DAVID ACHESON

THE WONDER BOOK OF GEOMETRY

a mathematical story

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Introduction

It all started at school, one cold winter morning in 1956, when I was ten.

Mr. Harding had been doing some maths at the blackboard, with chalk dust raining down everywhere, when he suddenly whirled round and told us all to draw a semicircle, with diameter AB.

Then we had to choose some point P on the semicircle, join it to A and B by straight lines, and measure the angle at P (Fig. 1).

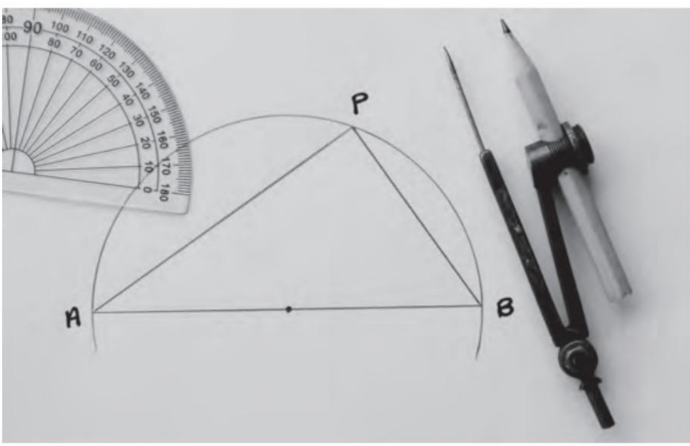


Fig. 1 Thales' theorem.

I duly got on with all this, casually assuming that the angle at P would depend on where P is, exactly, on the semicircle.

But it *doesn't*.

It's always 90°.

* * *

At the time, I had no idea that mathematics is full of surprises like this.

I had no idea, either, that this is one of the first great theorems of geometry, due to a mathematician called Thales, in ancient Greece. And according to Thales – so it is said – the key question is always not 'What do we know?' but rather '*How do we know it?*'

Why is it, then, that the angle in a semicircle is always 90°?

The short answer is that we can *prove* it, by a sequence of simple logical steps, from a few apparently obvious starting assumptions.

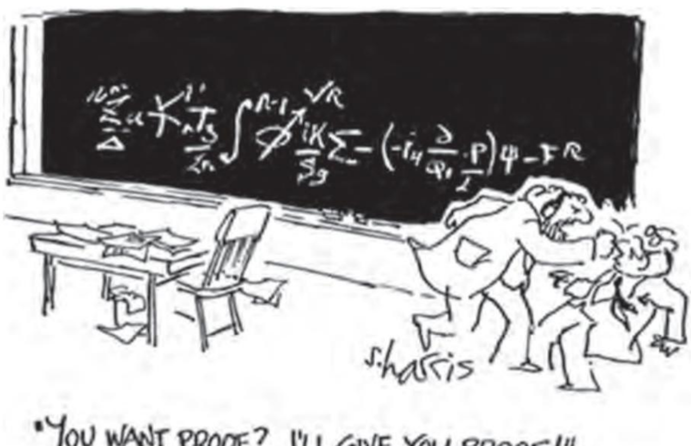


Fig. 2 The importance of proof.

And by doing just that, in the next few pages, I hope to not only lay some foundations for geometry, but do something far more ambitious.

For, with geometry, it is possible to see something of the whole nature and spirit of mathematics at its best, at almost any age, *within just half an hour of starting*.

And in case you don't quite believe me…

2

Getting Started

The first really major idea is that of *parallel lines*.

These are lines, in the same plane, which never meet, no matter how far they are extended.

And I will make two assumptions about them.

Parallel lines

Imagine, if you will, two lines crossed by a third line, producing the so-called *corresponding angles* of Fig. 3.

Fig. 3 Corresponding angles.

Then, throughout most of this book, I will assume that

(1) If two lines are parallel, the corresponding angles are equal.

(2) If corresponding angles are equal, the two lines are parallel.

These assumptions are rooted in the intuitive notion that parallel lines must be, so to speak, 'in the same direction', but however obvious (1) and (2) may seem, they *are* assumptions.

And, even at this early stage, it is worth noting that they amount to two very different statements.

