THE WORLD OF MATHEMATICS

Volume 1

EDITED BY
JAMES NEWMAN

PARTS I-IV

General Survey

Historical and Biographical

Arithmetic, Numbers
and the Art of Counting

Mathematics of Space and Motion

The World of MATHEMATICS

Volume 1

Edited by
James R. Newman

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PART I

General Survey

1. The Nature of Mathematics *by* PHILIP E. B. JOURDAIN

COMMENTARY ON

PHILIP E. B. JOURDAIN

PHILIP E. B. JOURDAIN (1879–1919), whose little book on the nature of mathematics is here reproduced in its entirety, was a logician, a philosopher and a historian of mathematics. To each of these subjects he brought a fresh outlook and a remarkably penetrating and creative intelligence. He was not yet forty when he died and from adolescence had been afflicted by a terrible paralytic ailment (Friedrick's ataxia) which gradually tightened its grip upon him. Yet he left behind a body of work that influenced the development of both mathematical logic and the history of science.

Jourdain, the son of a Derbyshire vicar, was educated at Cheltenham College and at Cambridge. The few years during which he was able to enjoy the normal pleasures of boyhood—long walks were his special delight—are described in a poignant memoir by his younger sister, Millicent, who suffered from the same hereditary disease. In 1900 the brother and sister went to Heidelberg to seek medical help. While at the hospital he began in earnest his study of the history of mathematics. "We had," wrote Millicent, "what was to be nearly our last bit of walking together here." The treatment was unavailing and when they returned to England, Jourdain could no longer walk or stand or even hold a pencil without difficulty. Nevertheless, he undertook with great energy and enthusiasm the first of a series of mathematical papers which established his reputation. Among his earlier writings were studies of Lagrange's use of differential equations, the work of Cauchy and Gauss in function theory, and conceptual problems of mathematical physics.[1] Between 1906 and 1912 he contributed to the *Archiv der Mathematik und Physik* a masterly group of papers on the mathematical theory of transfinite numbers, a subject in which he was always deeply interested. In the same period the *Quarterly Journal of Mathematics* published a group of essays on the development of the theories of mathematical logic and the principles of mathematics. Jourdain was an editor of *Isis* and the *Monist*, in whose pages appeared his articles on Leibniz, Napier, Hooke, Newton, Galileo, Poincaré and Dedekind. He edited reprints of works by De Morgan, Boole, Georg Cantor, Lagrange, Jacobi, Gauss and Ernst Mach; he wrote a brilliant and witty book, *The Philosophy of Mr. B*rtr*nd R*ss*ll*, dealing with Russell's analysis of the problems of logic and the foundations of mathematics; he took out a patent covering an invention of a "silent engine" (I have been unable to

[1] Bibliographies of Jourdain's writings appear in *Isis*, Vol. 5, 1923, pp. 134–136, and in the *Monist*, Vol. 30, 1920, pp. 161–182.

discover what this machine was) and he wrote poems and short stories which never got published. In 1914, at the height of his powers, he was producing enough "to keep two typists busy all day."

The distinctive qualities of Jourdain's thought were its independence and its cutting edge. He was renowned for his broad scholarship in the history and philosophy of science, but he was more than a scholar. Never content with comprehending all that others had said about a problem, he had to work it through in his own way and overcome its difficulties by his own methods. This led him to conclusions peculiarly his own. They are not always satisfactory but they always deserve close attention: Jourdain rarely failed to uncover points overlooked by less subtle and original investigators.

The Nature of Mathematics reflects his excellent grasp of the subject, his at times oblique but always rewarding approach to logic and mathematics, his wit and clear expression. He had sharpened his thinking on some of the hardest and most baffling questions of philosophy and had achieved an orderly understanding of them which he was fully capable of imparting to the attentive reader. The book is not a textbook collection of methods and examples, but an explanation of "how and why these methods grew up." It discusses concepts which are widely used even in elementary arithmetic, geometry and algebra—negative numbers, for example—but far from widely comprehended. It presents also a careful treatment of "the development of analytical methods and certain examinations of principles." There are at least two other excellent popularizations of mathematics, A. N. Whitehead's celebrated *Introduction to Mathematics*[2] and the more recent *Mathematics for the General Reader* by E. C. Titchmarsh.[3] Both books can be recommended strongly, the first as a characteristic, immensely readable work by one of the greatest of twentieth-century philosophers; the second as a first-class mathematician's lucid, unhurried account of the science of numbers from arithmetic through the calculus. Jourdain's book follows a somewhat different path of instruction in that it emphasizes the relation between mathematics and logic. It is the peer of the other two studies and has for the anthologist the additional appeal of being unjustly neglected and out of print. "I hope that I shall succeed," says Jourdain in his introduction, "in showing that the process of mathematical discovery is a living and a growing thing." In this attempt he did not fail.

[2] Oxford University Press, New York, 1948.
[3] Hutchinson's University Library, London, n. d.

Pure mathematics consists entirely of such asseverations as that, if such and such a proposition is true of anything, *then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true. . . . If our hypothesis is about* anything *and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.*

—BERTRAND RUSSELL

1 The Nature of Mathematics

By PHILIP E. B. JOURDAIN

CONTENTS

INTRODUCTION

AN eminent mathematician once remarked that he was never satisfied with his knowledge of a mathematical theory until he could explain it to the next man he met in the street. That is hardly exaggerated; however, we must remember that a satisfactory explanation entails duties on both sides. Any one of us has the right to ask of a mathematician, "What is the use of mathematics?" Any one may, I think and will try to show, rightly suppose that a satisfactory answer, if such an answer is anyhow possible, can be given in quite simple terms. Even men of a most abstract science, such as mathematics or philosophy, are chiefly adapted for the ends of ordinary life; when they think, they think, at the bottom, like other men. They are often more highly trained, and have a technical facility for thinking that comes partly from practice and partly from the use of the contrivances for correct and rapid thought given by the signs and rules for dealing with them that mathematics and modern logic provide. But there is no real reason why, with patience, an ordinary person should not understand, speaking broadly, what mathematicians do, why they do it, and what, so far as we know at present, mathematics is.

Patience, then, is what may rightly be demanded of the inquirer. And this really implies that the question is not merely a rhetorical one—an expression of irritation or scepticism put in the form of a question for the sake of some fancied effect. If Mr. A. dislikes the higher mathematics because he rightly perceives that they will not help him in the grocery business, he asks disgustedly, "What's the use of mathematics?" and does not wait for an answer, but turns his attention to grumbling at the lateness of his dinner. Now, we will admit at once that higher mathematics is of no more use in the grocery trade than the grocery trade is in the navigation of a ship; but that is no reason why we should condemn mathematics as entirely useless. I remember reading a speech made by an eminent surgeon, who wished, laudably enough, to spread the cause of elementary surgical instruction. "The higher mathematics," said he with great satisfaction to himself, "do not help you to bind up a broken leg!" Obviously they do not; but it is equally obvious that surgery does not help us to add up accounts; . . . or even to think logically, or to accomplish the closely allied feat of seeing a joke.

To the question about the use of mathematics we may reply by pointing out two obvious consequences of one of the applications of mathematics: mathematics prevents much loss of life at sea, and increases the commercial prosperity of nations. Only a few men—a few intelligent philosophers and more amateur philosophers who are not highly intelligent—would doubt if these two things were indeed benefits. Still, probably, all of us act as if we thought that they were. Now, I do not mean that mathematicians go about with life-belts or serve behind counters; they do not usually do so. What I mean I will now try to explain.

Natural science is occupied very largely with the prevention of waste of the labour of thought and muscle when we want to call up, for some purpose or other, certain facts of experience. Facts are sometimes quite useful. For instance, it is useful for a sailor to know the positions of the stars and sun on the nights and days when he is out of sight of land. Otherwise, he cannot find his whereabouts. Now, some people connected with a national institution publish periodically a *Nautical Almanac* which contains the positions of stars and other celestial things you see through telescopes, for every day and night years and years ahead. This *Almanac*, then, obviously increases the possibilities of trade beyond coasting-trade, and makes travel by ship, when land cannot be sighted, much safer; and there would be no *Nautical Almanac* if it were not for the science of astronomy; and there would be no practicable science of astronomy if we could not organise the observations we make of sun and moon and stars, and put hundreds of observations in a convenient form and in a little space—in short, if we could not economise our mental or bodily activity by remembering or carrying about two or three little formulæ instead of fat books full of details; and, lastly, we could not economise this activity if it were not for mathematics.

Just as it is with astronomy, so it is with all other sciences—both those of Nature and mathematical science: the very essence of them is the prevention of waste of the energies of muscle and memory. There are plenty of things in the unknown parts of science to work our brains at, and we can only do so efficiently if we organise our thinking properly, and consequently do not waste our energies.

The purpose of this little volume is not to give—like a text-book—a collection of mathematical methods and examples, but to do, firstly, what text-books do not do: to show how and why these methods grew up. All these methods are simply means, contrived with the conscious or unconscious end of economy of thought-labour, for the convenient handling of long and complicated chains of reasoning. This reasoning, when applied to foretell natural events, on the basis of the applications of mathematics, as sketched in the fourth chapter, often gives striking results. But the methods of mathematics, though often suggested by natural events, are purely logical. Here the word "logical" means something more than the traditional doctrine consisting of a series of extracts from the science of reasoning, made by the genius of Aristotle and frozen into a hard body of doctrine by the lack of genius of his school. Modern logic is a science which has grown up with mathematics, and, after a period in which it moulded itself on the model of mathematics, has shown that not only the reasonings but also conceptions of mathematics are logical in their nature.

In this book I shall not pay very much attention to the details of the elementary arithmetic, geometry, and algebra of the many text-books, but shall be concerned with the discussion of those conceptions—such as that of negative number—which are used and not sufficiently discussed in these books. Then, too, I shall give a somewhat full account of the development of analytical methods and certain examinations of principles.

I hope that I shall succeed in showing that the process of mathematical discovery is a living and a growing thing. Some mathematicians have lived long lives full of calm and unwavering faith—for faith in mathematics, as

I will show, has always been needed—some have lived short lives full of burning zeal, and so on; and in the faith of mathematicians there has been much error.

Now we come to the second object of this book. In the historical part we shall see that the actual reasonings made by mathematicians in building up their methods have often not been in accordance with logical rules. How, then, can we say that the reasonings of mathematics are logical in their nature? The answer is that the one word "mathematics" is habitually used in two senses, and so, as explained in the last chapter, I have distinguished between "mathematics," the methods used to discover certain truths, and "Mathematics," the truths discovered. When we have passed through the stage of finding out, by external evidence or conjecture, how mathematics grew up with problems suggested by natural events, like the falling of a stone, and then how something very abstract and intangible but very real separated out of these problems, we can turn our attention to the problem of the nature of Mathematics without troubling ourselves any more as to how, historically, it gradually appeared to us quite clearly that there is such a thing at all as Mathematics—something which exists apart from its application to natural science. History has an immense value in being suggestive to the investigator, but it is, logically speaking, irrelevant. Suppose that you are a mathematician; what you eat will have an important influence on your discoveries, but you would at once see how absurd it would be to make, say, the momentous discovery that 2 added to 3 makes 5 depend on an orgy of mutton cutlets or bread and jam. The methods of work and daily life of mathematicians, the connecting threads of suggestion that run through their work, and the influence on their work of the allied work of others, all interest the investigator because these things give him examples of research and suggest new ideas to him; but these reasons are psychological and not logical.

But it is as true as it is natural that we should find that the best way to become acquainted with new ideas is to study the way in which knowledge about them grew up. This, then, is what we will do in the first place, and it is here that I must bring my own views forward. Briefly stated, they are these. Every great advance in mathematics with which we shall be concerned here has arisen out of the needs shown in natural science or out of the need felt to connect together, in one methodically arranged whole, analogous mathematical processes used to describe different natural phenomena. The application of logic to our system of descriptions, which we may make either from the motive of satisfying an intellectual need (often as strong, in its way, as hunger) or with the practical end in view of satisfying ourselves that there are no hidden sources of error that may ultimately lead us astray in calculating future or past natural events, leads at once to those modern refinements of method that are regarded with disfavour by the old-fashioned mathematicians.

In modern times appeared clearly—what had only been vaguely suspected before—the true nature of Mathematics. Of this I will try to give some account, and show that, since mathematics is logical and not psychological in its nature, all those petty questions—sometimes amusing and often tedious—of history, persons, and nations are irrelevant to Mathematics in itself. Mathematics has required centuries of excavation, and the process of excavation is not, of course, and never will be, complete. But we see enough now of what has been excavated clearly to distinguish between it and the tools which have been or are used for excavation. This confusion, it should be noticed, was never made by the excavators themselves, but only by some of the philosophical onlookers who reflected on what was being done. I hope and expect that our reflections will not lead to this confusion.

CHAPTER I

THE GROWTH OF MATHEMATICAL SCIENCE IN ANCIENT TIMES

IN the history of the human race, inventions like those of the wheel, the lever, and the wedge were made very early—judging from the pictures on ancient Egyptian and Assyrian monuments. These inventions were made on the basis of an instinctive and unreflecting knowledge of the processes of nature, and with the sole end of satisfaction of bodily needs. Primitive men had to build huts in order to protect themselves against the weather, and, for this purpose, had to lift and transport heavy weights, and so on. Later, by reflection on such inventions themselves, possibly for the purposes of instruction of the younger members of a tribe or the newly-joined members of a guild, these isolated inventions were classified according to some analogy. Thus we see the same elements occurring in the relation of a wheel to its axle and the relation of the arm of a lever to its fulcrum—the same weights at the same distance from the axle or fulcrum, as the case may be, exert the same power, and we can thus class both instruments together in virtue of an analogy. Here what we call "scientific" classification begins. We can well imagine that this pursuit of science is attractive in itself; besides helping us to communicate facts in a comprehensive, compact, and reasonably connected way, it arouses a purely intellectual interest. It would be foolish to deny the obvious importance to us of our bodily needs; but we must clearly realise two things:—(1) The intellectual need is very strong, and is as much a fact as hunger or thirst; sometimes it is even stronger than bodily needs—Newton, for instance, often forgot to take food when he was engaged with his discoveries. (2) Practical results of value often follow from the satisfaction of intellectual needs. It was the satisfaction of certain intellectual needs in the cases of Maxwell and Hertz that ultimately led to wireless telegraphy; it was the satisfaction of some of Faraday's intellectual needs that made the dynamo and the electric telegraph possible. But many of the results of strivings after intellectual satisfaction have as yet no obvious bearing on the satisfaction of our bodily needs. However, it is impossible to tell whether or no they will always be barren in this way. This gives us a new point of view from which to consider the question, "What is the use of mathematics?" To condemn branches of mathematics because their results cannot obviously be applied to some practical purpose is short-sighted.

The formation of science is peculiar to human beings among animals. The lower animals sometimes, but rarely, make isolated discoveries, but never seem to reflect on these inventions in themselves with a view to rational classification in the interest either of the intellect or of the indirect furtherance of practical ends. Perhaps the greatest difference between man and the lower animals is that men are capable of taking circuitous paths for the attainment of their ends, while the lower animals have their minds so filled up with their needs that they try to seize the object they want, or remove that which annoys them, in a direct way. Thus, monkeys often vainly snatch at things they want, while even savage men use catapults or snares or the consciously observed properties of flung stones.

The communication of knowledge is the first occasion that compels distinct reflection, as everybody can still observe in himself. Further, that which the old members of a guild mechanically pursue strikes a new member as strange, and thus an impulse is given to fresh reflection and investigation.

