, it is my purpose to make explicit a number of mathematical ideas inherent in the story. My goal in this task is not to reduce the story in any capacity; rather it is to enrich and edify the reader by glossing the intellectual margins and substructures. Borges was a consummate synthesist; his lapidary prose sparkles and reveals unexpected depths when examined from any angle or perspective. I submit that because of his well-known affection for mathematics, exploring the story through the eyes of a mathematician is a dynamic, useful, and necessary addition to the body of Borgesian criticism.

In what follows, I assume no special mathematical knowledge. I only ask that the reader trust that I am a tour guide through a labyrinth, like that marble pathway on the floor of the cathedral at Chartres, not the gatekeeper of a Stygian maze without center or exit (figure 2). Beyond enhancing the story, the reader's reward will be an exposure to some intriguing and entrancing mathematical ideas.

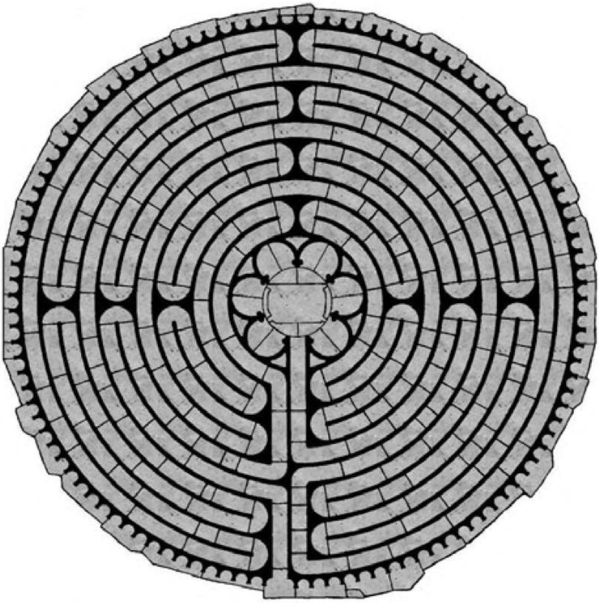

FIGURE 2. The labyrinth on the floor of the Chartres Cathedral. Movement in a labyrinth is constrained to only forwards or backwards motion. (Jeff Saward/Labyrinthos)

Borges was a master of understating ideas, allowing them the possibility of gathering heft and power, of generating their own gravity. I'm under no delusion that he traced out all the consequences of the dormant mathematics I uncover. I allow myself the ambition, though, to paraphrase what Borges wrote in a forward and hope that this book would have taught him many things about himself (see Barrenchea, p. vii).

I request a last indulgence from the reader. The introductory material, thus far, has been written in the friendly and confiding first person singular voice. Starting in the next paragraph, I will inhabit the first person plural for the duration of the mathematical expositions. This should not be construed as a "royal we." It has been a construct of the community of mathematicians for centuries and it traditionally signifies two ideas: that "we" are all in consultation with each other through space and time, making use of each other's insights and ideas to advance the ongoing human project of mathematics, and that "we"—the author and reader—are together following the sequences of logical ideas that lead to inexorable, and sometimes poetic, conclusions.

A word, too, about the language in the book. We started our college years intending to be some sort of creative writer. Beyond the insight mathematics offered into the natural world and epiphenomena of life, and beyond the aesthetic joy at understanding how the iron rules of logic crystallize a good proof into a work of art, one of the reasons we turned to math was the lilt and rhythm of the "if-then" syntax coupled with the musicality of words often repeated, such as "thus," "hence," "suppose," and "let." We hope our readers might develop an ear for this music, too.

We close the introduction by offering several related disclaimers. Mathematics, like any discipline, is not a monolith; it's a sprawling agglutination of overlapping and intersecting fields and specialties: one's talents, tastes, and beliefs determine individual focus. We carefully checked and rechecked our ideas, mathematics, and figures. To the best of our knowledge, there are no mistakes. However, a different mathematician might well expose divergent mathematical themes from the story and utilize different sets of ideas to explain them.

Furthermore, there's a natural tendency for an individual reaching across traditional boundaries to be perceived as a universal embodiment of the foreign, the other. Although our inductions and deductions are correct, some mathematicians might issue philosophic challenges to underlying assumptions, especially in the chapters "Real Analysis" and "More Combinatorics." Consequently, no one, including the author, should be seen as a Representative or Ambassador, speaking in one voice for an ideologically unified Entity of Mathematicians: such an Entity of Mathematicians simply doesn't exist. (Lest this be subject to misinterpretation, allow us to note that *all* mathematicians would agree on the centrality of logically consistent deductions and derivations from agreed-upon axioms.)