In effect, (1) helps us use parallel lines, while (2) helps us show that we have some.

Angles

We will measure angles in *degrees,* denoted by °, and the two parts of a straight line through some point P form an angle of 180° (Fig. 4).

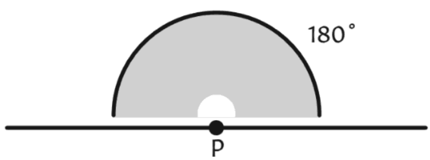


Fig. 4 A straight line.

A *right angle* is half this, i.e. an angle of 90°, and the two

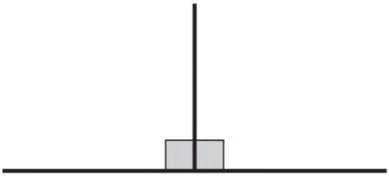


Fig. 5 Right angles.

lines forming it are then said to be perpendicular (Fig. 5).

Opposite angles

When two straight lines intersect, the so-called *opposite angles* are equal (Fig. 6).

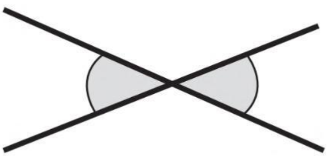


Fig. 6 Opposite angles.

Alternate angles

If two lines are parallel, and crossed by a third line, then the so-called *alternate angles* are equal (Fig. 7).

Fig. 7 Alternate angles.

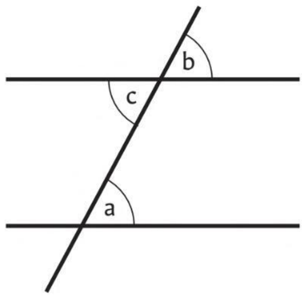


Fig. 8 Proof that alternate angles are equal.

This is because, in Fig. 8, $a = b$ (corresponding angles) *and* $b = c$ (opposite angles). So $a = c$.

The argument works 'in reverse', too, so that if alternate angles are equal, the two lines must be parallel.

And with these ideas in place, we are now ready to prove the first theorem which, in my view, is not obvious at all…

The angle-sum of a triangle

The three angles in any triangle add up to 180° (Fig. 9).

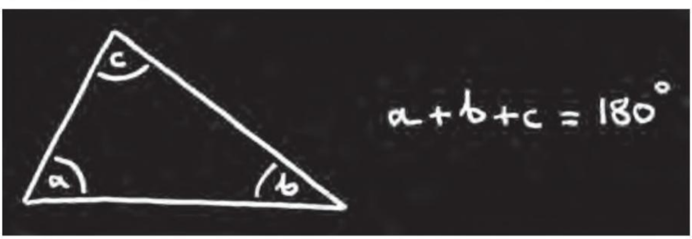


Fig. 9 Angles in a triangle.

To prove this, draw a straight line through one corner, parallel to the opposite side (Fig. 10).

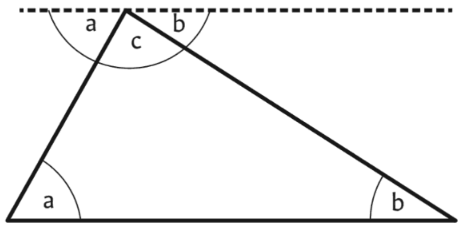


Fig. 10 Proof of the angle-sum of a triangle.

The angles *a* are then equal (alternate angles).

The angles *b* are also equal, for the same reason.

Finally, the new line is straight, so $a + b + c = 180°$, which completes the proof.

3

Euclid's Elements

The most famous example of geometry being presented in this concise, deductive, and carefully ordered way is the *Elements*, written by Euclid of Alexandria (Fig. 11), in about 300 BC.

Fig. 11 Euclid.

It is best to be clear from the outset, I think, that the precise theorems and proofs of Euclid's *Elements* (Fig. 12) are essentially about imaginary objects.