When we wish to bring to the knowledge of a person any phenomena or processes of nature, we have the choice of two methods: we may allow the person to observe matters for himself, when instruction comes to an end; or, we may describe to him the phenomena in some way, so as to save him the trouble of personally making anew each experiment. To describe an event—like the falling of a stone to the earth—in the most comprehensive and compact manner requires that we should discover what is constant and what is variable in the processes of nature; that we should discover the same law in the moulding of a tear and in the motions of the planets. This is the very essence of nearly all science, and we will return to this point later on.

We have thus some idea of what is known as "the economical function of science." This sounds as if science were governed by the same laws as the management of a business; and so, in a way, it is. But whereas the aims of a business are not, at least directly, concerned with the satisfaction of intellectual needs, science—including natural science, logic, and mathematics—uses business methods consciously for such ends. The methods are far wider in range, more reasonably thought out, and more intelligently applied than ordinary business methods, but the principle is the same. And this may strike some people as strange, but it is nevertheless true: there appears more and more as time goes on a great and compelling beauty in these business methods of science.

The economical function appears most plainly in very ancient and modern science. In the beginning, all economy had in immediate view the satisfaction simply of bodily wants. With the artisan, and still more so with the investigator, the most concise and simplest possible knowledge of a given province of natural phenomena—a knowledge that is attained with the least intellectual expenditure—naturally becomes in itself an aim; but though knowledge was at first a means to an end, yet, when the mental motives connected therewith are once developed and demand their satisfaction, all thought of its original purpose disappears. It is one great object of science to replace, or save the trouble of making, experiments, by the reproduction and anticipation of facts in thought. Memory is handier than experience, and often answers the same purpose. Science is communicated by instruction, in order that one man may profit by the experience of another and be spared the trouble of accumulating it for himself; and thus, to spare the efforts of posterity, the experiences of whole generations are stored up in libraries. And further, yet another function of this economy is the preparation for fresh investigation.[1]

The economical character of ancient Greek geometry is not so apparent as that of the modern algebraical sciences. We shall be able to appreciate this fact when we have gained some ideas on the historical development of ancient and modern mathematical studies.

The generally accepted account of the origin and early development of geometry is that the ancient Egyptians were obliged to invent it in order to restore the landmarks which had been destroyed by the periodical inundations of the Nile. These inundations swept away the landmarks in the valley of the river, and, by altering the course of the river, increased or decreased the taxable value of the adjoining lands, rendered a tolerably accurate system of surveying indispensable, and thus led to a systematic study of the subject by the priests. Proclus (412–485 A.D.), who wrote a summary of the early history of geometry, tells this story, which is also told by Herodotus, and observes that it is by no means strange that the invention of the sciences should have originated in practical needs, and that, further, the transition from perception with the senses to reflection,

[1] *Cf.* pp. 5, 13, 15, 16, 42, 71.

and from reflection to knowledge, is to be expected. Indeed, the very name "geometry"—which is derived from two Greek words meaning *measurement of the earth*—seems to indicate that geometry was not indigenous to Greece, and that it arose from the necessity of surveying. For the Greek geometricians, as we shall see, seem always to have dealt with geometry as an abstract science—to have considered *lines* and *circles* and *spheres* and so on, and not the rough pictures of these abstract ideas that we see in the world around us—and to have sought for propositions which should be absolutely true, and not mere approximations. The name does not therefore refer to this practice.

However, the history of mathematics cannot with certainty be traced back to any school or period before that of the Ionian Greeks. It seems that the Egyptians' geometrical knowledge was of a wholly practical nature. For example, the Egyptians were very particular about the exact orientation of their temples; and they had therefore to obtain with accuracy a north and south line, as also an east and west line. By observing the points on the horizon where a star rose and set, and taking a plane midway between them, they could obtain a north and south line. To get an east and west line, which had to be drawn at right angles to this, certain people were employed who used a rope ABCD, divided by knots or marks at B and C, so that the lengths AB, BC, CD were in the proportion 3:4:5. The length BC was placed along the north and south line, and pegs P and Q inserted at the knots B and C. The piece BA (keeping it stretched all the time) was then rotated round the peg P, and similarly the piece CD was rotated round the peg Q, until the ends A and D coincided; the point thus indicated was marked by a peg R. The result was to form a triangle PQR whose angle at P was a right angle, and the line PR would give an east and west line. A similar method is constantly used at the present time by practical engineers, and by gardeners in marking tennis courts, for measuring a right angle. This method seems also to have been known to the Chinese nearly three thousand years ago, but the Chinese made no serious attempt to classify or extend the few rules of arithmetic or geometry with which they were acquainted, or to explain the causes of the phenomena which they observed.

The geometrical theorem of which a particular case is involved in the method just described is well known to readers of the first book of Euclid's *Elements*. The Egyptians must probably have known that this theorem is true for a right-angled triangle when the sides containing the right angle are equal, for this is obvious if a floor be paved with tiles of that shape. But these facts cannot be said to show that geometry was then studied as a science. Our real knowledge of the nature of Egyptian geometry depends mainly on the Rhind papyrus.

The ancient Egyptian papyrus from the collection of Rhind, which was written by an Egyptian priest named Ahmes considerably more than a thousand years before Christ, and which is now in the British Museum, contains a fairly complete applied mathematics, in which the measurement of figures and solids plays the principal part; there are no theorems properly so called; everything is stated in the form of problems, not in general terms, but in distinct numbers. For example: to measure a rectangle the sides of which contain two and ten units of length; to find the surface of a circular area whose diameter is six units. We find also in it indications for the measurement of solids, particularly of pyramids, whole and truncated. The arithmetical problems dealt with in this papyrus—which, by the way, is headed "Directions for knowing all dark things"—contain some very interesting things. In modern language, we should say that the first part deals with the reduction of fractions whose numerators are 2 to a sum of fractions each of whose numerators is 1. Thus $2/29$ is stated to be the sum of $1/24$, $1/58$, $1/174$, and $1/232$. Probably Ahmes had no rule for forming the component fractions, and the answers given represent the accumulated experiences of previous writers. In one solitary case, however, he has indicated his method, for, after having asserted that $2/3$ is the sum of $1/2$ and $1/6$, he added that therefore two-thirds of one-fifth is equal to the sum of a half of a fifth and a sixth of a fifth, that is, to $1/10 + 1/30$.

That so much attention should have been paid to fractions may be explained by the fact that in early times their treatment presented considerable difficulty. The Egyptians and Greeks simplified the problem by reducing a fraction to the sum of several fractions, in each of which the numerator was unity, so that they had to consider only the various denominators: the sole exception to this rule being the fraction $2/3$. This remained the Greek practice until the sixth century of our era. The Romans, on the other hand, generally kept the denominator equal to twelve, expressing the fraction (approximately) as so many twelfths.

In Ahmes' treatment of multiplication, he seems to have relied on repeated additions. Thus, to multiply a certain number, which we will denote by the letter "a," by 13, he first multiplied by 2 and got $2a$, then he doubled the results and got $4a$, then he again doubled the result and got $8a$, and lastly he added together a, $4a$, and $8a$.

Now, we have used the sign "a" to stand for any number: not a particular number like 3, but *any* one. This is what Ahmes did, and what we learn to do in what we call "algebra." When Ahmes wished to find a number such that it, added to its seventh, makes 19, he symbolised the number by the sign we translate "heap." He had also signs for our "+," "−," and "=".[2] Nowadays we can write Ahmes' problem as: Find the number x

[2] In this book I shall take great care in distinguishing signs from what they signify. Thus 2 is to be distinguished from "2": by "2" I mean the *sign*, and the sign written without inverted commas indicates the thing signified. There has been, and is, much confusion, not only with beginners but with eminent mathematicians between a sign

such that $x + \frac{x}{7} = 19$. Ahmes gave the answer in the form $16 + \frac{1}{2} + \frac{1}{8}$.

We shall find that algebra was hardly touched by those Greeks who made of geometry such an important science, partly, perhaps, because the almost universal use of the abacus [3] rendered it easy for them to add and subtract without any knowledge of theoretical arithmetic. And here we must remember that the principal reason why Ahmes' arithmetical problems seem so easy to us is because of our use from childhood of the system of notation introduced into Europe by the Arabs, who originally obtained it from either the Greeks or the Hindoos. In this system an integral number is denoted by a succession of digits, each digit representing the product of that digit and a power of ten, and the number being equal to the sum of these products. Thus, by means of the local value attached to nine symbols and a symbol for zero, any number in the decimal scale of notation can be expressed. It is important to realise that the long and strenuous work of the most gifted minds was necessary to provide us with simple and expressive notation which, in nearly all parts of mathematics, enables even the less gifted of us to reproduce theorems which needed the greatest genius to discover. Each improvement in notation seems, to the uninitiated, but a small thing: and yet, in a calculation, the pen sometimes seems to be more intelligent than the user. Our notation is an instance of that great spirit of economy which spares waste of labour on what is already systematised, so that all our strength can be concentrated either upon what is known but unsystematised, or upon what is unknown.

Let us now consider the transformation of Egyptian geometry in Greek hands. Thales of Miletus (about 640–546 B.C.), who, during the early part of his life, was engaged partly in commerce and partly in public affairs, visited Egypt and first brought this knowledge into Greece. He discovered many things himself, and communicated the beginnings of many to his successors. We cannot form any exact idea as to how Thales presented his geometrical teaching. We infer, however, from Proclus that it consisted of a number of isolated propositions which were not arranged in a logical sequence, but that the proofs were deductive, so that the theorems were not a mere statement of an induction from a large number of special instances, as probably was the case with the Egyptian geometri-

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and what is signified by it. Many have even maintained that *numbers* are the *signs* used to represent them. Often, for the sake of brevity, I shall use the word in inverted commas (say "*a*") as short for "what we call '*a*,'" but the context will make plain what is meant.

[3] The principle of the abacus is that a number is represented by counters in a series of grooves, or beads strung on parallel wires; as many counters being put on the first groove as there are units, as many on the second as there are tens, and so on. The rules to be followed in addition, subtraction, multiplication, and division are given in various old works on arithmetic.

cians. The deductive character which he thus gave to the science is his chief claim to distinction. Pythagoras (born about 580 B.C.) changed geometry into the form of an abstract science, regarding its principles in a purely abstract manner, and investigated its theorems from the immaterial and intellectual point of view. Among the successors of these men, the best known are Archytas of Tarentum (428–347 B.C.), Plato (429–348 B.C.), Hippocrates of Chios (born about 470 B.C.), Menaechmus (about 375–325 B.C.), Euclid (about 330–275 B.C.), Archimedes (287–212 B.C.), and Apollonius (260–200 B.C.).

The only geometry known to the Egyptian priests was that of surfaces, together with a sketch of that of solids, a geometry consisting of the knowledge of the areas contained by some simple plane and solid figures, which they had obtained by actual trial. Thales introduced the ideal of establishing by exact reasoning the *relations* between the different parts of a figure, so that some of them could be found by means of others in a manner strictly rigorous. This was a phenomenon quite new in the world, and due, in fact, to the abstract spirit of the Greeks. In connection with the new impulse given to geometry, there arose with Thales, moreover, scientific astronomy, also an abstract science, and undoubtedly a Greek creation. The astronomy of the Greeks differs from that of the Orientals in this respect: the astronomy of the latter, which is altogether concrete and empirical, consisted merely in determining the duration of some periods or in indicating, by means of a mechanical process, the motions of the sun and planets; whilst the astronomy of the Greeks aimed at the discovery of the geometrical laws of the motions of the heavenly bodies.

Let us consider a simple case. The area of a right-angled field of length 80 yards and breadth 50 yards is 4000 square yards. Other fields which are not rectangular can be approximately measured by mentally dissecting them—a process which often requires great ingenuity and is a familiar problem to land-surveyors. Now, let us suppose that we have a circular field to measure. Imagine from the centre of the circle a large number of radii drawn, and let each radius make equal angles with the next ones on each side of it. By joining the points in succession where the radii meet the circumference of the circle, we get a large number of triangles of equal area, and the sum of the areas of all these triangles gives an approximation to the area of the circle. It is particularly instructive repeatedly to go over this and the following examples mentally, noticing how helpful the abstract ideas we call "straight line," "circle," "radius," "angle," and so on, are. We all of us know them, recognise them, and can easily feel that they are trustworthy and accurate ideas. We feel at home, so to speak, with the idea of a square, say, and can at once give details about it which are *exactly* true for it, and *very nearly* true for a field which we know is *very nearly* a square. This replacement in thought by an abstract geometrical object economises labour of thinking and imagining by leading us to concentrate our thoughts on that alone which is essential for our purpose.

Thales seems to have discovered—and it is a good thing to follow these discoveries on figures made with the help of compasses and ruler—the proof of what may be regarded as the obvious fact that the circle is divided into halves by its diameter, that the angles at the base of a triangle with two equal sides—an "isosceles" triangle—are equal, that all the triangles described in a semi-circle with two of their angular points at the ends of the diameter and the third anywhere on the circumference contain a right angle, and he measured the distance of vessels from the shore, presumably by causing two observers at a known distance apart to measure the two angles formed by themselves and the ship. This last discovery is an application of the fact that a triangle is determined if its base and base angles are given.

When Archytas and Menaechmus employed mechanical instruments for solving certain geometrical problems, "Plato," says Plutarch, "inveighed against them with great indignation and persistence as destroying and perverting all the good there is in geometry; for the method absconds from incorporeal and intellectual to sensible things, and besides employs again such bodies as require much vulgar handicraft: in this way *mechanics* was dissimilated and expelled from geometry, and, being for a long time looked down upon by philosophy, became one of the arts of war." In fact, manual labour was looked down upon by the Greeks, and a sharp distinction was drawn between the slaves, who performed bodily work and really observed nature, and the leisured upper classes, who speculated and often only knew nature by hearsay. This explains much of the naïve, hazy, and dreamy character of ancient natural science. Only seldom did the impulse to make experiments for oneself break through; but when it did, a great progress resulted, as was the case with Archytas and Archimedes. Archimedes, like Plato, held that it was undesirable for a philosopher to seek to apply the results of science to any practical use; but, whatever might have been his view of what ought to be the case, he did actually introduce a large number of new inventions.

We will not consider further here the development of mathematics with other ancient nations, nor the chief problems investigated by the Greeks; such details may be found in some of the books mentioned in the Bibliography at the end. The object of this chapter is to indicate the nature of the science of geometry, and how certain practical needs gave rise to investigations in which appears an abstract science which was worthy of being cultivated for its own sake, and which incidentally gave rise to advantages of a practical nature.

There are two branches of mathematics which began to be cultivated by the Greeks, and which allow a connection to be formed between the spirits of ancient and modern mathematics.

The first is the method of geometrical analysis to which Plato seems to have directed attention. The analytical method of proof begins by assuming that the theorem or problem is solved, and thence deducing some result. If the result be false, the theorem is not true or the problem is incapable of solution: if the result be true, if the steps be reversible, we get (by reversing them) a synthetic proof; but if the steps be not reversible, no conclusion can be drawn. We notice that the leading thought in analysis is that which is fundamental in algebra, and which we have noticed in the case of Ahmes: the calculation or reasoning with an unknown entity, which is denoted by a conventional sign, as if it were known, and the deduction at last, of some relation which determines what the entity must be.