It's important to bear in mind that the mathematical expositions contained herein are not rigorously developed, nor are they intended as comprehensive introductions to the various theories. Just as a stirring musical performance will not transform a concertgoer into a musician, composer, lyricist, musicologist, or music critic, so this book won't transform a reader into any kind of a mathematician. However, just as a concert may move, inspire, or transfigure a listener, so we hope that this book will stimulate, dazzle, and expand its readers.

Finally, about the title of the book: why the word "unimaginable"? By way of an answer, we note that in his sixth Meditation, Descartes makes clear the distinction between simply naming a thing and visualizing it in a clear, precise way that allows for mental manipulations.

> I note first the difference between imagination and pure intellection or conception. For example, when I imagine a triangle, I not only conceive it as a figure composed of three lines, but moreover consider these three lines as being present by the power and internal application of my mind, and that is properly what I call imagining. Now if I wish to think of a chiliagon, I indeed rightly conceive that it is a figure composed of a thousand sides, as easily as I conceive that a triangle is a figure composed of only three sides; but I cannot imagine the thousand sides of a chiliagon, as I do the three of a triangle, neither, so to speak, can I look upon them as present with the eyes of my mind.

Some of the ideas we'll talk about, such as titanic numbers and higher dimensions, are unimaginable in this sense. We can give names to the ideas, use metaphors to approach them, give simple examples to substitute in as models, and try to find a consistent set of rules and mathematical objects that encapsulate the essence of the ideas—but we will never be able to visualize them any more than we could Descartes' thousand-sided chiliagon. Indeed, our task as your guide is to trigger the processes by which you build intuition and insight into the Unimaginable.

The Unimaginable Mathematics of Borges' Library of Babel

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The Library of Babel

Jorge Luis Borges

By this art you may contemplate the variation of the 23 letters. . . .
—*Anatomy of Melancholy*, Pt. 2, Sec. II, Mem. IV

THE UNIVERSE (WHICH OTHERS CALL THE Library) is composed of an indefinite, perhaps infinite number of hexagonal galleries. In the center of each gallery is a ventilation shaft, bounded by a low railing. From any hexagon one can see the floors above and below—one after another, endlessly. The arrangement of the galleries is always the same: Twenty bookshelves, five to each side, line four of the hexagon's six sides; the height of the bookshelves, floor to ceiling, is hardly greater than the height of a normal librarian. One of the hexagon's free sides opens onto a narrow sort of vestibule, which in turn opens onto another gallery, identical to the first—identical in fact to all. To the left and right of the vestibule are two tiny compartments. One is for sleeping, upright; the other, for satisfying one's physical necessities. Through this space, too, there passes a spiral staircase, which winds upward and downward into the remotest distance. In the vestibule there is a mirror, which faithfully duplicates appearances. Men often infer from this mirror that the Library is not infinite—if it were, what need would there be for that illusory replication? I prefer to dream that burnished surfaces are a figuration and promise of the infinite. . . . Light is provided by certain spherical fruits that bear the name "bulbs." There are two of these bulbs in each hexagon, set crosswise. The light they give is insufficient, and unceasing.

Like all the men of the Library, in my younger days I traveled; I have journeyed in quest of a book, perhaps the catalog of catalogs. Now that my eyes can hardly make out what I myself have written, I am preparing to die, a few leagues from the hexagon where I was born. When I am dead, compassionate hands will throw me over the railing; my tomb will be the unfathomable air, my body will sink for ages, and will decay and dissolve in the wind engendered by my fall, which shall be infinite. I declare that the Library is endless. Idealists argue that the hexagonal rooms are the necessary shape of absolute space, or at least of our *perception* of space. They argue that a triangular or pentagonal chamber is inconceivable. (Mystics claim that their ecstasies reveal to them a circular chamber containing an enormous circular book with a continuous spine that goes completely around the walls. But their testimony is suspect, their words obscure. That cyclical book is God.) Let it suffice for the moment that I repeat the classic dictum: *The Library is a sphere whose exact center is any hexagon and whose circumference is unattainable.*

Each wall of each hexagon is furnished with five bookshelves; each bookshelf holds thirty-two books identical in format; each book contains four hundred ten pages; each page, forty lines; each line, approximately eighty black letters. There are also letters on the front cover of each book; those letters neither indicate nor prefigure what the pages inside will say. I am aware that that lack of correspondence once struck men as mysterious. Before summarizing the solution of the mystery (whose discovery, in spite of its tragic consequences, is perhaps the most important event in all history), I wish to recall a few axioms.