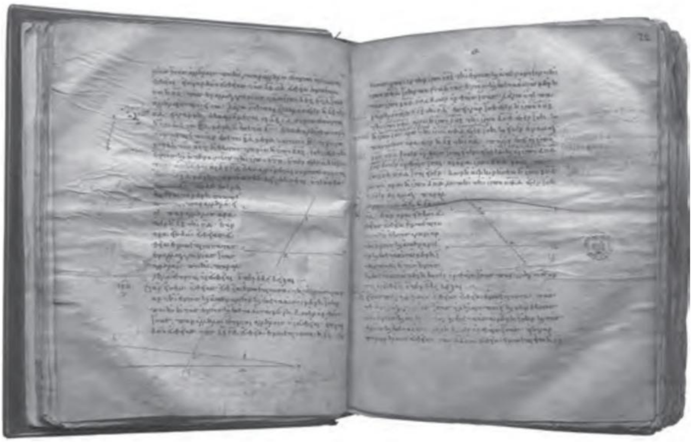


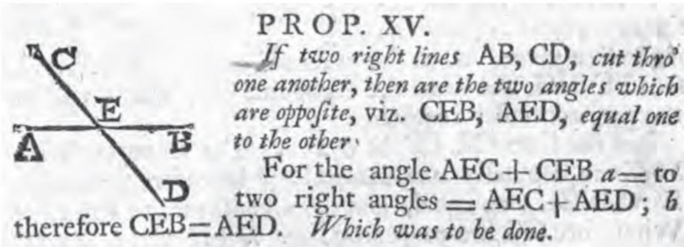
Fig. 12 The oldest surviving copy of Euclid's *Elements*, MS D'Orville 301, copied by Stephen the Clerk for Arethas of Patras, in Constantinople in AD 888.

A Euclidean straight line, for instance, isn't just 'perfectly' straight—it has *zero thickness*. So even if I could draw one properly, you wouldn't be able to see it.

And a point isn't a blob of small dimension—it has no dimension at all. Or, as Euclid put it:

> A point is that which has no part.

It should be said, too, that Euclid makes no use of what we would call 'measurement units' for length. And there are no degrees in Euclid; the nearest he comes to having a unit for angle is the concept of *right angle*, which he uses a great deal (Fig. 13).



PROP. XV.

If two right lines AB, CD, *cut thro' one another, then are the two angles which are opposite*, viz. CEB, AED, *equal one to the other*.

For the angle AEC+CEB *a* = to two right angles = AEC+AED; *b* therefore CEB = AED. *Which was to be done.*

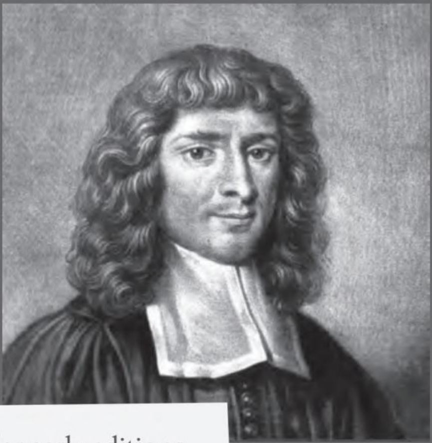
Fig. 13 Proof that opposite angles are equal, from a 1732 edition of Euclid's *Elements*.

In spite of this, and the austere style of exposition, the *Elements* has had more influence, and more editions, than almost any other book in human history.

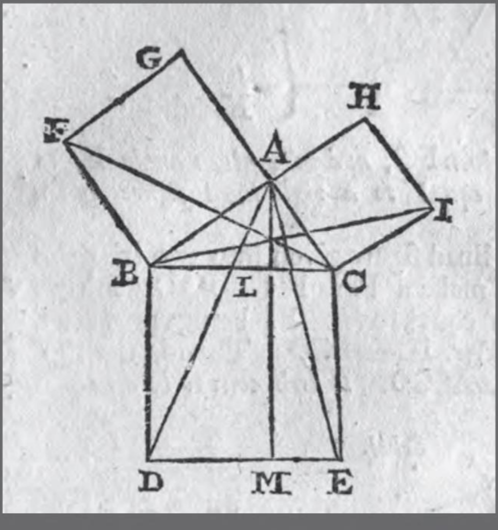
In the end, however, there can be no single 'best' way of doing geometry, and we all have to find our own path into the subject.