And this brings us to the second branch spoken of: algebra with the later Greeks. Diophantus of Alexandria, who probably lived in the early half of the fourth century after Christ, and probably was the original inventor of an algebra, used letters for unknown quantities in arithmetic and treated arithmetical problems analytically. Juxtaposition of symbols represented what we now write as "+," and "−" and "=" were also represented by symbols. All these symbols are mere abbreviations for words, and perhaps the most important advantage of symbolism—the power it gives of carrying out a complicated chain of reasoning almost mechanically—was not made much of by Diophantus. Here again we come across the economical value of symbolism: it prevents the wearisome expenditure of mental and bodily energy on those processes which can be carried out mechanically. We must remember that this economy both emphasises the unsubjugated—that is to say, unsystematised—problems of science, and has a charm—an æsthetic charm, it would seem—of its own.

Lastly, we must mention "incommensurables," "loci," and the beginnings of "trigonometry."

Pythagoras was, according to Eudemus and Proclus, the discoverer of "incommensurable quantities." Thus, he is said to have found that the diagonal and the side of a square are "incommensurable." Suppose, for example, that the side of the square is one unit in length; the diagonal is longer than this, but it is not two units in length. The excess of the length of the diagonal over one unit is not an integral submultiple of the unit. And, expressing the matter arithmetically, the remainder that is left over after each division of a remainder into the preceding divisor is not an integral submultiple of the remainder used as divisor. That is to say, the rule given in text-books on arithmetic and algebra for finding the greatest common measure does not come to an end. This rule, when applied to integer numbers, always comes to an end; but, when applied to certain lengths, it does not. Pythagoras proved, then, that if we start with a line of any length, there are other lines whose lengths do not bear to the first length the ratio of one integer to another, no matter if we have all the integers to choose from. Of course, any two fractions have the ratio of two integers to one another. In the above case of the diagonal, if the diagonal *were* in length some number x of units, we should have $x^2 = 2$, and it can be proved that no fraction, when "multiplied"—in the sense to be given in the next chapter—by itself gives 2 exactly, though there are fractions which give this result more and more approximately.

On this account, the Greeks drew a sharp distinction between "numbers," and "magnitudes" or "quantities" or measures of lengths. This distinction was gradually blotted out as people saw more and more the advantages of identifying numbers with the measures of lengths. The invention of analytical geometry, described in the third chapter, did most of the blotting out. It is in comparatively modern times that mathematicians have adequately realised the importance of this logically valid distinction made by the Greeks. It is a curious fact that the abandonment of strictly logical thinking should have led to results which transgressed what was then known of logic, but which are now known to be readily interpretable in the terms of what we now know of Logic. This subject will occupy us again in the sixth chapter.

The question of *loci* is connected with geometrical analysis, and is difficult to dissociate from a mental picture of a point in motion. Think of a point under restrictions to move only in some curve. Thus, a point may move so that its distance from a fixed point is constant; the peak of an angle may move so that the arms of the angle pass—slipping—through two fixed points, and the angle is always a right angle. In both cases the moving point keeps on the circumference of a certain circle. This curve is a "locus." It is evident how thinking of the locus a point can describe may help us to solve problems.

We have seen that Thales discovered that a triangle is determined if its base and base angles are given. When we have to make a survey of either an earthly country or part of the heavens, for the purpose of map-making, we have to measure angles—for example, by turning a *sight*, like those used on guns, through an angle measured in a circular arc of metal —to fix the relative directions of the stars or points on the earth. Now, for terrestrial measurements, a piece of country is approximately a flat surface, while the heavens are surveyed as if the stars were, as they seem to be, scattered on the inside of a sphere at whose centre we are. Secondly, it is a network of *triangles*—plane or spherical—of which we measure the angles and sometimes the sides: for, if the angles of a triangle are known, the *proportionality* of the sides is known; and this proportionality cannot be concluded from a knowledge of the angles of a rectangle, say. Hipparchus (born about 160 B.C) seems to have invented this practical science of the complete measurement of triangles from certain data, or, as it is called, "trigonometry," and the principles laid down by him were worked out by Ptolemy of Alexandria (died 168 A.D.) and also by the Hindoos and Arabians. Usually, only angles can be measured with accuracy, and so the question arises: given the magnitude of the angles, what can be concluded as to the kind of proportionality of the sides. Think of a circle described round the centre *O*, and let *AP* be the arc of this circle which measures the angle *AOP*. Notice that the ratio of *AP* to the radius is the same for the angle *AOP* whatever value the radius may have. Draw *PM* perpendicular to *OA*. Then the figure *OPMAP* reminds one of a stretched bow, and hence are derived the names "sine of the arc *AP*" for the line *PM*, and "cosine" for *OM*. Tables of sines and cosines of arcs (or of angles, since the arc fixes the angle if the radius is known) were drawn up, and thus the sides *PM* and *OM* could be found in terms of the radius, when the arc was known. It is evident that this contains the essentials for the finding of the proportions of the sides of plane triangles. Spherical trigonometry contains more complicated relations which are directly relevant to the position of an astronomer and his measurements.

Mathematics did not progress in the hands of the Romans: perhaps the genius of this people was too practical. Still, it was through Rome that mathematics came into medieval Europe. The Arab mathematical textbooks and the Greek books from Arab translations were introduced into Western Europe by the Moors in the period 1150–1450, and by the end of the thirteenth century the Arabic arithmetic had been fairly introduced into Europe, and was practised by the side of the older arithmetic founded on the work of Boethius (about 475–526). Then came the Renascence. Mathematicians had barely assimilated the knowledge obtained from the Arabs, including their translations of Greek writers, when the refugees who escaped from Constantinople after the fall of the Eastern Empire (1453) brought the original works and the traditions of Greek science into Italy. Thus by the middle of the fifteenth century the chief results of Greek and Arabian mathematics were accessible to European students.

The invention of printing about that time rendered the dissemination of discoveries comparatively easy.

CHAPTER II

THE RISE AND PROGRESS OF MODERN MATHEMATICS—ALGEBRA

MODERN mathematics may be considered to have begun approximately with the seventeenth century. It is well known that the first 1500 years of the Christian era produced, in Western Europe at least, very little knowledge of value in science. The spirit of the Western Europeans showed itself to be different from that of the ancient Greeks, and only slightly less so from that of the more Easterly nations; and, when Western mathematics began to grow, we can trace clearly the historical beginnings of the use, in a not quite accurate form, of those conceptions—*variable* and *function*—which are characteristic of modern mathematics. We may say, in anticipation, that these conceptions, thoroughly analysed by reasoning as they are now, make up the difference of our modern views of Mathematics from, and have caused the likeness of them to, those of the ancient Greeks. The Greeks seem, in short, to have taken up a very similar position towards the mathematics of their day to that which logic forces us to take up towards the far more general mathematics of to-day. The generality of character has been attained by the effort to put mathematics more into touch with natural sciences—in particular the science of motion. The main difficulty was that, to reach this end, the way in which mathematicians expressed themselves was illegitimate. Hence philosophers, who lacked the real sympathy that must inspire all criticism that hopes to be relevant, never could discover any reason for thinking that what the mathematicians said was true, and the world had to wait until the mathematicians began logically to analyse their own conceptions. No body of men ever needed this sympathy more than the mathematicians from the revival of letters down to the middle of the nineteenth century, for no science was less logical than mathematics.

The ancient Greeks never used the conception of *motion* in their systematic works. The idea of a *locus* seems to imply that some curves could be thought of as generated by moving points; the Greeks discovered some things by helping their imaginations with imaginary moving points, but they never introduced the use of motion into their final proofs. This may have been because the Eleatic school, of which one of the principal representatives was Zeno (495–435 B.C.), invented some exceedingly subtle puzzles to emphasize the difficulty there is in the conception of motion. We shall return in some detail to these puzzles, which have not been appreciated in all the ages from the time of the Greeks till quite modern times. Owing to this lack of subtlety, the conception of variability was freely introduced into mathematics. It was the conceptions of *constant*, *variable*, and *function*, of which we shall, from now on, often have occasion to speak, which were generated by ideas of motion, and which, when they were logically purified, have made both modern mathematics and modern logic, to which they were transferred by mathematical logicians—Leibniz, Lambert, Boole, De Morgan, and the numerous successors of Boole and De Morgan from about 1850 onwards—into a science much more general than, but bearing some close analogies with, the ideal of Greek mathematical science. Later on will be found a discussion of what can be meant by a "moving point."

Let us now consider more closely the history of modern mathematics. Modern mathematics, like modern philosophy and like one part—the speculative and not the experimental part—of modern physical science, may be considered to begin with René Descartes (1596–1650). Of course, as we should expect, Descartes had many and worthy predecessors. Perhaps the greatest of them was the French mathematician François Viète (1540–1603), better known by his Latinized name of "Vieta." But it is simpler and shorter to confine our attention to Descartes.

Descartes always plumed himself on the independence of his ideas, the breach he made with the old ideas of the Aristotelians, and the great clearness and simplicity with which he described his ideas. But we must not underestimate the part that "ideas in the air" play; and, further, we know now that Descartes' breach with the old order of things was not as great as he thought.

Descartes, when describing the effect which his youthful studies had upon him when he came to reflect upon them, said:

"I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings: but I had not as yet a precise knowledge of their true use; and, thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations so strong and solid should have had no loftier superstructure reared on them."

And again:

"Among the branches of philosophy, I had, at an earlier period, given some attention to logic, and, among those of the mathematics, to geometrical analysis and algebra—three arts or sciences which ought, as I conceived, to contribute something to my design. But, on examination, I found that, as for logic, its syllogisms and the majority of its other precepts are of avail rather in the communication of what we already know, or even in speaking without judgment of things of which we are ignorant, than in the investigation of the unknown: and although this science contains indeed a number of correct and very excellent precepts, there are, nevertheless, so many others, and these either injurious or superfluous, mingled with the former, that it is almost quite as difficult to effect a severance of the true from the false as it is to extract a Diana or a Minerva from a rough block of marble. Then as to the analysis of the ancients and the algebra of the moderns: besides that they embrace only matters highly abstract, and, to appearance, of no use, the former is so exclusively restricted to the consideration of figures that it can exercise the understanding only on condition of greatly fatiguing the imagination; and, in the latter, there is so complete a subjection to certain rules and formulas, that there results an art full of confusion and obscurity, calculated to embarrass, instead of a science fitted to cultivate the mind. By these considerations I was induced to seek some other method which would comprise the advantages of the three and be exempt from their defects. . . .

"The long chains of simple and easy reasonings by means of which geometers are accustomed to reach the conclusions of their most difficult demonstrations had led me to imagine that all things to the knowledge of which man is competent are mutually connected in the same way, and that there is nothing so far removed from us as to be beyond our reach, or so hidden that we cannot discover it, provided only that we abstain from accepting the false for the true, and always preserve in our thoughts the order necessary for the deduction of one truth from another. And I had little difficulty in determining the objects with which it was necessary to begin, for I was already persuaded that it must be with the simplest and easiest to know, and, considering that, of all those who have hitherto sought truth in the sciences, the mathematicians alone have been able to find any demonstrations, that is, any certain and evident reasons, I did not doubt but that such must have been the rule of their investigations. I resolved to begin, therefore, with the examination of the simplest objects, not anticipating, however, from this any other advantage than that to be found in accustoming my mind to the love and nourishment of truth and to a distaste for all such reasonings as were unsound. But I had no intention on that account of attempting to master all the particular sciences commonly denominated 'mathematics'; but observing that, however different their objects, they all agree in considering only the various relations or proportions subsisting among those objects, I thought it best for my purpose to consider these proportions in the most general form possible, without referring them to any objects in particular, except such as would most facilitate the knowledge of them, and without by any means restricting them to these, that afterwards I might thus be the better able to apply them to every other class of objects to which they are legitimately applicable. Perceiving further that, in order to understand these relations, I should have sometimes to consider them one by one and sometimes only to bear in mind or embrace them in the aggregate, I thought that, in order the better to consider them individually, I should view them as subsisting between straight lines, than which I could find no objects more simple or capable of being more distinctly represented to my imagination and senses; and, on the other hand, that, in order to retain them in the memory, or embrace an aggregate of many, I should express them by certain characters the briefest possible. In this way I believed that I could borrow all that was best both in geometrical analysis and in algebra, and correct all the defects of the one by help of the other."

Let us, then, consider the characteristics of algebra and geometry.

We have seen, when giving an account, in the first chapter, of the works of Ahmes and Diophantus, that mathematicians early saw the advantage of representing an unknown number by a letter or some other sign that may denote various numbers ambiguously, writing down—much as in geometrical analysis—the relations which they bear, by the conditions of the problem, to other numbers, and then considering these relations. If the problem is determinate—that is to say, if there are one or more definite solutions which can be proved to involve only numbers already fixed upon—this consideration leads, by the use of certain rules of calculation, to the determination—actual or approximate—of this solution or solutions. Under certain circumstances, even if there is a solution, depending on a variable, we can find it and express it in a quite general way, by rules, but that need not occupy us here.

Suppose that you know my age, but that I do not know yours, but wish to. You might say to me: "I was eight years old when you were born." Then I should think like this. Let x be the (unknown) number of years in your age at this moment and, say, 33 the number of years in my age at this moment; then in essentials your statement can be translated by the equation "$x - 8 = 33$." The meaning of the signs "$-$," "$=$," and "$+$" are supposed to be known—as indeed they are by most people nowadays quite sufficiently for our present purpose. Now, one of the rules of algebra is that any term can be taken from one side of the sign "$=$" to the other if only the "$+$" or "$-$" belonging to it is changed into "$-$" or "$+$," as the case may be. Thus, in the present case, we have: "$x = 33 + 8 = 41$." This absurdly simple case is chosen intentionally. It is essential in mathematics to remember that even apparently insignificant economies of thought add up to make a long and complicated calculation readily performed. This is the case, for example, with the convention introduced by Descartes of using the last letters of the alphabet to denote unknown numbers, and the first letters to denote known ones. This convention is adopted, with a few exceptional cases, by algebraists to-day, and saves much trouble in explaining and in looking for unknown and known quantities in an equation. Then, again, the signs "$+$," "$-$," "$=$" have great merits which those unused to long calculations cannot readily understand. Even the saving of space made by writing "xy" for "$x \times y$" ("x multiplied by y") is important, because we can obtain by it a shorter and more readily surveyed formula. Then, too, Descartes made a general practice of writing "powers" or "exponents" as we do now; thus "x^3" stands for "xxx" and "x^5" for some less suggestive symbol representing the continued multiplication of five x's.

One great advantage of this notation is that it makes the explanation of logarithms, which were the great and laborious discovery of John Napier (1550–1617), quite easy. We start from the equation "$x^m x^n = x^{m+n}$." Now, if $x^p = y$, and we call p the "logarithm of y to the base x"; in signs: "$p = \log_x y$"; the equation from which we started gives, if we denote x^m by "u" and x^n by "v," so that $m = \log_x u$ and $n = \log_x v$, that $\log_x (uv) = \log_x u + \log_x v$. Thus, if the logarithms of numbers to a given base (say $x = 10$) are tabulated, calculations with large numbers are made less arduous, for *addition replaces multiplication*, when logarithms are found. Also subtraction of logarithms gives the logarithm of the quotient of two numbers.

Let us now shortly consider the history of algebra from Diophantus to Descartes.