First: *The Library has existed* ab æternitate. That truth, whose immediate corollary is the future eternity of the world, no rational mind can doubt. Man, the imperfect librarian, may be the work of chance or of malevolent demiurges; the universe, with its elegant appointments—its bookshelves, its enigmatic books, its indefatigable staircases for the traveler, and its water closets for the seated librarian—can only be the handiwork of a god. In order to grasp the distance that separates the human and the divine, one has only to compare these crude trembling symbols which my fallible hand scrawls on the cover of a book with the organic letters inside—neat, delicate, deep black, and inimitably symmetrical.

Second: *There are twenty-five orthographic symbols.*[1] That discovery enabled mankind, three hundred years ago, to formulate a general theory of the Library and thereby satisfactorily solve the riddle that no conjecture had been able to divine—the formless and chaotic nature of virtually all books. One book, which my father once saw in a hexagon in circuit 15–94, consisted of the letters M C V perversely repeated from the first line to the last. Another (much consulted in this zone) is a mere labyrinth of letters whose penultimate page contains the phrase *O Time thy pyramids.* This much is known: For every rational line or forthright statement there are leagues of senseless cacophony, verbal nonsense, and incoherency. (I know of one semibarbarous zone whose librarians repudiate the "vain and superstitious habit" of trying to find sense in books, equating such a quest with attempting to find meaning in dreams or in the chaotic lines of the palm of one's hand. . . . They will acknowledge that the inventors of writing imitated the twenty-five natural symbols, but contend that that adoption was fortuitous, coincidental, and that books in themselves have no meaning. That argument, as we shall see, is not entirely fallacious.)

For many years it was believed that those impenetrable books were in ancient or far-distant languages. It is true that the most ancient peoples, the first librarians, employed a language quite different from the one we speak today; it is true that a few miles to the right, our language devolves into dialect and that ninety floors above, it becomes incomprehensible. All of that, I repeat, is true—but four hundred ten pages of unvarying M C V's cannot belong to any language, however dialectal or primitive it may be. Some have suggested that each letter influences the next, and that the value of M C V on page 71, line 3, is not the value of the same series on another line of another page, but that vague thesis has not met with any great acceptance. Others have mentioned the possibility of codes;

[1] The original manuscript has neither numbers nor capital letters; punctuation is limited to the comma and the period. Those two marks, the space, and the twenty-two letters of the alphabet are the twenty-five sufficient symbols that our unknown author is referring to. [Ed. note.]

that conjecture has been universally accepted, though not in the sense in which its originators formulated it.

Some five hundred years ago, the chief of one of the upper hexagons[2] came across a book as jumbled as all the others, but containing almost two pages of homogeneous lines. He showed his find to a traveling decipherer, who told him that the lines were written in Portuguese; others said it was Yiddish. Within the century experts had determined what the language actually was: a Samoyed-Lithuanian dialect of Guaraní, with inflections from classical Arabic. The content was also determined: the rudiments of combinatory analysis, illustrated with examples of endlessly repeating variations. Those examples allowed a librarian of genius to discover the fundamental law of the Library. This philosopher observed that all books, however different from one another they might be, consist of identical elements: the space, the period, the comma, and the twenty-two letters of the alphabet. He also posited a fact which all travelers have since confirmed: *In all the Library, there are no two identical books*. From those incontrovertible premises, the librarian deduced that the Library is "total"—perfect, complete, and whole—and that its bookshelves contain all possible combinations of the twenty-two orthographic symbols (a number which, though unimaginably vast, is not infinite)—that is, all that is able to be expressed, in every language. *All*—the detailed history of the future, the autobiographies of the archangels, the faithful catalog of the Library, thousands and thousands of false catalogs, the proof of the falsity of those false catalogs, a proof of the falsity of the *true* catalog, the gnostic gospel of Basilides, the commentary upon that gospel, the commentary on the commentary on that gospel, the true story of your death, the translation of every book into every language, the interpolations of every book into all books, the treatise Bede could have written (but did not) on the mythology of the Saxon people, the lost books of Tacitus.