And if, in this book, I unashamedly assume more than Euclid does, it is because I want to proceed more quickly to interesting and surprising results…

Euclid, 1732



One of the most popular early editions of Euclid was by **Isaac Barrow**. It was first published in 1660, but my own copy dates from 1732.



EUCLIDE's ELEMENTS;

The whole FIFTEEN BOOKS compendiously Demonstrated:

WITH

ARCHIMEDES's Theorems of the Sphere and Cylinder Investigated by the *Method of Indivisibles*.

By ISAAC BARROW, D.D. *late Master of* Trinity *College in* Cambridge.

To which is Annex'd,

EUCLIDE's *Data*, and a brief *Treatise of Regular Solids*.

The Whole revis'd with great Care, and some Hundreds of Errors of the former Impression corrected.

By THOMAS HASELDEN, *Teacher of the Mathematicks*.

Καθαρμοὶ ψυχῆς λογικῆς εἰσιν αἱ μαθηματικαὶ ἐπιστῆμαι.

LONDON: Printed for *Daniel Midwinter* and *Aaron Ward* in *Little-Britain*; *Arthur Bettesworth* and *Charles Hitch* in *Pater-noster-row*; and *Thomas Page* and *William Mount* on *Tower-Hill*. 1732.

4

Thales' Theorem

Thales' theorem says that the angle in a semicircle is always 90°.

And, to prove it, we need just one or two more key ideas.

Congruent triangles

Congruent triangles are ones which have *exactly the same size and shape.*

And the most obvious way of fixing the exact size and shape of a triangle is, perhaps, to specify the lengths of two sides and the angle between them.

This leads to a very simple test for congruence, known informally as 'side-angle-side', or SAS (Fig. 14).

Fig. 14 Congruence by SAS.

Isosceles triangles

An isosceles triangle is one in which two sides are equal.

Triangles of this kind play a major part in geometry, largely because *the 'base' angles of an isosceles triangle are equal* (Fig. 15).

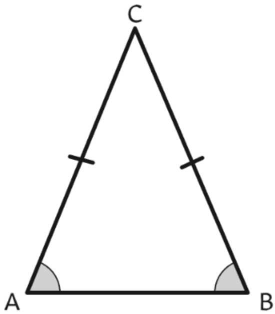


Fig. 15 An isosceles triangle.

Many people, I think, find this particular result rather obvious. After all, if we 'nip round the back' of an isosceles triangle it will look exactly the same.

A more formal way of proving the result is to introduce the line CD bisecting the angle at C (Fig. 16).

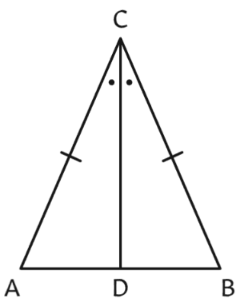


Fig. 16 Proof that the base angles of an isosceles triangle are equal.

The triangles ACD and BCD are then congruent by SAS, and one is, in fact, a 'mirror image' or 'overturned' version of the other. In particular, then, the angles at A and B must be equal.

(If *all three* sides of a triangle happen to be equal it is said to be *equilateral*. The triangle is then isosceles in three different ways, so all three of its angles are equal.)

Circles

The defining property of a circle is that all its points are the same distance from one particular point, called the *centre*, O.

Some other common terminology is introduced in Fig. 17.

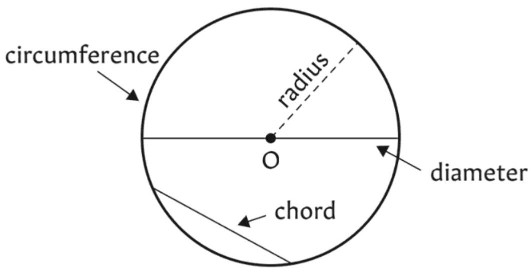


Fig. 17 The circle.

And this gives us all we need to prove Thales' theorem.

Thales' theorem

We want to prove that if P is any point on the semicircle in Fig. 18, then $\angle APB = 90°$, where $\angle APB$ denotes the angle between AP and PB.