The word "algebra" is the European corruption of an Arabic phrase which means *restoration and reduction*—the first word referring to the fact that the same magnitude may be added to or subtracted from both sides of an equation, and the last word meaning the process of simplification. The science of algebra was brought among the Arabs by Mohammed ben Musa (Mahomet the son of Moses), better known as Alkarismi, in a work written about 830 A.D., and was certainly derived by him from the Hindoos. The algebra of Alkarismi holds a most important place in the history of mathematics, for we may say that the subsequent Arab and the early medieval works on algebra were founded on it, and also that through it the Arabic or Indian system of decimal numeration was introduced into the West. It seems that the Arabs were quick to appreciate the work of others—notably of the Greek masters and of the Hindoo mathematicians—but, like the ancient Chinese and Egyptians, they did not systematically develop a subject to any considerable extent.

Algebra was introduced into Italy in 1202 by Leonardo of Pisa (about 1175–1230) in a work based on Alkarismi's treatise, and into England by Robert Record (about 1510–1558) in a book called the *Whetstone of Witte* published in 1557. Improvements in the method or notations of algebra were made by Record, Albert Girard (1595–1632), Thomas Harriot (1560–1621), Descartes, and many others.

In arithmetic we use *symbols of number.* A symbol is any sign for a quantity, which is not the quantity itself. If a man counted his sheep by pebbles, the pebbles would be symbols of the sheep. At the present day, when most of us can read and write, we have acquired the convenient habit of using marks on paper, "1, 2, 3, 4," and so on, instead of such things as pebbles. Our "1 + 1" is abbreviated into "2," "2 + 1" is abbreviated into "3," "3 + 1" into "4," and so on. When "1," "2," "3," &c., are used to abbreviate, rather improperly, "1 mile," "2 miles," "3 miles," &c., for instance, they are called signs for *concrete* numbers. But when we shake off all idea of "1," "2," &c., meaning one, two, &c., of anything in particular, as when we say, "six and four make ten," then the numbers are called *abstract* numbers. To the latter the learner is first introduced in treatises on arithmetic, and does not always learn to distinguish rightly between the two. Of the operations of arithmetic only addition and subtraction can be performed with concrete numbers, and without speaking of more than one sort of 1. Miles can be added to miles, or taken from miles. Multiplication involves a new sort of 1, 2, 3, &c., standing for *repetitions* (or *times*, as they are called). Take 6 miles 5 times. Here are two kinds of units, 1 mile and 1 time. In multiplication, one of the units must be a number of repetitions or times, and to talk of multiplying 6 feet by 3 feet would be absurd. What notion can be formed of 6 feet taken "3 feet" times? In solving the following question, "If 1 yard cost 5 shillings, how much will 12 yards cost?" we do not multiply the 12 *yards* by the 5 *shillings*; the process we go through is the following: Since each yard costs 5 shillings, the buyer must put down 5 shillings as often (as many times) as the seller uses a one-yard measure; that is, 5 shillings is taken 12 times. In division we must have the idea either of repetition or of *partition*, that is, of cutting a quantity into a number of equal parts. "Divide 18 miles by 3 *miles*" means, find out how many *times* 3 miles must be repeated to give 18 miles: but "divide 18 miles by 3" means, cut 18 miles into 3 equal parts, and find how many miles are in each part.

The symbols of arithmetic have a *determinate connection*; for instance, 4 is always 2 + 2, whatever the things mentioned may be, miles, feet, acres, &c. In algebra we take symbols for numbers which have no determinate connection. As in arithmetic we draw conclusions about 1, 2, 3, &c., which are equally true of 1 foot, 2 feet, &c., 1 minute, 2 minutes, &c.; so in algebra we reason upon numbers in general, and draw conclusions which are equally true of all numbers. It is true that we also use, in kinds of algebra which have been developed within the last century, letters to represent things other than numbers—for example, *classes* of individuals with a certain property, such as "horned animals," for logical purposes; or certain geometrical or physical things with directions in space, such as "forces"—and signs like "+" and "−" to represent ways of combination of the things, which are analogous to, but not identical with, addition and subtraction. If "*a*" denotes "the class of horned animals" and "*b*" denotes "the class of beasts of burden," the sign "*ab*" has been used to denote "the class of horned beasts of burden." We see that here $ab = ba$, just as in the multiplication of numbers, and the above operation has been called, partly for this reason, "logical multiplication," and denoted in the above way. Here we meet the practice of mathematicians—and of all scientific men—of using words in a wider sense for the sake of some analogy. This habit is all the more puzzling to many people because mathematicians are often not conscious that they do it, or even talk sometimes as if they thought that they were generalising *conceptions* instead of words. But, when we talk of a "family tree," we do not indicate a widening of our conception of trees of the roadside.

We shall not need to consider these modern algebras, but we shall be constantly meeting what are called the "generalisations of number" and transference of methods to analogous cases. Indeed, it is hardly too much to say that in this lies the very spirit of discovery. An example of this is given by the extension of the word "numbers" to include the names of *fractions as well.* The occasion for this extension was given by the use of arithmetic to express such quantities as distances. This had been done by Archimedes and many others, and had become the usual method of procedure in the works of the mathematicians of the sixteenth century, and plays a great part in Descartes' work.

Mathematicians, ever since they began to apply arithmetic to geometry, became alive to the fact that it was convenient to represent points on a straight line by numbers, and numbers by points on a straight line. What is meant by this may be described as follows. If we choose a unit of length, we can mark off points in a straight line corresponding to 0 units —which means that we select a point, called "the origin," to start from,— 1 unit, 2 units, 3 units, and so on, so that "the point *m*," as we will call it for short, is at a distance of *m* units from the origin. Then we can divide up the line and mark points corresponding to the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{8}$, $\frac{1}{13}$, $\frac{5}{3}$, or the point between 1 and 2 which is the same distance from 1 as $\frac{2}{3}$ is from 0, and so on. Now, there is nothing here to distinguish fractions from numbers. Both are treated exactly in the same way; the results of addition, subtraction, multiplication, and division [4] are interpretable, in much the same way as new points whether the "*a*" and "*b*" in

[4] The operation of what is called, for the sake of analogy, "multiplication" of fractions is defined in the manner indicated in the following example. If $\frac{3}{4}$ of a yard costs 10d., how much does $\frac{7}{8}$ of a yard cost? The answer is $\frac{10 \times 4 \times 7}{3 \times 8}$ pence, and we define $\frac{4 \times 7}{3 \times 8}$ as $\frac{1}{\frac{3}{4}}$ "multiplied by" $\frac{7}{8}$, by analogy with what would happen if $\frac{3}{4}$ were 1 and $\frac{7}{8}$ were, say, 3.

"$a + b$," "$a - b$," "ab," and so on, stand for numbers or fractions, and we have, for example,

$$a + b = b + a,\ ab = ba,\ a\,(b + c) = ab + ac,$$

always. Because of this very strong analogy, mathematicians have called the fractions "numbers" too, and they often speak and write of "generalisations of numbers," of which this is the first example, as if the conception of number were generalised, and not merely the *name* "number," in virtue of a great and close and important analogy.

When once the points of a line were made to represent numbers, there seemed to be no further difficulty in admitting certain "irrational numbers" to correspond to the end-points of the incommensurable lines which had been discovered by the Greeks. This question will come up again at a later stage: there are necessary discussions of principle involved, but mathematicians did not go at all deeply into questions of principle until fairly modern times. Thus it has happened that, until the last sixty years or so, mathematicians were nearly all bad reasoners, as Swift remarked of the mathematicians of Laputa in *Gulliver's Travels*, and were unpardonably hazy about first principles. Often they appealed to a sort of faith. To an intelligent and doubting beginner, an eminent French mathematician of the eighteenth century said: "Go on, and faith will come to you." It is a curious fact that mathematicians have so often arrived at truth by a sort of instinct.

Let us now return to our numerical algebra. Take, say, the number 8, and the fraction, which we will now call a "number" also, ⅛. Add 1 to both; the greater contains the less exactly 8 times. Now this property is possessed by *any* number, and not 8 alone. In fact, if we denote the number we start with by "a," we have, by the rules of algebra, $\dfrac{a+1}{{}^1/_a + 1} = a$. This is an instance of a general property of numbers proved by algebra.

Algebra contains many rules by which a complicated algebraical expression can be reduced to its simplest terms. Owing to the suggestive and compact notation, we can easily acquire an almost mechanical dexterity in dealing with algebraical symbols. This is what Descartes means when he speaks of algebra as not being a science fitted to cultivate the mind. On the other hand, this art is due to the principle of the economy of thought, and the mechanical aspect becomes, as Descartes foresaw, very valuable if we could use it to solve geometrical problems without the necessity of fatiguing our imaginations by long reasonings on geometrical figures.

I have already mentioned that the valuable notation "x^m" was due to Descartes. This was published, along with all his other improvements in algebra, in the third part of his *Geometry* of 1637. I shall speak in the next chapter of the great discovery contained in the first two parts of this work; here I will resume the improvements in notation and method made by Descartes and his predecessors, which make the algebraical part of the *Geometry* very like a modern book on algebra.

It is still the custom in arithmetic to indicate addition by juxtaposition: thus "2½" means "2 + ½." In algebra, we always, nowadays, indicate addition by the sign "+" and multiplication by juxtaposition or, more rarely, by putting a dot or the sign "×" between the signs of the numbers to be multiplied. Subtraction is indicated by "−".

Here we must digress to point out—what is often, owing to confusion of thought, denied in text-books—that, where "a" and "b" denote numbers, "$a - b$" can only denote a number if a is equal to or greater than b. If a is equal to b, the number denoted is zero; there is really no good reason for denying, say, that the numbers of Charles II.'s foolish sayings and wise deeds are equal, if a well-known epitaph be true. Here again we meet the strange way in which mathematics has developed. For centuries mathematicians used "negative" and "positive" numbers, and identified "positive" numbers with signless numbers like 1, 2, and 3, without any scruple, just as they used fractionary and irrational "numbers." And when logically-minded men objected to these wrong statements, mathematicians simply ignored them or said: "Go on; faith will come to you." And the mathematicians were right, and merely could not give correct reasons—or at least always gave wrong ones—for what they did. We have, over again, the fact that criticism of the mathematicians' procedure, if it wishes to be relevant, must be based on thorough sympathy and understanding. It must try to account for the rightness of mathematical views, and bring them into conformity with logic. Mathematicians themselves never found a competent philosophical interpreter, and so nearly all the interesting part of mathematics was left in obscurity until, in the latter half of the nineteenth century, mathematicians themselves began to cultivate philosophy—or rather logic.

Thus we must go out of the historical order to explain what "negative numbers" means. First, we must premise that when an algebraical expression is enclosed in brackets, it signifies that the whole result of that expression stands in the same relation to surrounding symbols as if it were one letter only. Thus, "$a - (b - c)$" means that from a we are to take $b - c$, or what is left after taking c from b. It is not, therefore, the same as $a - b - c$. In fact we easily find that $a - (b - c)$ is the same as $a - b + c$. Note also that "$(a + b)\ (c + d)$" means $(a + b)$ multiplied by $(c + d)$.

Now, suppose a and b are numbers, and a is greater than b. Let $a - b$ be c. To get c from a, we carry out the operation of taking away b. *This operation, which is the fulfilment of the order: "Subtract b," is a "negative number."* Mathematicians call it a "number" and denote it by "$-b$" simply because of analogy: the same rules for calculation hold for "negative numbers" and "positive numbers" like "$+b$," whose meaning is now clear too, as do for our signless numbers; when "addition," "subtraction," &c., are redefined for these operations. The way in which this redefinition must take place is evident when we represent integers, fractions, and positive and negative numbers by points on a straight line. To the right of 0 are the integers and fractions, to the left of 0 are the negative numbers, and to the right of 0 stretch the series of positive numbers, $+a$ coinciding with a and being symmetrically placed with $-a$ as regards 0. Also we determine that the operations of what we call "addition," &c., of these new "numbers" *must lead to the same results* as the former operations of the same name. Thus the same symbol is used in different senses, and we write

$$a + b - b = a + 0 = (+a) + (+b) + (-b) = +a = a.$$

This is a remarkable sequence of quick changes.

We have used the sign of equality, "=". It means originally, "is the same as." Thus $3 + 1 = 4$. But we write, by the above convention, "$a = +a$," and so we sacrifice exactness, which sometimes looks rather pedantic, for the sake of keeping our analogy in view, and for brevity.

Let us bear this, at first sight, puzzling but, at second sight, justifiable peculiarity of mathematicians in mind. It has always puzzled intelligent beginners and philosophers. The laws of calculation and convenient symbolism are *the* things a mathematician thinks of and aims at. He seems to identify different things if they both satisfy the same laws which are important to him, just as a magistrate may think that there is not much difference between Mr. A., who is red-haired and a tinker and goes to chapel, and Mr. B., who is a brown-haired horse-dealer and goes to church, if both have been found out committing petty larceny. But their respective ministers of religion or wives may still be able to distinguish them.

Any two expressions connected by the sign of equality form an "equation." Here we must notice that the words "Solve the equation $x^2 + ax = b$" mean: find the value or values of x such that, a and b being given numbers, $x^2 + ax$ becomes b. Thus, if $a = 2$ and $b = -1$, the solution is $x = -1$.

As we saw above, Descartes fixed the custom of employing the letters at the beginning of the alphabet to denote known quantities, and those at the end of the alphabet to denote unknown quantities. Thus, in the above example, a and b are some numbers supposed to be given, while x is sought. The question is solved when x is found in terms of a and b and fixed numbers (like 1, 2, 3); and so, when to a and b are attributed any fixed values, x becomes fixed. The signs "a" and "b" denote ambiguously, not uniquely like "2" does; and "x" does not always denote ambiguously when a and b are fixed. Thus, in the above case, when $a = 2$, $b = -1$, "x"

denotes the one negative number -1. What is meant is this: In each member of the class of problems got by giving a and b fixed values independently of one another, there is an unknown x, which may or may not denote different numbers, which only becomes known when the equation is solved. Consider now the equation $ax + by = c$, where a, b, and c are known quantities and x and y are unknown. We can find x in terms of a, b, c, and y, or y in terms of a, b, c, and x; but x is only fixed when y is fixed, or y when x is fixed. Here in each case of fixedness of a, b, and c, x is undetermined and "variable," that is to say, it may take any of a whole class of values. Corresponding to each x, one y belongs; and y also is a "variable" depending on the "independent variable" x. The idea of "variability" will be further illustrated in the next chapter; here we will only point out how the notion of what is called by mathematicians the "functional dependence" of y on x comes in. The variable y is said to be a "function" of the variable x if to every value of x corresponds one or more values of y. This use has, to some extent, been adopted in ordinary language. We should be understood if we were to say that the amount of work performed by a horse is a function of the food that he eats.

Descartes also adopted the custom—if he did not arrive at it independently—advocated by Harriot of transferring all the terms of an equation to the same side of the sign of equality. Thus, instead of "$x = 1$," "$ax + b = c$," and "$3x^2 + g = hx$," we write respectively "$x - 1 = 0$," "$ax + (b - c) = 0$," and "$3x^2 - hx + g = 0$." The point of this is that all equations of the same degree in the unknown—we shall have to consider cases of more unknowns than one in the next chapter—that is to say, equations in which the highest power of x (x or x^2 or x^3 . . .) is the same, are easily recognisable. Further, it is convenient to be able to speak of the expression which is equated to 0 as well as of the equation. The equations in which x^2, and no higher power of x, appears are called "quadratic" equations—the result of equating a "quadratic" function to 0; those in which x^3, and no higher power, appears are called "cubic"; and so on for equations "of the fourth, fifth . . ." degrees. Now the quadratic equations, $3x^2 + g = 0$, $ax^2 + bx + c = 0$, $x^2 - 1 = 0$, for example, are different, but the differences are unimportant in comparison with this common property of being of the same degree: all can be solved by modifications of one general method.