When it was announced that the Library contained all books, the first reaction was unbounded joy. All men felt themselves the possessors of an intact and secret treasure. There was no personal problem, no

[2] In earlier times, there was one man for every three hexagons. Suicide and diseases of the lung have played havoc with that proportion. An unspeakably melancholy memory: I have sometimes traveled for nights on end, down corridors and polished staircases, without coming across a single librarian.

world problem, whose eloquent solution did not exist—somewhere in some hexagon. The universe was justified; the universe suddenly became congruent with the unlimited width and breadth of humankind's hope. At that period there was much talk of The Vindications—books of *apologiæ* and prophecies that would vindicate for all time the actions of every person in the universe and that held wondrous arcana for men's futures. Thousands of greedy individuals abandoned their sweet native hexagons and rushed downstairs, upstairs, spurred by the vain desire to find their Vindication. These pilgrims squabbled in the narrow corridors, muttered dark imprecations, strangled one another on the divine staircases, threw deceiving volumes down ventilation shafts, were themselves hurled to their deaths by men of distant regions. Others went insane.... The Vindications do exist (I have seen two of them, which refer to persons in the future, persons perhaps not imaginary), but those who went in quest of them failed to recall that the chance of a man's finding his own Vindication, or some perfidious version of his own, can be calculated to be zero.

At that same period there was also hope that the fundamental mysteries of mankind—the origin of the Library and of time—might be revealed. In all likelihood those profound mysteries can indeed be explained in words; if the language of the philosophers is not sufficient, then the multiform Library must surely have produced the extraordinary language that is required, together with the words and grammar of that language. For four centuries, men have been scouring the hexagons.... There are official searchers, the "inquisitors." I have seen them about their tasks: they arrive exhausted at some hexagon, they talk about a staircase that nearly killed them—rungs were missing—they speak with the librarian about galleries and staircases, and, once in a while, they take up the nearest book and leaf through it, searching for disgraceful or dishonorable words. Clearly, no one expects to discover anything.

That unbridled hopefulness was succeeded, naturally enough, by a similarly disproportionate depression. The certainty that some bookshelf in some hexagon contained precious books, yet that those precious books were forever out of reach, was almost unbearable. One blasphemous sect proposed that the searches be discontinued and that all men shuffle letters and symbols until those canonical books, through some improbable stroke of chance, had been constructed. The authorities were forced to issue strict orders. The sect disappeared, but in my childhood I have seen old men who for long periods would hide in the latrines with metal disks and a forbidden dice cup, feebly mimicking the divine disorder.

Others, going about it in the opposite way, thought the first thing to do was eliminate all worthless books. They would invade the hexagons, show credentials that were not always false, leaf disgustedly through a volume, and condemn entire walls of books. It is to their hygienic, ascetic rage that we lay the senseless loss of millions of volumes. Their name is execrated today, but those who grieve over the "treasures" destroyed in that frenzy overlook two widely acknowledged facts: One, that the Library is so huge that any reduction by human hands must be infinitesimal. And two, that each book is unique and irreplaceable, but (since the Library is total) there are always several hundred thousand imperfect facsimiles—books that differ by no more than a single letter, or a comma. Despite general opinion, I daresay that the consequences of the depredations committed by the Purifiers have been exaggerated by the horror those same fanatics inspired. They were spurred on by the holy zeal to reach—someday, through unrelenting effort—the books of the Crimson Hexagon—books smaller than natural books, books omnipotent, illustrated, and magical.

We also have knowledge of another superstition from that period: belief in what was termed the Book-Man. On some shelf in some hexagon, it was argued, there must exist a book that is the cipher and perfect compendium *of all other books*, and some librarian must have examined that book; this librarian is analogous to a god. In the language of this zone there are still vestiges of the sect that worshiped that distant librarian. Many have gone in search of Him. For a hundred years, men beat every possible path—and every path in vain. How was one to locate the idolized secret hexagon that sheltered Him? Someone proposed searching by regression: To locate book A, first consult book B, which tells where book A can be found; to locate book B, first consult book C, and so on, to infinity. . . . It is in ventures such as these that I have squandered and spent my years. I cannot think it unlikely that there is such a total book[3] on some shelf in the universe. I pray to the unknown gods

[3] I repeat: In order for a book to exist, it is sufficient that it be *possible*. Only the impossible is excluded. For example, no book is also a staircase, though there are no doubt books that discuss and deny and prove that possibility, and others whose structure corresponds to that of a staircase.

that some man—even a single man, tens of centuries ago—has perused and read that book. If the honor and wisdom and joy of such a reading are not to be my own, then let them be for others. Let heaven exist, though my own place be in hell. Let me be tortured and battered and annihilated, but let there be one instant, one creature, wherein thy enormous Library may find its justification.