Now, the simplest way of using the fact that P lies on the semicircle, surely, is to draw in the line OP and observe that

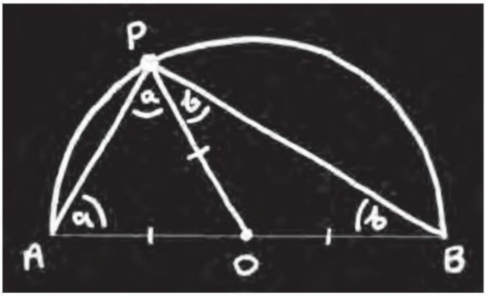


Fig. 18 Proof of Thales' theorem.

OP = OA = OB, because all points on a circle are the same distance from its centre.

Suddenly, then, we have two *isosceles triangles*, AOP and BOP.

The two 'base angles' *a* are therefore equal, and so are the two base angles *b*.

Finally, the three angles of the large triangle APB must add up to 180°, so

$$a+(a+b)+b=180°$$

and therefore $a + b = 90°$. In consequence, $\angle APB = 90°$, which proves the theorem.

And in all the years since I first saw this proof, on a cold winter morning in 1956, I have never forgotten it.

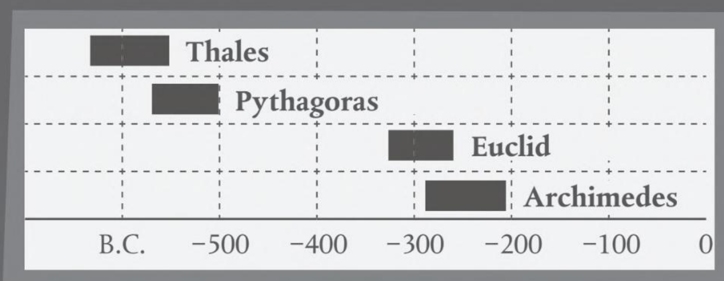
After all, the result is, at first sight, rather difficult to believe, yet just a few minutes later we find ourselves saying, arguably: 'Oh, it's sort of obvious, really, isn't it—*when you look at it the right way*.'

And in my experience, at least, this is often one of the hallmarks of mathematics at its best.

the mathematical world of ANCIENT GREECE

Thales lived in Miletus.

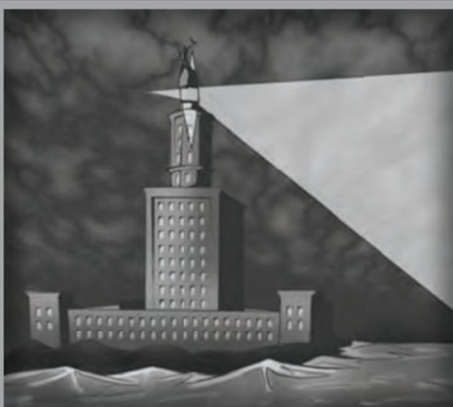
Pythagoras came from the island of Samos, but later moved to Crotona.



Plato's Academy in Athens had this famous inscription over its entrance:

ΑΓΕΩΜΕΤΡΗΤΟΣ
ΜΗΔΕΙΣ ΕΙΣΙΤΩ

"Let no one ignorant of geometry enter here"



The Pharos Lightouse, Alexandria

Euclid wrote *The Elements* in Alexandria.

Archimedes lived and worked in Syracuse.

5

Geometry in Action

Throughout history there have been practical applications of geometry, and one of the earliest was Thales' attempt to calculate the height of the Great Pyramid in Egypt.

Thales and similar triangles

Thales measured the shadow of the Great Pyramid cast by the Sun, and by adding half the pyramid's base determined the distance *L* in Fig. 19.

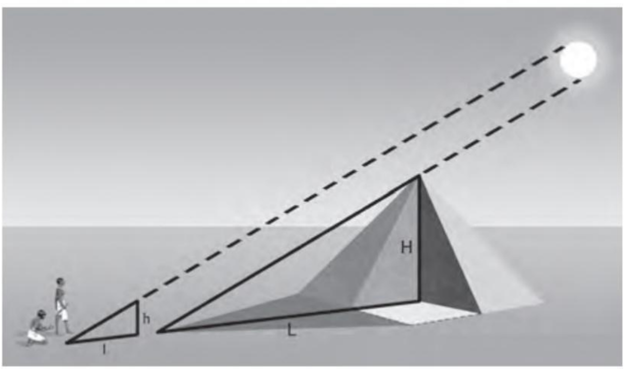


Fig. 19 Similar triangles.