Here it is convenient again to depart from the historical order and briefly consider the meaning of what are called "imaginary" expressions. If we are given the equation $x^2 - 1 = 0$, its solutions are evidently $x = +1$ or $x = -1$, for the square roots of $+1$ are $+1$ and -1. But if we are given the equation $x^2 + 1 = 0$, analogy would lead us to write down the two solutions $x = +\sqrt{-1}$ and $x = -\sqrt{-1}$. But there is no positive or negative "number" which we have yet come across which, when multiplied by itself, gives a negative "number." Thus "imaginary numbers" were rejected by Descartes and his followers. Thus $x^2 - 1 = 0$ had two solutions, but $x^2 + 1 = 0$ none; further, $x^3 + x^2 + x + 1 = 0$ had one solution ($x = -1$), while $x^3 - x^2 - x + 1 = 0$ had two ($x = 1$, $x = -1$), and $x^3 - 2x^2 - x + 2 = 0$ had three ($x=1$, $x=-1$, $x=2$). Now, suppose, for a moment, that we can have "imaginary" roots and $(\sqrt{-1})\ (\sqrt{-1}) = -1$, and also that we can speak of *two* roots when the roots are identical in a case like the equation $x^2 + 2x + 1 = 0$, or $(x + 1)^2 = 0$, which has two identical roots $x = -1$. Then, in the above five equations, the first two quadratic ones have two roots each ($+1$, -1, and $+\sqrt{-1}$, $-\sqrt{-1}$ respectively), and the three cubics have three each (-1, $+\sqrt{-1}$, $-\sqrt{-1}$; $+1$, -1, $+1$; and $+1$, -1, $+2$ respectively). In the general case, the theorem has been proved that every equation has as many roots as (and not merely "no more than," as Descartes said) its degree has units. For this and for many other reasons like it in enabling theorems to be stated more generally, "imaginary numbers" came to be used almost universally. This was greatly helped by one puzzling circumstance: true theorems can be discovered by a process of calculation with imaginaries. The case is analogous to that which led mathematicians to introduce and calculate with "negative numbers."

For the case of imaginaries, let a, b, c, and d be any numbers, then

$$\begin{aligned}(a^2 + b^2)(c^2 + d^2) &= (a + b\sqrt{-1})(a - b\sqrt{-1})\\ &\qquad (c + d\sqrt{-1})(c - d\sqrt{-1})\\ &= (a + b\sqrt{-1})(c + d\sqrt{-1})\\ &\qquad (a - b\sqrt{-1})(c - d\sqrt{-1})\\ &= [(ac - bd) + \sqrt{-1}(ad + bc)]\\ &\qquad [(ac - bd) - \sqrt{-1}(ad + bc)]\\ &= (ac - bd)^2 + (ad + bc)^2.\end{aligned}$$

We get, then, an interesting and easily verifiable theorem on numbers by calculation with imaginaries, and imaginaries disappear from the conclusion. Mathematicians thought, then, that imaginaries, though apparently uninterpretable and even self-contradictory, *must* have a logic. So they were used with a faith that was almost firm and was only justified much later. Mathematicians indicated their growing security in the use of $\sqrt{-1}$ by writing "i" instead of "$\sqrt{-1}$" and calling it "the complex unity," thus denying, by implication, that there is anything really imaginary or impossible or absurd about it.

The truth is that "i" is not uninterpretable. It represents an operation, just as the negative numbers do, but is of a different kind. It is geometrically interpretable also, though not in a straight line, but in a plane. For this we must refer to the Bibliography; but here we must point out that, in this "generalisation of number" again, the words "addition," "multiplication," and so on, do not have exactly the same, but an analogous, meaning to those which they had before, and that "complex numbers" form a domain like a plane in which a line representing the integers, fractions, and irrationals is contained. But we must leave the further development of these questions.

It must be realised that the essence of algebra is its generality. In the most general case, every symbol and every statement of a proposition in algebra is interpretable in terms of certain operations to be undertaken with abstract things such as numbers or classes or propositions. These operations merely express the relations of these things to one another. If the results at any stage of an algebraical process can be interpreted—and this interpretation is often suggested by the symbolism—say, not as operations with operations with integers, but as other operations with integers, they express true propositions. Thus $(a+b)^2 = a^2 + 2ab + b^2$ expresses, for example, a relation holding between those operations with integers that we call "fractionary numbers," or an analogous relation between integers. The language of algebra is a wonderful instrument for expressing shortly, perspicuously, and suggestively, the exceedingly complicated relations in which abstract things stand to one another. The motive for studying such relations was originally, and is still in many cases, the close analogy of relations between certain abstract things to relations between certain things we see, hear, and touch in the world of actuality round us, and our minds are helped in discovering such analogies by the beautiful picture of algebraical processes made in space of two or of three dimensions made by the "analytical geometry" of Descartes, described in the next chapter.

CHAPTER III

THE RISE AND PROGRESS OF MODERN MATHEMATICS—ANALYTICAL GEOMETRY AND THE METHOD OF INDIVISIBLES

WE will now return to the consideration of the first two sections of Descartes' book *Geometry* of 1637.

In Descartes' book we have to glean here and there what we now recognise as the essential points in his new method of treating geometrical questions. These points were not expressly stated by him. I shall, however, try to state them in a small compass.

Imagine a curve drawn on a plane surface. This curve may be considered as a *picture* of an algebraical equation involving x and y in the following way. Choose any point on the curve, and call "x" and "y" the numbers that express the perpendicular distances of this point, in terms of a unit of length, from two straight lines (called "axes") drawn at right angles to one another in the plane mentioned. Now, as we move from point to point of the curve, x and y both vary, *but there is an unvarying relation which connects x and y, and this relation can be expressed by an algebraical equation called "the equation of the curve," and which contains, in germ as it were, all the properties of the curve considered.* This constant relation between x and y is a relation like $y^2 = 4ax$. We must distinguish carefully between a constant relation between variables and a relation between constants. We are always coming across the former kind of relation in mathematics; we call such a relation a "function" of x and y —the word was first used about fifty years after Descartes' *Geometry* was published, by Leibniz—and write a function of x and y in general as "$f(x, y)$." In this notation, no hint is given as to any particular relation x and y may bear to each other, and, in such a particular function as $y^2 - 4ax$, we say that "the *form* of the function is constant," and this is only another way of saying that the relation between x and y is fixed. This may be also explained as follows. If x is fixed, there is fixed one or more values of y, and if y is fixed, there is fixed one or more values of x. Thus the equation $ax + by + c = 0$ gives one y for each x and one x for each y; the equation $y^2 - 4ax = 0$ gives two y's for each x and one x for each y.[5]

Consider the equation $ax + by + c = 0$, or, say, the more definite instance $x + 2y - 2 = 0$. Draw axes and mark off points; having fixed on a unit of length, find the point $x = 1$ on the x-axis, on the perpendicular to this axis measure where the corresponding y, got by substituting $x = 1$ in the above equation, brings us. We find $y = ½$. Take $x = ⅓$, then $y = ⅚$; and so on. We find that all the points on the parallels to the y-axis lie on one straight line. This straight line is determined by the equation $x + 2y - 2 = 0$; every point off that straight line is such that its x and y are not connected by the relation $x + 2y - 2 = 0$, and every point of it is such that its x and y are connected by the relation $x + 2y - 2 = 0$. Similarly we can satisfy ourselves that every point on the circumference of a circle of radius c units of length, described round the point where the axes cross, is such that $x^2 + y^2 = c^2$, and every point not on this circumference does not have an x and y such that the constant relation $x^2 + y^2 = c^2$ is satisfied for it.

There are two points to be noticed in the above general statement. Firstly, I have said that the curve "*may* be expressed," and so on. By this I mean that it is possible—and not necessarily always true—that the curve

[5] We also denote a function of x by "$f(x)$" or "$F(x)$" or "$\phi(x)$," &c. Here "f" is a sign for "function of," not for a number, just as later we shall find "sin" and "Δ" and "d" standing for functions and not numbers. This may be regarded as an extension of the language of early algebra. The equation $y = f(x)$ is in a good form for graphical representation in the manner explained below.

may be so considered. We can imagine curves that cannot be represented by a finite algebraical equation. Secondly, about the fundamental lines of reference—the "axes" as they are called. One of these axes we have called the "x-axis," and the distance measured by the number x is sometimes called "the abscissa"; while the line of length y units which is perpendicular to the end of the abscissa farthest from the origin, and therefore parallel to the other axis ("the y-axis") is called "the ordinate." The name "ordinate" was used by the ancient Roman surveyors. The lines measured by the numbers x and y are called the "co-ordinates" of the point determining and determined by them. Sometimes the numbers x and y themselves are called "co-ordinates," and we will adopt that practice here.

Sometimes the axes are not chosen at right angles to one another, but it is nearly always far simpler to do so, and in this book we always assume that the axes are rectangular. The whole plane is divided by the axes into four partitions, the co-ordinates are measured from the point—called "the origin"—where the axes cross. Here the interpretation in geometry of the "negative quantities" of algebra—which so often seems so puzzling to intelligent beginners—gives us a means of avoiding the ambiguity arising from the fact that there would be a point with the same co-ordinates in each quadrant into which the plane is divided.

Consider the x-axis. Measure lengths on it from the origin, so that to the origin (O) corresponds the number 0. Let OA, measured from left to right along the axis, be the unit of length; then to the point A corresponds the number 1. Then let lengths AB, BC, and so on, all measured from left to right, be equal to OA in length; to the points B, C, and so on, correspond the numbers 2, 3, and so on. Further to the point that bisects OA let the fraction $\frac{1}{2}$ correspond; and so on for the other fractions. In this way half of the x-axis is nearly filled up with points. But there are points, such as the point P, such that OP is the length of the circumference of a circle, say of unit diameter. For picturesqueness, we may imagine this point P got by rolling the circle along the x-axis from O through one revolution. The point P will fall a little to the left of the point $3\frac{1}{7}$ and a little to the right of the point $3\frac{7}{50}$, and so on; the point P is not one of the points to which names of fractions have been assigned by the process sketched above. This can be proved rigidly. If it were not true, it would be very easy to "square the circle."

There are many other points like this. There is no fraction which, multiplied by itself, gives 2; but there is a length—the diagonal of a square of unit side—which is such that, if *we were to assume that a number corresponded to every point on OX*, it would be a number a such that $a^2 = 2$. We will return to this important question of the correspondence of points and lines to numbers, and will now briefly recall that "negative numbers" are represented, in Descartes' analytical geometry, on the x-axis, by the points to the left of the origin, and, on the y-axis, by the points below the origin. This was explained in the second chapter.

Algebraical geometry gave us a means of classifying curves. All straight lines determine equations of the first degree between x and y, and all such equations determine straight lines; all equations of the second degree between x and y, that is to say, of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

determine curves which the ancient Greeks had studied and which result from cutting a solid circular cone, or two equal cones with the same axis, whose only point of contact is formed by the vertices. It is somewhat of a mystery why the Greek geometricians should have pitched upon these particular curves to study, and we can only say that it seems, from the present standpoint, an exceedingly lucky chance. For these "conic sections"—of which, of course, the circle is a particular case—are all the curves, and those only, which are represented by the above equation of the second degree. The three great types of curves—the "parabola," the "ellipse," and the "hyperbola"—all result from the above equation when the coefficients a, b, c, d, e, f satisfy certain special conditions. Thus, the equation of a circle—which is a particular kind of ellipse—is always of the form got from the above equation by putting $b = 0$ and $c = a$.

It may be mentioned that, long after these curves were introduced as sections of a cone, Pappus discovered that they could all be defined in a plane as loci of a point P which moves so that the proportion that the distance of P from a fixed point (S) bears to the perpendicular distance of P (PN) to a fixed straight line is constant. As this proportion is less than equal to or greater than 1, the curve is an ellipse, parabola, or hyperbola, respectively.

It will not be expected that a detailed account should here be given of the curves which result from the development of equations of the second or higher degrees between x and y. I will merely again emphasize some points which are, in part, usually neglected or not clearly stated in text-books. The letters "$a, b, \ldots x, y,$" here stand for "numbers" in the extended sense. We have seen in what sense we may, with the mathematicians, speak of fractionary, positive, and negative "numbers," and identify, say, the positive number $+2$ and the fraction $2/1$ with the signless integer 2. Well, then, the above letters stand for numbers of that class which includes in this sense the fractionary, irrational, positive and negative numbers, but excludes the imaginary numbers. We call the numbers of this class "real" numbers. The question of irrational numbers will be discussed at greater length in the sixth chapter, but enough has been said to show how they were introduced. In mathematics it has, I think, always happened that conceptions have been used long before they were formally introduced, and used long before this use could be logically justified or its nature clearly explained. The history of mathematics is the history of a faith whose justification has been long delayed, and perhaps is not accomplished even now.

These numbers are the measurements of length, in terms of a definite unit, like the inch, of the abscissæ and ordinates of certain points. We speak of such points simply by naming their co-ordinates, and say, for example, that "the distance of the point (x, y) from the point (a, b) is the positive square root of $(x-a)^2+(y-b)^2$."

Notice that x^2, for example, is the length of a *line*. It is natural to make, as algebraists before Descartes did, x^2 stand *primarily* for the number of square units in a square whose sides are x units in length, but there is no necessity in this. We shall often use the latter kind of measurement in the fourth and fifth chapters.

The equation of a straight line can be made to satisfy two given conditions. We can write the equation in the form

$$x+\frac{b}{a}y+\frac{c}{a}=0,$$

and thus have two ratios, $\frac{b}{a}$ and $\frac{c}{a}$, that we can determine according to the conditions. The equation $ax+by+c=0$ has apparently *three* "arbitrary constants," as they are called, but we see that this greater generality is only apparent. Now we can so fix these constants that two conditions are fulfilled by the straight line in question. Thus, suppose that one of these conditions is that the straight line should pass through the origin—the point $(0, 0)$. This means simply that when $x=0$, then $y=0$. Putting them, $x=0$ and $y=0$ in the above equation, we get $\frac{c}{a}=0$, and thus one of the constants is determined. The other is determined by a new condition that, say, the line also passes through the point $(⅓, 2)$. Substituting, then, in the above equation, we have, as $\frac{c}{a}=0$ as we know already, $⅓+\frac{2b}{a}=0$, whence $\frac{b}{a}=-⅙$. Hence the equation of the line passing through $(0, 0)$ and $(⅓, 2)$ is $x-⅙y=0$, or $y-6x=0$. Instead of having to pass through a certain point, a condition may be, for example, that the perpendicular from the origin on the straight line should be of a certain length, or that the line should make a certain angle with the x-axis, and so on.

Similarly, the circle whose equation is written in the form

$$(x-a)^2+(y-b)^2=c^2$$

is of radius c and centre (a, b). It can be determined to pass through any *three* points, or, say, to have a determined length of radius and position of centre. Fixation of centre is equivalent to two conditions. Thus, suppose the radius is to be of unit length: the above equation is $(x-a)^2+(y-b)^2=1$. Then, if the centre is to be the origin, both a and b are determined to be 0, and this may also be effected by determining that the circle is to pass through the points $(\frac{1}{2}, 0)$ and $(-\frac{1}{2}, 0)$, for example.