Infidels claim that the rule in the Library is not "sense," but "non-sense," and that "rationality" (even humble, pure coherence) is an almost miraculous exception. They speak, I know, of "the feverish Library, whose random volumes constantly threaten to transmogrify into others, so that they affirm all things, deny all things, and confound and confuse all things, like some mad and hallucinating deity." Those words, which not only proclaim disorder but exemplify it as well, prove, as all can see, the infidels' deplorable taste and desperate ignorance. For while the Library contains all verbal structures, all the variations allowed by the twenty-five orthographic symbols, it includes not a single absolute piece of nonsense. It would be pointless to observe that the finest volume of all the many hexagons that I myself administer is titled *Combed Thunder*, while another is titled *The Plaster Cramp*, and another, *Axaxaxas mlö*. Those phrases, at first apparently incoherent, are undoubtedly susceptible to cryptographic or allegorical "reading"; that reading, that justification of the words' order and existence, is itself verbal and, *ex hypothesi*, already contained somewhere in the Library. There is no combination of characters one can make—*dhcmrlchtdj*, for example—that the divine Library has not foreseen and that in one or more of its secret tongues does not hide a terrible significance. There is no syllable one can speak that is not filled with tenderness and terror, that is not, in one of those languages, the mighty name of a god. To speak is to commit tautologies. This pointless, verbose epistle already exists in one of the thirty volumes of the five bookshelves in one of the countless hexagons—as does its refutation. (A number *n* of the possible languages employ the same vocabulary; in some of them, the *symbol* "library" possesses the correct definition "everlasting, ubiquitous system of hexagonal galleries," while a library—the thing—is a loaf of bread or a pyramid or something else, and the six words that define it themselves have other definitions. You who read me—are you certain you understand my language?)

Methodical composition distracts me from the present condition of humanity. The certainty that everything has already been written annuls us, or renders us phantasmal. I know districts in which the young people prostrate themselves before books and like savages kiss their pages, though they cannot read a letter. Epidemics, heretical discords, pilgrimages that inevitably degenerate into brigandage have decimated the population. I believe I mentioned the suicides, which are more and more frequent every year. I am perhaps misled by old age and fear, but I suspect that the human species—the *only* species—teeters at the verge of extinction, yet that the Library—enlightened, solitary, infinite, perfectly unmoving, armed with precious volumes, pointless, incorruptible, and secret—will endure.

I have just written the word "infinite." I have not included that adjective out of mere rhetorical habit; I hereby state that it is not illogical to think that the world is infinite. Those who believe it to have limits hypothesize that in some remote place or places the corridors and staircases and hexagons may, inconceivably, end—which is absurd. And yet those who picture the world as unlimited forget that the number of possible books is *not*. I will be bold enough to suggest this solution to the ancient problem: *The Library is unlimited but periodic.* If an eternal traveler should journey in any direction, he would find after untold centuries that the same volumes are repeated in the same disorder—which, repeated, becomes order: the Order. My solitude is cheered by that elegant hope.[4]

Mar del Plata, 1941

[4] Letizia Alvarez de Toledo has observed that the vast Library is pointless; strictly speaking, all that is required is *a single volume*, of the common size, printed in nine- or ten-point type, that would consist of an infinite number of infinitely thin pages. (In the early seventeenth century, Cavalieri stated that every solid body is the super-position of an infinite number of planes.) Using that silken *vademecum* would not be easy: each apparent page would open into other similar pages; the inconceivable middle page would have no "back."

ONE

Combinatorics

Contemplating Variations of the 23 Letters

There are some, King Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its multitude.