He then measured the shadow ℓ cast by a vertical pole of height h.

Assuming the Sun's rays to be parallel, he reasoned that the two triangles in Fig. 19, though of very different size, would have exactly the same *shape*, and that corresponding sides would therefore be in the same proportion.

In particular, then, he reasoned that

$$\frac{h}{H} = \frac{\ell}{L},$$

and so, having measured the other three lengths, he was able to determine the Great Pyramid's height H.

Fig. 20 Thales, on a Greek postage stamp of 1994.

Today we use the term *similar* to describe triangles which have exactly the same shape, and, as we will see later, they play a major part in some of the most striking theorems of geometry.

Measuring the Earth

According to its Greek roots, the word 'geometry' means, quite literally, 'Earth measurement'. So it seems appropriate to look next at a famous attempt to measure the circumference of the Earth, by Eratosthenes of Alexandria, in about 240 BC.

And, as it happens, he too used the Sun's rays, but in a rather different way.

Eratosthenes knew that, at noon on the longest day of the year, the Sun was directly overhead at his birthplace Syene (modern-day Aswan), because it illuminated the bottom of a deep well there.

He also knew that, at the same time, the Sun made an angle of 7.2° with the vertical at Alexandria, which he took to be 5000 *stades* due north of Syene.

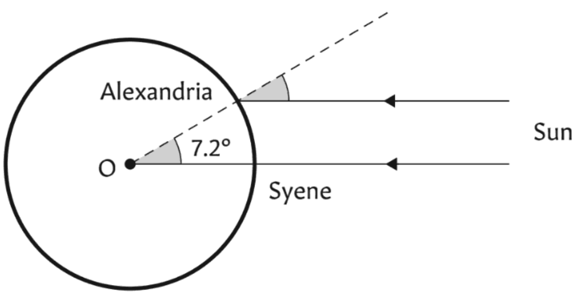


Fig. 21 Measuring the Earth.

Eratosthenes assumed that the Sun was so far from the Earth that its rays arrived parallel. The two shaded angles in

Fig. 21 are then corresponding angles, so the angle at O, the centre of the Earth, must also be 7.2°.

Now, 7.2° is one-fiftieth of 360°, so he reasoned that the circumference of the Earth must be 50 times the distance between Alexandria and Syene, i.e. 250,000 stades.

In truth, this is probably an oversimplification of what Eratosthenes actually did. Moreover, what a *stade* was, as a unit of distance, is also lost; estimates by subsequent scholars put it between 0.15 and 0.2 km, leading to a result for the Earth's circumference somewhere between 37,500 and 50,000 km. (The actual value is about 40,000 km.)

'Practical work', 1929

There is, of course, another, quite different aspect to practical geometry, namely the actual construction of geometrical figures using ruler, compasses, and other tools of the trade.

When doing this kind of work, however, we have to be continually mindful of what physicists would call 'experimental error'; otherwise, things can get a bit ridiculous.

On my shelves at home, for instance, there is an old geometry exercise book, dating from 1929, that once belonged to a pupil at a primary school in the north of England.

The book itself has considerable charm, and consists mostly of simplified Euclid, carried out neatly and well. On the very last page, however, and without any warning at all, we suddenly meet something called 'Practical Work', involving—apparently—some actual *measurement* (Fig. 22).