Now, if we are to find the points of intersection of the straight line $2x+2y=1$ and the circle $x^2+y^2=1$, we seek those points which are common to both curves, that is to say, all the pairs of values of x and y which satisfy *both* the above equations. Thus we need not trouble about the geometrical picture, but we only have to apply the rules of algebra for finding the values of x and y which satisfy two "simultaneous" equations in x and y. In the above case, if (X, Y) is a point of intersection, we have $Y=\dfrac{1-2X}{2}$, and therefore, by substitution in the other equation, $X^2+\left(\dfrac{1-2X}{2}\right)^2=1$. This gives a quadratic equation

$$8X^2-4X-3=0$$

for X, and, by rules, we find that X must be either $\frac{1}{4}(1+\sqrt{7})$ or $\frac{1}{4}(1-\sqrt{7})$. Hence there are *two* values of the abscissa which are given when we ask what are the co-ordinates of the points of intersection; and the value of y which corresponds to each of these x's is given by substitution in the equation $2x+2y=1$.

Thus we find again the fact, obvious from a figure, that a straight line cuts [a circle] at two points at most. We can determine the points of intersection of any two curves whose equations can be expressed algebraically, but of course the process is much more complicated in more general cases. Here we will consider an important case of intersection of a straight line.

Think of a straight line cutting a circle at two points. Imagine one point fixed and the other point moved up towards the first. The intersecting line approaches more and more to the position of the tangent to the circle at the first point, and, by making the movable point approach the other closely enough, the secant will approach the tangent in position as nearly as we wish. Now, a tangent to a curve at a certain point was defined by the Greeks as a straight line through the point such that between it and the curve no other *straight line* could be drawn. Note that other *curves* might be drawn: thus various circles may have the same tangent at a common point on their circumference, but no circle—and no curve met with in elementary mathematics—has more than one tangent at a point. Descartes and many of his followers adopted different forms of definition which really involve the idea of a *limit,* an idea which appears boldly in the infinitesimal calculus. A tangent is the *limit* of a secant as the points of intersection approach infinitely near to one another; it is a produced side of the polygon with infinitesimal sides that the curve is supposed to be; it is the direction of motion at an instant of a point moving in the curve considered. The equation got from that of the curve by substituting for y from the equation of the intersecting straight line has, if this straight line is a tangent, two equal roots. In the above case, this equation was quadratic. In the case of a circle, we can easily deduce the well-known property of a tangent of being perpendicular to the radius; and see that this property has no analogue in the case of other curves.

We must remember that, just as *plane* curves determine and are determined by equations with *two* independent variables x and y, so surfaces—spheres, for instance—in three-dimensional space determine and are determined by equations with *three* independent variables, x, y, and z. Here x, y, and z are the co-ordinates of a point in space; that is to say, the numerical measures of the distances of this point from three fixed planes at right angles to each other. Thus, the equation of a sphere of radius d and centre at (a, b, c) is $(x-a)^2+(y-b)^2+(z-c)^2=d^2$.

We may look at analytical geometry from another point of view which we shall find afterwards to be important, and which even now will suggest to us some interesting thoughts. The essence of Descartes' method also appears when we represent *loci* by the method. Consider a circle; it is the locus of a point (P) which moves in a plane so as to preserve a constant distance from a fixed point (O). Here we may think of P as varying in position, and make up a very striking picture of what we call a *variable* in mathematics. We must, however, remember that, by what we call a "variable" for the sake of picturesqueness, we do not necessarily mean something which varies. Think of the point of a pen as it moves over a sheet of writing paper; it occupies different positions with respect to the paper at different times, and we understandably say that the pen's point moves. But now think of a point in space. A geometrical point—which is not the bit of space occupied by the end of a pen or even an "atom" of matter—is merely a mark of position. We cannot, then, speak of a point moving; the very essence of point is to *be* position. The motion of a point of *space,* as distinguished from a point of matter, is a fiction, and is the supposition that a given point can now be one point and now another. Motion, in the ordinary sense, is only possible to matter and not to space.

Thus, when we speak of a "variable position," we are speaking absurdly if we wish our words to be taken literally. But we do not really so wish when we come to think about it. What we are doing is this: we are using a picturesque phrase for the purpose of calling up an easily imagined thought which helps us to visualise roughly a mathematical proposition which can only be described accurately by a prolix process. The ancient Greeks allowed prolixity, and it was only objected to by the uninitiated. Modern mathematics up to about sixty years ago successfully warred against prolixity; hence the obscurity of its fundamental notions and processes and its great conquests. The great conquests were made by sacrificing very much to analogy: thus, entities like the integer 2, the ratio 2/1, and the real number which is denoted by "2" were identified, as we have seen, because of certain close analogies that they have. This seems to have been the chief reason why the procedure of the mathematicians has been so often condemned by logicians and even by philosophers. In fact, when mathematicians began to try to find out the nature of Mathematics, they had to examine their entities and the methods which they used to deal with them with the minutest care, and hence to look out for the points when the analogies referred to break down, and distinguish between what mathematicians had usually failed to distinguish. Then the people who do not mind a bit what Mathematics *is,* and are only interested in what it *does,* called these earnest inquirers "pedants" when they should have said "philosophers," and "logic-choppers"—whatever they may be—when they should have said "logicians." We have tried to show why ratios or fractions, and so on, are called "numbers," and apparently said to be something which they are not; we must now try to get at the meaning of the words "constant" and "variable."

By means of algebraic formulæ, rules for the reconstruction of great numbers—sometimes an infinity—of facts of nature may be expressed very concisely or even embodied in a *single* expression. The essence of the formula is that it is an expression of a *constant* rule among *variable* quantities. These expressions "constant" and "variable" have come down into ordinary language. We say that the number of miles which a certain man walks per day is a "variable quantity"; and we do not mean that, on a particular day, the number was not fixed and definite, but that on different days he walked, generally speaking, different numbers of miles. When, in mathematics, we speak of a "variable," what we mean is that we are considering a class of definite objects—for instance, the class of men alive at the present moment—and want to say something about *any one* of them indefinitely. Suppose that we say: "If it rains, Mr. A. will take his umbrella out with him"; the letter "A" here is what we call the sign of the "variable." We do not mean that the above proposition is about a *variable man*. There is no such thing; we say that a man varies in health and so *in time,* but, whether or not such a phrase is strictly correct, the meaning we would have to give the phrase "a variable" in the above sentence is not one and the same man at different periods of his own existence, but one and the same man who is different men in turn. What we mean is that if "A" denotes any man, and not Smith or Jones or Robinson *alone,* then he takes out his umbrella on certain occasions. The statement is not always true; it depends on A. If "A" stands for a bank manager, the statement may be true; if for a tramp or a savage, it probably is not. Instead of "A," we may put "B" or "C" or "X"; the kind of mark on paper does not really matter in the least. But we attach, by convention, certain meanings to certain signs; and so, if we wrote down a mark of exclamation for the sign of a variable, we might be misunderstood and even suspected of trying to be funny. We shall see, in the seventh chapter, the importance of the variable in logic and mathematics.

"Laws of nature" express the dependence upon one another of two or more variables. This idea of dependence of variables is fundamental in all scientific thought, and reaches its most thorough examination in mathematics and logic under the name of "functionality." On this point we must refer back to the second chapter. The ideas of function and variable were not prominent until the time of Descartes, and names for these ideas were not introduced until much later.

The conventions of analytical geometry as to the signs of co-ordinates in different quadrants of the plane had an important influence in the transformation of trigonometry from being a mere adjunct to a practical science. In the same notation as that used at the end of the first chapter, we may conveniently call the number $\frac{AP}{OP}$, which is the same for all lengths of OP, by the name "u," for short, and define $\frac{PM}{OP}$ and $\frac{OM}{OP}$ as the "sine of u," and the "cosine of u" respectively. Thus "sin u" and "cos u," as we write them for short, stand for numerical functions of u. Considering O as the origin of a system of rectangular co-ordinates of which OA is the x-axis, so that u measures the angle POA and $\frac{x}{r}$ and $\frac{y}{r}$ are cos u and sin u respectively. Now, even if u becomes so great that POA is successively obtuse, more than two right angles . . . , these definitions can be preserved, if we pay attention to the signs of x and y in the various quadrants. Thus sin u and cos u become separated from geometry, and appear as numerical functions of the variable u, whose values, as we see on reflection, repeat themselves at regular intervals as u becomes larger and larger. Thus, suppose that OP turns about O in a direction opposite to that in which the hands of a clock move. In the first quadrant, sin u

and cos u are $\frac{y}{r}$ and $\frac{x}{r}$; in the second they are $\frac{y}{r}$ and $\frac{-x}{r}$; in the third they are $\frac{-y}{r}$ and $\frac{-x}{r}$; in the fourth they are $\frac{-y}{r}$ and $\frac{x}{r}$; in the fifth they are $\frac{y}{r}$ and $\frac{x}{r}$ again; and so on. Trigonometry was separated from geometry mainly by John Bernoulli and Euler, whom we shall mention later.

We will now turn to a different development of mathematics.

The ancient Greeks seem to have had something approaching a general method for finding areas of curvilinear figures. Indeed, infinitesimal methods, which allow indefinitely close approximation, naturally suggest themselves. The determination of the area of any rectilinear figure can be reduced to that of a rectangle, and can thus be completely effected. But this process of finding areas—this "method of quadratures"—failed for areas or volumes bounded by curved lines or surfaces respectively. Then the following considerations were applied. When it is impossible to find the exact solution of a question, it is natural to endeavour to approach to it as nearly as possible by neglecting quantities which embarrass the combinations, if it be foreseen that these quantities which have been neglected cannot, by reason of their small value, produce more than a trifling error in the result of the calculation. For example, as some properties of curves with respect to areas are with difficulty discovered, it is natural to consider the curves as polygons of a great number of sides. If a regular polygon be supposed to be inscribed in a circle, it is evident that these two figures, although always different, are nevertheless more and more alike according as the number of the sides of the polygon increases. Their perimeters, their areas, the solids formed by their revolving round a given axis, the angles formed by these lines, and so on, are, if not respectively equal, at any rate so much the nearer approaching to equality as the number of sides becomes increased. Whence, by supposing the number of these sides very great, it will be possible, without any perceptible error, to assign to the circumscribed circle the properties that have been found belonging to the inscribed polygon. Thus, if it is proposed to find the area of a given circle, let us suppose this curve to be a regular polygon of a great number of sides: the area of any regular polygon whatever is equal to the product of its perimeter into the half of the perpendicular drawn from the centre upon one of its sides; hence, the circle being considered as a polygon of a great number of sides, its area ought to equal the product of the circumference into half the radius. Now, this result is exactly true. However, the Greeks, with their taste for strictly correct reasoning, could not allow themselves to consider curves as polygons of an "infinity" of sides. They were also influenced by the arguments of Zeno, and thus regarded the use of "infinitesimals" with suspicion.

Zeno showed that we meet difficulties if we hold that time and space are infinitely divisible. Of the arguments which he invented to show this, the best known is the puzzle of Achilles and the Tortoise. Zeno argued that, if Achilles ran ten times as fast as a tortoise, yet, if the tortoise has (say) 1000 yards start, it could never be overtaken. For, when Achilles had gone the 1000 yards, the tortoise would still be 100 yards in front of him; by the time he had covered these 100 yards, it would still be 10 yards in front of him; and so on for ever; thus Achilles would get nearer and nearer to the tortoise, but never overtake it. Zeno invented some other subtle puzzles for much the same purpose, and they could only be discussed really satisfactorily by quite modern mathematics.

To avoid the use of infinitesimals, Eudoxus (408–355 B.C.) devised a method, exposed by Euclid in the Twelth Book of his *Elements* and used by Archimedes to demonstrate many of his great discoveries, of verifying results found by the doubtful infinitesimal considerations. When the Greeks wished to discover the area bounded by a curve, they regarded the curve as the fixed boundary to which the inscribed and circumscribed polygons approach continually, and as much as they pleased, according as they increased the number of their sides. Thus they exhausted in some measure the space comprised between these polygons and the curve, and doubtless this gave to this operation the name of "the method of exhaustion." As these polygons terminated by straight lines were known figures, their continual approach to the curve gave an idea of it more and more precise, and "the law of continuity" serving as a guide, the Greeks could eventually arrive at the exact knowledge of its properties. But it was not sufficient for geometricians to have observed, and, as it were, guessed at these properties; it was necessary to verify them in an unexceptionable way; and this they did by proving that every supposition contrary to the existence of these properties would necessarily lead to some contradiction: thus, after, by infinitesimal considerations, they had found the area (say) of a curvilinear figure to be a, they verified it by proving that, if it is not a, it would yet be greater than the area of some polygon inscribed in the curvilinear figure whose area is palpably greater than that of the polygon.

In the seventeenth century, we have a complete contrast with the Grecian spirit. The method of discovery seemed much more important than correctness of demonstration. About the same time as the invention of analytical geometry by Descartes came the invention of a method for finding the areas of surfaces, the positions of the centres of gravity of variously shaped surfaces, and so on. In a book published in 1635, and in certain later works, Bonaventura Cavalieri (1598–1647) gave his "method of indivisibles" in which the cruder ideas of his predecessors, notably of Kepler (1571–1630), were developed. According to Cavalieri, a line is made up of an infinite number of points, each without magnitude, a surface of an infinite number of lines, each without breadth, and a volume of an infinite number of surfaces, each without thickness. The use of this idea may be illustrated by a single example. Suppose it is required to find the area of a right-angled triangle. Let the base be made up of n points (or indivisibles), and similarly let the side perpendicular to the base be made of na points, then the ordinates at the successive points of the base will contain $a, 2a \ldots, na$ points. Therefore the number of points in the areas is $a + 2a + \ldots + na$; the sum of which is $\frac{1}{2}(n^2a + na)$. Since n is very large, we may neglect $\frac{1}{2}na$, for it is inconsiderable compared with $\frac{1}{2}n^2a$. Hence the area is composed of a number $\frac{1}{2}(na)n$ of points, and thus the area is measured in square units by multiplying half the linear measure of the altitude by that of the base. The conclusion, we know from other facts, is exactly true.

Cavalieri found by this method many areas and volumes and the centres of gravity of many curvilinear figures. It is to be noticed that both Cavalieri and his successors held quite clearly that such a supposition that lines were composed of points was literally absurd, but could be used as a basis for a direct and concise method of abbreviation which replaced with advantage the indirect, tedious, and rigorous methods of the ancient Greeks. The logical difficulties in the principles of this and allied methods were strongly felt and commented on by philosophers—sometimes with intelligence; felt and boldly overcome by mathematicians in their strong and not unreasonable faith; and only satisfactorily solved by mathematicians—not the philosophers—in comparatively modern times.

The method of indivisibles—whose use will be shown in the next chapter in an important question of mechanics—is the same in principle as "the integral calculus." The integral calculus grew out of the work of Cavalieri and his successors, among whom the greatest are Roberval (1602–1675), Blaise Pascal (1623–1662), and John Wallis (1616–1703), and mainly consists in the provision of a convenient and suggestive notation for this method. The discovery of the infinitesimal calculus was completed by the discovery that the inverse of the problem of finding the areas of figures enclosed by curves was the problem of drawing tangents to these curves, and the provision of a convenient and suggestive notation for this inverse and simpler method, which was, for certain historical reasons, called "the differential calculus."