—Archimedes, *The Sand Reckoner*

WE BEGIN WITH A PAEAN TO THE MODERN method of denoting numbers, especially the convention of exponential notation, employed first by Descartes in 1637, then extended over the next few decades, primarily by Napier and Newton. (These days, it's commonly also called scientific notation.) In one of his most famous works, Archimedes, a singularly brilliant intellect of the classical world, needed approximately 12 pages (in English translation) to create names of numbers and methods of multiplication to produce an *upper bound*—a maximal estimate, a cap—on the number of grains of sand in the world. By using modern notation, particularly the idea of the exponential, it will take us less than one paragraph to produce an upper bound on the number of grains of sand in the *universe*. Furthermore, in short order these exponential conventions confer the power to accomplish a task that might well have stymied Archimedes: calculating the precise number of distinct books in the Library.

A positive integer *exponent* signifies, "the amount of times some number is multiplied by itself." For example,

$$5^3 = 5 \cdot 5 \cdot 5 \quad \text{and} \quad 2^{4,781} = \underbrace{2 \cdot 2 \cdot \ \ldots \ \cdot 2 \cdot 2}_{4,781 \text{ times}}$$

are concise ways to express a "small" number

$$5^3 = 125$$

and a very large number.* There are only two rules regarding the manipulation of exponentials that concern us. The first:

> **Rule 1:** Multiplying numbers written in exponential notation is equivalent to adding the exponents.

For example:

$$5^3 \cdot 5^{14} = (5 \cdot 5 \cdot 5) \cdot \underbrace{(5 \cdot 5 \cdot \ldots \cdot 5 \cdot 5)}_{14 \text{ times}} = \underbrace{(5 \cdot 5 \cdot \ldots \cdot 5 \cdot 5)}_{17 \text{ times}} = 5^{17}.$$

The second rule nicely complements the first.

* 16765220490415253625065478163110488777596070684631829708120311409986396665091758868942316900907777384574090574408577882732061772110931659947399568714591497545824796138075835421197279779754323576490572256786468422800398414001130840404432159220567873647879819752992180116091963070003460102877057133859986460838201334698105992713225457349777667823840107714018295679082043307285550872688827887567010456660198813317308577461625092980751975955442225426797719393203367532575001211842556594519778330069767047797344180140352992420259949470026323167037321871020156554080028628985372035016289304847323104057902026971342243620895518683161620610971532819079644261674019733075609639725425948141117929760571410501529175736939057142480970571052799564262028069719662143027579309322592780037655988299492532761268919600892082956363896640596815107919370351679897793541041087048548047318020669269646014131957475053716230240145815191289468390501752049291549261025060765820082045923357997387162458153303902782719259482207647732608090999484600968177752900336140864517350814719001366340483051936550164732484666637269545402336941985560597412463505491361370778907853996319948651214328189127063348723482046097851696224594521840433253736095156882633878161655155708353469566551811184159038072931547810565363280312371971406298562246478087376179917052553905525688581305925549191324529576300043914446535610319757557673115929921792891932243531101879093801244638169577763635

Rule 2: Dividing numbers written in exponential notation is equivalent to subtracting the denominator's exponent from the numerator's.

For example:

$$\frac{2^{4,781}}{2^{14}} = \frac{\overbrace{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}^{4,781 \text{ times}}}{\underbrace{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}_{14 \text{ times}}} = \frac{\overbrace{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}^{14 \text{ times}}}{\underbrace{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}_{14 \text{ times}}} \times \frac{\overbrace{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}^{4,767 \text{ times}}}{1}$$

$$= \overbrace{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}^{4,767 \text{ times}} = 2^{4,767}.$$

The second rule leads to the useful convention of using a *negative* exponent to represent a power in the denominator, for instance,

$$\frac{1}{2^{14}} = 2^{-14}.$$

Thus the previous example may concisely be written

$$\frac{2^{4,781}}{2^{14}} = \left(2^{4,781}\right)\left(2^{-14}\right) = 2^{4,781+(-14)} = 2^{4,767}.$$

It is remarkable that such relatively simple notation can transform relatively complicated tasks, multiplication and division, into the relatively easy and intuitive computations of addition and subtraction.

While pondering previous critical responses to "The Library of Babel," we discovered that a number of people either calculated the number of books or gave some indication of how one might go about it.[1] Our intent in providing the lightning review of exponential notation is to demystify the calculation, and then, more importantly, to give a sense of the enormity of the Library. Then, after the calculation, we tease out a previously overlooked detail from the story and use it to set a new lower bound on the number of books in the Library. (For us, a *lower bound* will be number that says, "We guarantee that *there are at least this many* books in the Library.")