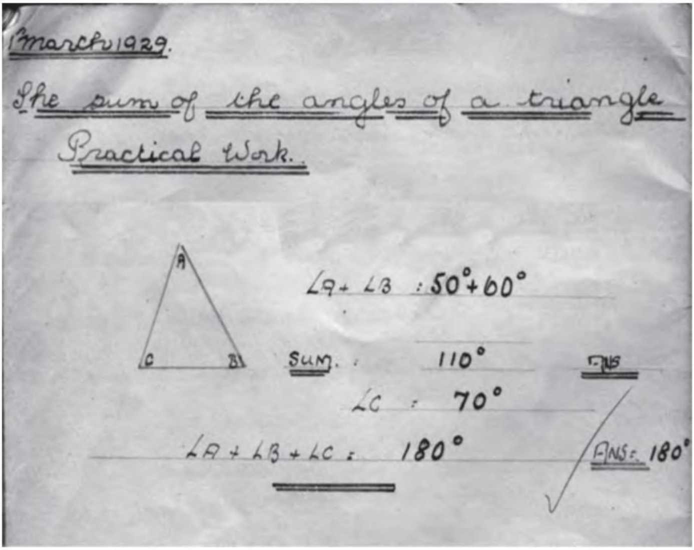


Fig. 22 From a 1929 school exercise book.

Yet, despite the tick of approval from the teacher, there is something faintly absurd about this particular piece of work.

So far as I can determine, the angle A is closer to 45° than 50°, and it looks to me as if the various numbers have simply been cooked up, quite unashamedly, so that the angle-sum comes out 'right'.

Area

Perhaps the oldest geometrical idea of real practical importance is that of *area,* driven largely by problems concerning land.

We begin with a *square* of side 1 unit, and it quickly becomes evident how to calculate the area of a rectangle with sides which are whole numbers (Fig. 23).

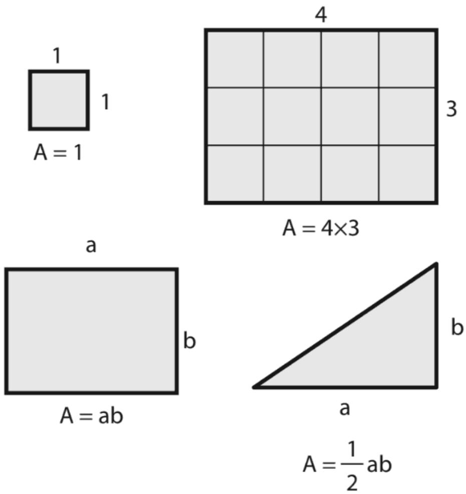


Fig. 23 Area.

This leads us to define the area of a rectangle, more generally, as

$$A = ab,$$

where the side lengths *a* and *b* may now be fractional or even irrational.

Introducing a diagonal then bisects the rectangle itself, giving $\frac{1}{2}ab$ as the area of a right-angled triangle.

And, improbable as it may seem, these elementary ideas of area are enough to let us take a first look at one of the most famous—and far-reaching—theorems of all…

6

Pythagoras' Theorem

There is a surprisingly simple relationship between the lengths of the sides of *any* right-angled triangle (Fig. 24).

And, like so much that is best in mathematics, it is this *generality* that gives the theorem its power.

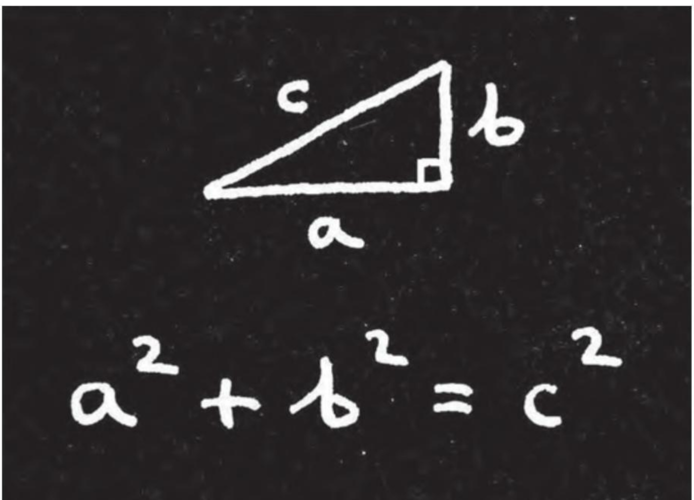


Fig. 24 Pythagoras' theorem.

In Fig. 24, *c* denotes the length of the *hypotenuse*—meaning the side opposite the right angle—while *a* and *b* denote the lengths of the other two sides.