Both analytical geometry and the infinitesimal calculus are enormously powerful instruments for solving geometrical and physical problems. The secret of their power is that long and complicated reasonings can be written down and used to solve problems almost mechanically. It is the merest superficiality to despise mathematicians for busying themselves, sometimes even consciously, with the problem of economising thought. The powers of even the most god-like intelligences amongst us are extremely limited, and none of us could get very far in discovering any part whatever of the Truth if we could not make trains of reasoning which we have thought through and verified, very ready for and easy in future application by being made as nearly mechanical as possible. In both analytical geometry and the infinitesimal calculus, all the essential properties of very many of the objects dealt with in mathematics, and the essential features of very many of the methods which had previously been devised for dealing with them are, so to speak, packed away in a well-arranged (and therefore readily got at) form, and in an easily usable way.

CHAPTER IV

THE BEGINNINGS OF THE APPLICATION OF MATHEMATICS TO NATURAL SCIENCE—THE SCIENCE OF DYNAMICS

THE end of very much mathematics—and of the work of many eminent men—is *the simple and, as far as may be, accurate description* of things in the world around us, of which we become conscious through our senses.

Among these things, let us consider, say, a particular person's face, and a billiard ball. The appearance to the eye of the ball is obviously much easier to describe than that of the face. We can call up the image—a very accurate one—of a billiard ball in the mind of a person who has never seen it by merely giving the colour and radius. And, unless we are engaged in microscopical investigations, this description is usually enough. The description of a face is a harder matter: unless we are skilful modellers, we cannot do this even approximately; and even a good picture does not attempt literal accuracy, but only conveys a correct impression—often better than a model, say in wax, does.

Our ideal in natural science is to build up a working model of the universe out of the sort of ideas that all people carry about with them everywhere "in their heads," as we say, and to which ideas we appeal when we try to teach mathematics. These ideas are those of *number, order,* the numerical measures of *times* and *distances,* and so on. One reason why we strive after this ideal is a very practical one. If we have a working model of, say, the solar system, we can tell, in a few minutes, what our position with respect to the other planets will be at all sorts of far future times, and can thus *predict certain future events.* Everybody can see how useful this is: perhaps those persons who see it most clearly are those sailors who use the *Nautical Almanac.* We cannot make the earth tarry in its revolution round its axis in order to give us a longer day for finishing some important piece of work; but, by finding out the unchanging laws concealed in the phenomena of the motions of earth, sun, and stars, the mathematician can construct the model just spoken of. And the mathematician is completely master of his model; he can repeat the occurrences in his universe as often as he likes; something like Joshua, he can make his "sun" stand still, or hasten, in order that he may publish the *Nautical Almanac* several years ahead of time. Indeed, the "world" with which we have to deal in theoretical or mathematical mechanics is but a mathematical scheme, the function of which it is to imitate, by logical consequences of the properties assigned to it by definition, certain processes of nature as closely as possible. Thus our "dynamical world" may be called a model of reality, and must not be confused with the reality itself.

That this model of reality is constructed solely out of logical conceptions will result from our conclusion that mathematics is based on logic, and on logic alone; that such a model is possible is really surprising on reflection. The need for completing facts of nature in thought was, no doubt, first felt as a *practical* need—the need that arises because we feel it convenient to be able to predict certain kinds of future events. Thus, with a purely mathematical model of the solar system, we can tell, with an approximation which depends upon the completeness of the model, the relative positions of the sun, stars, and planets several years ahead of time; this it is that enables us to publish the *Nautical Almanac,* and makes up to us, in some degree, for our inability "to grasp this sorry scheme of things entire . . . and remould it nearer to the heart's desire."

Now, what is called "mechanics" deals with a very important part of the structure of this model. We spoke of a billiard ball just now. Everybody gets into the way, at an early age, of abstracting from the colour, roughness, and so on, of the ball, and forming for himself the conception of a *sphere.* A sphere can be exactly described; and so can what we call a "square," a "circle" and an "ellipse," in terms of certain conceptions such as those called "point," "distance," "straight line," and so on. Not so easily describable are certain other things, like a person or an emotion. In the world of moving and what we roughly class as *inanimate* objects—that is to say, objects whose behaviour is not perceptibly complicated by the phenomena of what we call "life" and "will"—people have sought from very ancient times, and with increasing success, to discover rules for the motions and rest of given systems of objects (such as a lever or a wedge) under given circumstances (pulls, pressures, and so on). Now, this discovery means: the discovery of an ideal, exactly describable motion which should approximate as nearly as possible to a natural motion or class of motions. Thus Galileo (1564–1642) discovered the approximate law of bodies falling freely, or on an inclined plane, near the earth's surface; and Newton (1642–1727) the still more accurate law of the motions of any number of bodies under any forces.

Let us now try to think clearly of what we mean by such a rule, or, as it is usually called, a "scientific" or "natural law," and why it plays an important part in the arrangement of our knowledge in such a convenient way that we can at once, so to speak, lay our hand on any particular fact the need of which is shown by practical or theoretical circumstances.

For this purpose, we will see how Galileo, in a work published in 1638, attacked the problem of a falling body. Consider a body falling freely to the earth: Galileo tried to find out, not *why* it fell, but *how* it fell—that is to say, in what mathematical form the distance fallen through and the velocity attained depends on the time taken in falling and the space fallen through. Freely falling bodies are followed with more difficulty by the eye the farther they have fallen; their impact on the hand receiving them is, in like measure, sharper; the sound of their striking louder. The velocity accordingly increases with the time elapsed and the space traversed. Thus, the modern inquirer would ask: What function is the number (v) representing the velocity of those (s and t) representing the distance fallen through and the time of falling? Galileo asked, in his primitive way: Is v proportional to s; or again, is v proportional to t? Thus he made assumptions, and then *ascertained by actual trial the correctness or otherwise of these assumptions.*

One of Galileo's assumptions was, thus, that the velocity acquired in the descent is proportional to the time of the descent. That is to say, if a body falls once, and then falls again during twice as long an interval of time as it first fell, it will attain in the second instance double the velocity it acquired in the first. To find by experiment whether or not this assumption accorded with observed facts, as it was difficult to prove by any direct means that the velocity acquired was proportional to the time of descent, but easier to investigate by what law the distance increased with the time, *Galileo deduced from his assumption the relation that obtained between the distance and the time.* This very important deduction he effected as follows.

On the straight line *OA*, let the abscissæ *OE, OC, OG,* and so on, represent in length various lengths of time elapsed from a certain instant represented by *O*, and let the ordinates *EF, CD, GH,* and so on, corresponding to these abscissæ, represent in length the magnitude of the velocities acquired at the time represented by the respective abscissæ.

We observe now that, by our assumption, *O, F, D, H, lie in a straight line OB,* and so: (1) At the instant *C*, at which one-half *OC* of the time of descent *OA* has elapsed, the velocity *CD* is also one-half of the final velocity AB; (2) If E and G are equally distant in opposite directions on OA from C, the velocity GH exceeds the mean velocity CD by the same amount that the velocity EF falls short of it; and for every instant antecedent to C there exists a corresponding one subsequent to C and equally distant from it. Whatever loss, therefore, as compared with uniform motion with half the final velocity, is suffered in the first half of the motion, such loss is made up in the second half. The distance fallen through we may consequently regard as having been uniformly described with half the final velocity.

In symbols, if the number of units of velocity acquired in t units of time is v, and suppose that v is proportional to t, the number s of units of space descended through is proportional to $\frac{1}{2}t^2$. In fact, s is given by $\frac{1}{2}vt$, and, as v is proportional to t, s is proportional to $\frac{1}{2}t^2$.

Now, Galileo verified this relation between s and t experimentally. The motion of free falling was too quick for Galileo to observe accurately with the very imperfect means—such as water-clocks—at his disposal. There were no mechanical clocks at the beginning of the seventeenth century; they were first made possible by the dynamical knowledge of which Galileo laid the foundations. Galileo, then, made the motion slower, so that s and t were big enough to be measured by rather primitive apparatus in which the moving balls ran down grooves in inclined planes. That the spaces traversed by the ball are proportional to the squares of the measures of the times in free descent as well as in motion on an inclined plane, Galileo verified by experimentally proving that a ball which falls through the height of an inclined plane attains the same final velocity as a ball which falls through its length. This experiment was an ingenious one with a pendulum whose string, when half the swing had been accomplished, caught on a fixed nail so placed that the remaining half of the swing was with a shorter string than the other half. This experiment showed that the bob of the pendulum rose, in virtue of the velocity acquired in its descent, *just as high* as it had fallen. This fact is in agreement with our instinctive knowledge of natural events; for if a ball which falls down the length of an inclined plane could attain a greater velocity than one which falls through its height, we should only have to let the body pass with the acquired velocity to another more inclined plane to make it rise to a greater vertical height than that from which it had fallen. Hence we can deduce, from the acceleration on an inclined plane, the acceleration of free descent, for, since the final velocities are the same and $s = \frac{1}{2}vt$, the lengths of the sides of the inclined plane are simply proportional to the times taken by the ball to pass over them.

The motion of falling that Galileo found actually to exist is, accordingly, a motion of which the velocity increases proportionally to the time.

Like Galileo, we have started with the notions familiar to us (through the practical arts, for example), such as that of *velocity*. Let us consider this motion more closely.

If a motion is *uniform* and c feet are travelled over in every second, at the end of t seconds it will have travelled ct feet. Put $ct = s$ for short. Then we call the "velocity" of the moving body the distance traversed in a unit of time so that it is $\frac{s}{t}$ units of length per second, the number which is the measure of the distance divided by the number which is the measure of the time elapsed. Galileo, now, attained to the conception of a motion in which the velocity increases proportionally to the time. If we draw a diagram and set off, from the origin O along the x-axis OA, a series of abscissæ which represent the times in length, and erect the corresponding ordinates to represent the velocities, the ends of these ordinates will lie on a line OB, which, in the case of the "uniformly accelerated motion" to which Galileo attained, is *straight*, as we have already seen. But if the ordinates represent *spaces* instead of *velocities*, the straight line OB becomes a curve. We see the distinction between the "curve of spaces" and "the curve of velocities," with times as abscissæ in both cases. If the velocity is uniform, the curve of spaces is a straight line OB drawn from the origin O, and the curve of velocities is a straight line parallel to the x-axis. If the velocity is variable, the curve of spaces is never a straight line; but if the motion is uniformly accelerated, the curve of velocities is a straight line like OB. The relations between the curve of spaces, the curve of velocities, and the areas of such curves AOB are, as we shall see, relations which are at once expressible by the "differential and integral calculus"—indeed, it is mainly because of this important illustration of the calculus that the elementary problems of dynamics have been treated here. And the measurement of velocity in the case where the velocity varies from time to time is an illustration of the formation of the fundamental conception of the differential calculus.

It may be remarked that the finding of the velocity of a particle at a given instant and the finding of a tangent to a curve at a given point are both of them the same kind of problem—the finding of the "differential quotient" of a function. We will now enter into the matter more in detail.

Consider a curve of spaces. If the motion is uniform, the number measuring *any* increment of the distance divided by the number measuring the corresponding increment of the time gives the same value for the measure of the velocity. But if we were to proceed like this where the velocity is variable, we should obtain widely differing values for the velocity. However, the smaller the increment of the time, the more nearly does the bit of the curve of spaces which corresponds to this increment approach straightness, and hence uniformity of increase (or decrease) of s. Thus, if we denote the increment of t by "Δt,"—where "Δ" does not stand for a number but for the phrase "the increment of,"—and the corresponding increment (or decrement) of s by "Δs," we may define the measure of average velocity in this element of the motion as $\frac{\Delta s}{\Delta t}$. But, however small Δt is, the line represented by Δs is not, usually at least, quite straight, and the velocity at the instant t, which, in the language of Leibniz's differential calculus, is defined as the quotient of "infinitely small" increments and symbolised by $\frac{ds}{dt}$, —the Δ's being replaced by d's when we consider "infinitesimals,"—appears to be only defined approximately. We have met this difficulty when considering the method of indivisibles, and will meet it again when considering the infinitesimal calculus, and will only see how it is overcome when we have become familiar with the conception of a "limit."

This new notion of velocity includes that of uniform velocity as a particular case. In fact, the rules of the infinitesimal calculus allow us to conclude, from the equation $\frac{ds}{dt} = a$, where a is some constant, the equation $s = at + b$, where b is another constant. We must remember that all this was not *expressly* formulated until about fifty years after Galileo had published his investigations on the motion of falling.

If we consider the curve of velocities, uniformly accelerated motion occupies in it exactly the same place as uniform velocity does in the curve of spaces. If we denote by v the numerical measure of the velocity at the end of t units of time, the acceleration, in the notation of the differential calculus, is measured by $\frac{dv}{dt}$, and the equation $\frac{dv}{dt} = h$, where h is some constant, is the equation of uniformly accelerated motion. In Newtonian dynamics, we have to consider *variably* accelerated motions, and this is where the infinitesimal calculus or some practically equivalent calculus such as Newton's "method of fluxions" becomes so necessary in theoretical mechanics.

We will now consider the curve of spaces for uniformly accelerated motion. On this diagram—the arcs being t and s—we will draw the curve

$$s = \frac{gt^2}{2},$$

where g denotes a constant. Of course, this is the same thing as drawing the curve $y = \frac{gx^2}{2}$ in a plane divided up by the x-axis and the y-axis of

Descartes. This curve is a parabola passing through the origin. An interesting thing about this curve is that it is the curve that would be described by a body projected obliquely near the surface of the earth if the air did not resist, and is very nearly the path of such a projectile in the resisting atmosphere. A free body, according to Galileo's view, always falls towards the earth with a uniform vertical acceleration measured by the above number g. If we project a body vertically upwards with the initial velocity of c units, its velocity at the end of t units of time is $-c + gt$ units, for if the direction downwards (of g) is reckoned positive, the direction upwards (of c) must be reckoned negative. If we project a body horizontally with the velocity of a units, and neglect the resistance of the air, Galileo recognised that it would describe, in the horizontal direction, a distance of at units in t units of time, while *simultaneously* it would fall a distance of $\frac{gt^2}{2}$ units. The two motions are to be considered as going on *independently* of each other. Thus also, oblique projection may be considered as compounded of a horizontal and a vertical projection. In all these cases the path of the projectile is a parabola; in the case of the horizontal projection, its equation in x and y co-ordinates is got from the two equations $x = at$ and $y = \frac{gt^2}{2}$, and is thus $y = \frac{gx^2}{2a^2}$.

Now, suppose that the velocity is neither uniform nor increases uniformly, but is different and increases at a different rate at different points of time. Then in the curve of velocities, the line OB is no longer straight. *In the former case, the number s was the number of square units in the area of the triangle AOB.* In this case the figure AOB is not a triangle, though we shall find that its area is the s units we seek, although v does not increase uniformly from O to A.

Notice again that if, on OA, we take points C and E very close together, the little arc DF is very nearly straight, and the figure DGF very nearly a rectilinear triangle. Note that we are only trying, in this, to get a first approximation to the value of s, so that, instead of the continuously changing velocities we know—or think we know—from our daily experience, we are considering a fictitious motion in which the velocity increases (or decreases) so as to be the same as that of the motion thought of at a large number of points at minute and equal distances, and between successive points increases (or decreases) uniformly.

Note also that we are assuming (what usually happens with the curves with which we shall have to do) that the arc DF which corresponds to CE becomes as straight as we wish if we take C and E close enough together.

And now let us calculate s approximately. Starting from O, in the first small interval OH the rectilinear triangle OHK, where HK is the ordinate at H, represents approximately the space described. In the next small interval HL, where the length of HL is equal to that of OH, the space described is represented by the rectilinear figure $KHLM$. The rectangle KL is the space passed over with the uniform velocity HK in time HL; and the triangle KNM is the space passed over by a motion in which the velocity increases from zero to MN. And so on for other intervals beyond HL. Thus s is ultimately given (approximately) as the number of square units in a polygon which closely approximates to the figure AOB.