For the purposes of this book, *combinatorics* is the branch of mathematics that counts the number of ways objects can be combined or ordered. Before using combinatorics to calculate the number of the books, let's consider 10 familiar orthographic objects, the symbols we use as representations for digits: 3, 8, 9, 1, 6, 2, 0, 5, 7, 4. We deliberately disordered them to help you see them not as you usually do, as *numbers*, but rather as symbolic representatives of the numbers 0 through 9.

Using these symbols, we'd like to occupy exactly one slot with one symbol, and so we ask: how many distinct ways can we fill one slot? Hopefully, the answer is clear—there are 10 ways to fill one slot with one of the symbols.

1. $\underline{0}$
2. $\underline{1}$
3. $\underline{2}$
4. $\underline{3}$
5. $\underline{4}$
6. $\underline{5}$
7. $\underline{6}$
8. $\underline{7}$
9. $\underline{8}$
10. $\underline{9}$

Now, how many distinct ways are there to fill *two* slots, such that each slot contains one symbol? One complete list of answers, ordered in a familiar way, reads: $\underline{00}$, $\underline{01}$, $\underline{02}$, $\underline{03}$, . . . , $\underline{97}$, $\underline{98}$, $\underline{99}$. So we see that there are 100 ways to fill the two slots, given that each slot contains one symbol and that repetition is allowed (enabling such combinations as 00, 11, 22, 33, etc.). Deliberately blurring the distinction between the orthographic symbols and the numbers they represent, we note that there are

$$100 = 10 \cdot 10 = 10^2$$

ways to fill the two slots. If we ask how many distinct ways there are to fill *three* slots, such that repetition is allowed and each slot contains one symbol, we generalize our work from above and produce a complete list that reads: $\underline{000}$, $\underline{001}$, $\underline{002}$, $\underline{003}$, . . . , $\underline{997}$, $\underline{998}$, $\underline{999}$. This time, we see that there are 1,000 ways to fill the three slots. Continuing to blur the distinction between the orthographic symbols and the numbers they represent, it follows that there are

$$1{,}000 = 10 \cdot 10 \cdot 10 = 10^3$$

distinct ways to fill the three slots. By seizing on these ideas, by sensing that a simple pattern has been established and can be used to predict what we couldn't possibly list, we may ask how many distinct ways there are to fill, for example, 36 slots, where each slot contains one of our 10 allowed orthographic symbols and repetition of symbols is allowed. By applying the reasoning we established above, we see that there must be 10^{36} ways; that is, a 1 followed by thirty-six 0s—a thousand billion, billion, billion, billion, billion, billion ways:

$$10^{36} = 1{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000.$$

Just for a lark, here are the first few and last few slot-fillings of the usual way one would list the fillings.

1. 000000000000000000000000000000000000
2. 000000000000000000000000000000000001
3. 000000000000000000000000000000000002

(Quite a few more!)

$(10^{36} - 2)$. 999999999999999999999999999999999997
$(10^{36} - 1)$. 999999999999999999999999999999999998
10^{36}. 999999999999999999999999999999999999

And that's the end of the list.

In an article in the academic journal *Variaciones Borges*, our ideal reader, Umberto Eco, argues that the exact number of distinct volumes in the Library is irrelevant to both the story and to the reader. To the extent that the numbers of pages, lines, and letters in each book were chosen arbitrarily by Borges, we agree with him. (See the beginning of the chapter "Geometry and Graph Theory" for a quote from Borges regarding this matter.) However, we assert that understanding the combinatorial process that produces the exact number of distinct volumes is both important and relevant to an understanding of the story. So let's apply these ideas to the story and, given the numbers and constraints Borges provides, use them to calculate the number of distinct volumes in the Library.

In "The Library of Babel," Borges writes:

> . . . each book contains four hundred ten pages; each page, forty lines; each line, approximately eighty black letters. There are also letters on the front cover of each book; these letters neither indicate nor prefigure what the pages inside will say.

From these lines, we conclude each book consists of $410 \cdot 40 \cdot 80 = 1{,}312{,}000$ orthographic symbols; that is, we may consider a book as consisting of 1,312,000 slots to be filled with orthographic symbols. Here a few more excerpts from the next few paragraphs:

> *There are twenty-five orthographic symbols*. That discovery enabled mankind, three hundred years ago, to formulate a general theory of the Library and thereby satisfactorily resolve the riddle that no conjecture had been able to divine—the formless and chaotic nature of virtually all books. . .
>
> Some five hundred years ago, the chief of one of the upper hexagons came across a book as jumbled as all the others, but containing almost two pages of homogeneous lines. He showed his find to a traveling decipherer, who told him the lines were written in Portuguese; others said it was Yiddish. Within the century experts had determined what the language actually was: a Samoyed-Lithuanian dialect of Guaraní, with inflections from classical Arabic. The content was also determined: the rudiments of combinatory analysis, illustrated with examples of endlessly repeating variations. These examples allowed a librarian of genius to discover the fundamental law of the Library.

> This philosopher observed that all books, however different from one another they might be, consist of identical elements: the space, the period, the comma, and the twenty-two letters of the alphabet. He also posited a fact which all travelers have since confirmed: *In all the Library, there are no two identical books.* From those incontrovertible premises, the librarian deduced that the Library is "total"—perfect, complete, and whole—and that its bookshelves contain all possible combinations of the twenty-two orthographic symbols (a number which, though unimaginably vast, is not infinite)—that is, all that is able to be expressed, in every language.

How many distinct books constitute the Library? Each book has 1,312,000 slots, each of which may be filled with 25 orthographic symbols—this is the "variations with unlimited repetition" mentioned above. Again, by employing the ideas outlined above, there are

25 ways to fill one slot,
$25 \cdot 25 = 25^2$ ways to fill two slots,
$25 \cdot 25 \cdot 25 = 25^3$ ways to fill three slots,
and so on,
and so on for 1,312,000 slots.

It follows immediately that there are

$$25^{1,312,000}$$

distinct books in the Library. That's it.

Somehow, it feels all too easy, even anticlimactic, as though instead we should have had to write pages and pages of dense, technical, high-level mathematics, overcoming one complex puzzle after another, before arriving at the answer. But most of the beauty—the elegance—of mathematics is this: applying potent ideas and clean notation to a problem much as the precise taps of a diamond-cutter cleave and husk the dispensable parts of the crystal, ultimately revealing the fire within. (Perhaps we should have ended the calculation by writing "That's it!" instead of "That's it.")

Our new twist on these calculations involves what Hurley translates as the "letters on the front cover of each book." For the sake of precision, we note that the Spanish reads "el dorso de cada libro," which translates literally as "the back of the book." Idiomatically and bibliographically, however, the sense of this phrase is that the letters are on the *spine* of the Library's books. As such, the interpretation we use for the rest of this book is that the letters are on the spine.

Now, the number $25^{1,312,000}$ we calculated above doesn't account for these spinal letters. It strikes us as likely that, within the imaginary universe of the Library, a book with the letters *The Plaster Cramp* written on the spine, whose 1,312,000 slots are filled by the repeated sequence of orthographic symbols MCV, should be considered as a book distinct from one with the exact same pages which is instead imprinted with the letters *Axaxaxas Mlö* on the spine.[2] Scanning through the original Spanish version, "La biblioteca de Babel," we find a book described with the 19 orthographic symbols *El calambre de yeso* on its spine. This means that there are a minimum of 19 slots to fill on each spine, and accounting for these variations with repetition expands the Library by a factor of *at least*

$$25^{19} = 363,797,880,709,171,295,166,015,625.$$

We write this number out explicitly to re-echo the vastness of the numbers woven through the Library. Simply adding 19 orthographic symbols on the spine magnifies the Library more than 300 septillion times. For comparison, this number is roughly the number of microscopic plant cells comprising a grove of 364 oak trees.[3] So if the Library of $25^{1,312,000}$ books is considered as *one* imperceptible plant cell, accounting for differing symbols on the spine multiplies the Library into a *grove* of 364 giant oak trees.

However, since we cannot be sure of either the maximum number of symbols on the spine of each book or of Borges' intent, we restrict ourselves to $25^{1,312,000}$ books. This number, so easy to write, is, in a powerful sense, utterly unimaginable. To see that we can't see it, let's begin by converting this number to a *power of 10,* which puts it in a more familiar context.

$25^{1,312,000}$ is just a little bit larger than $10^{1,834,097}$;

which is, of course, a 1 followed by one million, eight hundred thirty-four thousand, and ninety-seven 0s. We accomplish this conversion to a power