We must now say a few words about the meaning of the letters in geometrical and mechanical *equations* which, following Descartes, we use instead of the proportions used by Galileo and even many of his contemporaries and followers. It seems better, when beginning mechanics, to think in proportions, but afterwards, for convenience in dealing with the symbolism of mathematical data, it is better to think in equations.

A typical proportion is: Final velocities are to one another as the times; or, in symbols,

$$\text{“}V : V' : : T : T'.\text{”}$$

Here "V" (for example) is just short for "the velocity attained at the end of the period of time" (reckoned from some fixed instant) denoted by "T," and $V : V'$, and $T : T'$, are just *numbers* (real numbers); and the proportion states the equality of these numbers. Hence the proportion is sometimes written "$V : V' = T : T'$." If, now, v is the numerical measure, merely, of V, v' that of V', and so on, we have $\frac{v}{v'} = \frac{t}{t'}$ or $vt' = v't$.

In the last equation, the letters v and t have a mnemonic significance, as reminding us that we started from *velocities* and *times, but we must carefully avoid the idea that we are "multiplying"* (or can do so) *velocities by times;* what we *are* doing is multiplying the numerical measures of them. People who write on geometry and mechanics often say inaccurately, simply for shortness, "Let s denote the distance, t the time," and so on; whereas, by a tacit convention, small italics are usually employed to denote *numbers*. However, in future, for the sake of shortness, I shall do as the writers referred to, and speak of v as "the velocity." Equations in mechanics, such as "$s = \frac{gt^2}{2}$." are only possible if the left-hand side is of the same kind as the right-hand side: we cannot equate spaces and times, for example.

Suppose that we have fixed on the unit of length as one inch and the unit of time as one second. As unit of velocity we might choose the velocity with which, say, a inches are described uniformly in one second. If we did this, we should express the relation between the s units of space passed over by a body with a given velocity (v units) in a given time (t units) as "$s = avt$"; whereas, if we defined the unit of velocity as the velocity with which the unit of length is travelled over in the unit of time, we should write "$s = vt$."

Among the units derived from the fundamental units—such as those of length and time—the simplest possible relations are made to hold. Thus, as the unit of area and the unit of volume, the square and the cube of unit sides are respectively used, the unit of velocity is the uniform rate at which unit of length is travelled over in the unit of time, the unit of acceleration is the gain of unit velocity in unit time, and so on.

The derived units depend on the fundamental units, and the *function* which a given derived unit is of its fundamental units is called its "dimensions." Thus the velocity v is got by dividing the length s by the time t. The dimensions of a velocity are written

$$\text{“}[V] = \frac{[L]}{[T]}\text{,”}$$

and those of an acceleration—denoted "F"—

$$\text{“}[F] = \frac{[V]}{[T]} = \frac{[L]}{[T]^2}\text{.”}$$

These equations are merely mnemonic; the letters do not mean numbers. The mnemonic character comes out when we wish to pass from one set of units to another. Thus, if we pass to a unit of length b times greater and one of time c times greater, the acceleration f with the old units is related to that (f') with the new units by the equation

$$f'\left(\frac{c^2}{b}\right) = f.$$

As the units become greater, f' becomes less; and, since the dimensions of F are $\frac{[L]}{[T]^2}$, the factor $\frac{c^2}{b}$ is obviously suggested to us—the symbol "$[T]^2$" suggesting a squaring of the number measuring the time.

From Galileo's work resulted the conclusion that, where there is no change of *velocity in a straight line,* there is no force. The state of a body unacted upon by force is uniform rectilinear motion; and rest in a special case of this motion where the velocity is and remains zero. This "law of inertia" was exactly opposite to the opinion, derived from Aristotle, that force is requisite to keep up a uniform motion, and may be roughly verified by noticing the behaviour of a body projected with a given velocity and moving under little resistance—as a stone moving on a sheet of ice. Newton and his contemporaries saw how important this law was in the explanation of the motion of a planet—say, about the sun. Think of a simple case, and imagine the orbit to be a circle. The planet tends to move along the tangent with uniform velocity, but the attraction of the sun simultaneously draws the planet towards itself, and the result of this continual combination of two motions is the circular orbit. Newton succeeded in calculating the shapes of the orbits for different laws of attraction, and found that, when attraction varies inversely as the square of the distance, the shapes are conic sections, as had been observed in the case of our solar system.

The problem of the solar system appeared, then, in a mathematical dress; various things move about in space, and this motion is completely described if we know the geometrical relations—distances, positions, and angular distances—between these things at some moment, the velocities at this moment, and the accelerations at *every* moment. Of course, if we knew all the positions of all the things at all the instants, our description would be complete; it happens that the *accelerations* are usually simpler to find directly than the positions: thus, in Galileo's case the acceleration was simply constant. Thus, we are given functional relations between these positions and their rates of change. We have to determine the positions from these relations.

It is the business of the "method of fluxions" or the "infinitesimal calculus" to give methods for finding the relations between variables from relations between their rates of change or between them and these rates. This shows the importance of the calculus in such physical questions.

Mathematical physics grew up—perhaps too much so—on the model of theoretical astronomy, its first really extensive conquest. There are signs that mathematical physics is freeing itself from its traditions, but we need not go further into the subject in this place.

Roberval devised a method of tangents which is based on Galileo's conception of the composition of motions. The tangent is the direction of the resultant motion of a point describing the curve. Newton's method, which is to be dealt with in the fifth chapter, is analogous to this, and the idea of velocity is fundamental in his "method of fluxions."

CHAPTER V

THE RISE OF MODERN MATHEMATICS —THE INFINITESIMAL CALCULUS

IN the third chapter we have seen that the ancient Greeks were sometimes occupied with the theoretically exact determination of the areas enclosed by curvilinear figures, and that they used the "method of exhaustion," and, to demonstrate the results which they got, an indirect method. We have seen, too, a "method of indivisibles," which was direct and seemed to gain in brevity and efficiency from a certain lack of correctness in expression and perhaps even a small inexactness in thought. We shall find the same merits and demerits—both, especially the merits, intensified —in the "infinitesimal calculus."

By the side of researches on quadratures and the finding of volumes and centres of gravity developed the methods of drawing tangents to curves. We have begun to deal with this subject in the third chapter: here we shall illustrate the considerations of Fermat (1601–1665) and Barrow (1630–1677)—the intellectual descendants of Kepler—by a simple example.

Let it be proposed to draw a tangent at a given point P in the circumference of a circle of centre O and equation $x^2 + y^2 = 1$. Let us take the circle to be a polygon of a great number of sides; let PQ be one of these sides, and produce it to meet the x-axis at T. Then PT will be the tangent in question. Let the co-ordinates of P be X and Y; those of Q will be $X + e$ and $Y + a$, where e and a are infinitely small increments, positive or negative. From a figure in which the ordinates and abscissæ of P and Q are drawn, so that the ordinate of P is PR, we can see, by a well-known property of triangles, that TR is to RP (or Y) as e is to a. Now, X and Y are related by the equation $X^2 + Y^2 = 1$, and, since Q is also on the locus $x^2 + y^2 = 1$, we have $(X + e)^2 + (Y + a)^2 = 1$. From the two equations in which X and Y occur, we conclude that $2eX + e^2 + 2aY + a^2 = 0$, and hence $\frac{e}{a}(X + \frac{e}{2}) + Y + \frac{a}{2} = 0$. But $\frac{e}{a} = \frac{TR}{Y}$; hence $TR = \frac{-Y(Y + {}^a/_2)}{X + {}^e/_2}$. Now, a and e may be neglected in comparison with X and Y, and thus we can say that, at any rate *very* nearly, we have $TR = \frac{Y^2}{X}$. But this is *exactly right,* for, since TP is at right angles to OP, we know that OR is to RP as PR is to RT. Here X and Y are constant, but we can say that the abscissa of the point where the tangent at *any* point (say y) of the circle cuts the x-axis is given by adding $\dfrac{y^2}{x}$ to x.

Thus, we can find tangents by considering the ratios of infinitesimals to one another. The method obviously applies to other curves besides circles; and Barrow's method and nomenclature leads us straight to the notation and nomenclature of Leibniz. Barrow called the triangle PQS, where S is where a parallel to the x-axis through Q meets PR, the "differential triangle," and Leibniz denoted Barrow's a and e by dy and dx (short for the "differential of y" and "the differential of x," so that "d" does not denote a number but "dx" altogether stands for an "infinitesimal") respectively, and called the collection of rules for working with his signs the "differential calculus."

But before the notation of the differential calculus and the rules of it were discovered by Gottfried Wilhelm von Leibniz (1646–1716), the celebrated German philosopher, statesman, and mathematician, he had invented the notation and found some of the rules of the "integral calculus": thus, he had used the now well-known sign "$\int$" or long "s" as short for "the sum of," when considering the sum of an infinity of infinitesimal elements as we do in the method of indivisibles. Suppose that we propose to determine the area included between a certain curve $y = f(x)$, the x-axis, and two fixed ordinates whose equations are $x = a$ and $x = b$; then, if we make use of the idea and notation of differentials, we notice that the area in question can be written as

$$\text{“}\int y \,.\, dx,\text{”}$$

the summation extending from $x = a$ to $x = b$. We will not here further concern ourselves about these boundaries. Notice that in the above expression we have put a dot between the "y" and the "dx": this is to indicate that y is to multiply dx. Hitherto we have used juxtaposition to denote multiplication, but here d is written close to x with another end in view; and it is desirable to emphasise the difference between "d" used in the sense of an adjective and "d" used in the sense of a multiplying number, at least until the student can easily tell the difference by the context. If, then, we imagine the abscissa divided into equal infinitesimal parts, each of length dx, corresponding to the constituents called "points" in the method of indivisibles, $y \,.\, dx$ is the area of the little rectangle of sides dx and y which stand at the end of the abscissa x. If, now, instead of extending to $x = b$, the summation extends to the ordinate at the indeterminate or "variable" point x, $y \,.\, dx$ becomes a function of x.

Now, if we think what must be the differential of this sum—the infinitesimal increment that it gets when the abscissa of length x, which is one of the boundaries, is increased by dx—we see that it must be $y \,.\, dx$. Hence

$$d(\int y \,.\, dx) = y \,.\, dx,$$

and hence the sign of "d" destroys, so to speak, the effect of the sign "$\int$". We also have $\int dx = x$, and find that this summation is the inverse process to differentiation. *Thus the problems of tangents and quadratures are inverses of one another.* This vital discovery seems to have been first made by Barrow without the help of any technical symbolism. The quantity which by its differentiation produces a proposed differential, is called the "integral" of this differential; since we consider it as having been formed by infinitely small continual additions: each of these additions is what we have named the differential of the increasing quantity, it is a fraction of it: and the sum of all these fractions is the entire quantity which we are in search of. For the same reason we call "integrating" or "taking the sum of" a differential the finding the integral of the sum of all the infinitely small successive additions which form the series, the differential of which, properly speaking, is the general term.

It is evident that two variables which constantly remain equal increase the one as much as the other during the same time, and that consequently their differences are equal: and the same holds good even if these two quantities had differed by any quantity whatever when they began to vary; provided that this primitive difference be always the same, their differentials will always be equal.

Reciprocally, it is clear that two variables which receive at each instant infinitely small equal additions must also either remain constantly equal to one another, or always differ by the same quantity—that is, the integrals of two differentials which are equal can only differ from each other by a constant quantity. For the same reason, if any two quantities whatever differ in an infinitely small degree from each other, their differentials will also differ from one another infinitely little: and reciprocally if two differential quantities differ infinitely little from one another, their integrals, putting aside the constant, can also differ but infinitely little one from the other.

Now, some of the rules for differentiation are as follows. If $y = f(x)$, $dy = f(x + dx) - f(x)$, in which higher powers of differentials added to lower ones may be neglected. Thus, if $y = x^2$, then $dy = (x + dx)^2 - x^2 = 2x \,.\, dx + (dx)^2 = 2x \,.\, dx$. Here it is well to refer back to the treatment of the problem of tangents at the beginning of this chapter. Again, if $y = a \,.\, x$, where a is constant, $dy = a \,.\, dx$. If $y = x \,.\, z$, then $dy = (x + dx)(z + dz) - x \,.\, z = x \,.\, dz + z \,.\, dx$. If $y = \dfrac{x}{z}$, $x = y \,.\, z$, so $dx = y \,.\, dz + z \,.\, dy$; hence $dy = \dfrac{dx - y \,.\, dz}{z}$. Since the integral calculus is the inverse of the differential calculus, we have at once

$$\int 2x \,.\, dx = x^2,\ \int a \,.\, dx = a\int dx,$$
$$\int x \,.\, dz + \int z \,.\, dx = xz,$$

and so on. More fully, from $d(x^3) = 3x^2 \,.\, dx$, we conclude, not that $\int x^2 \,.\, dx = \frac{1}{3}x^3$, but that $\int x^2 \,.\, dx = \frac{1}{3}x^3 + c$, where "$c$" denotes some constant depending on the fixed value for x from which the integration starts.

Consider a parabola $y^2 = ax$; then $2y \,.\, dy = a \,.\, dx$, or $dx = \dfrac{2y \,.\, dy}{a}$. Thus the area from the origin to the point x is $\displaystyle\int \frac{2y^2 \,.\, dy}{a} + c$; but $d\dfrac{2y^3}{3a} = \dfrac{2y^2 \,.\, dy}{a}$; thus the area is $\dfrac{2y^3}{3a} + c$, or, since $y^2 = ax$, $\frac{2}{3}x \,.\, y + c$. To determine c when we measure the area from 0 to x, we have the area zero when $x = 0$; hence the above equation gives $c = 0$. This whole result, now quite simple to us, is one of the greatest discoveries of Archimedes.

Let us now make a few short reflections on the infinitesimal calculus. First, the extraordinary power of it in dealing with complicated questions lies in that the question is split up into an infinity of *simpler* ones which can all be dealt with at once, thanks to the wonderfully economical fashion in which the calculus, like analytical geometry, deals with variables. Thus, a *curvilinear* area is split up into *rectangular* elements, all the rectangles are added together at once when it is observed that integral is the inverse of the easily acquired practice of differentiation. We must never lose sight of the fact that, when we differentiate y or integrate $y \,.\, dx$, we are considering, not a particular x or y, but *any* one of an infinity of them. Secondly, we have seen that what in the first place had been regarded but as a simple method of approximation, leads at any rate in certain cases to results perfectly exact. The fact is that the exact results are due to a compensation of errors: the error resulting from the false supposition made, for example, by regarding a curve as a polygon with an infinite number of sides each infinitely small and which when produced is a tangent of the curve, is corrected or compensated for by that which springs from the very processes of the calculus, according to which we retain in differentiation infinitely small quantities of the same order alone. In fact, after having introduced these quantities into the calculation to facilitate the expression of the conditions of the problem, and after having regarded them as absolutely zero in comparison with the proposed quantities, with a view to simplify these equations, in order to banish the errors that they had occasioned, and to obtain a result perfectly exact, there remains but to eliminate these same quantities from the equations where they may still be.

But all this cannot be regarded as a strict proof. There *are* great difficulties in trying to determine what infinitesimals are: at one time they are treated like finite numbers and at another like zeros or as "ghosts of departed quantities," as Bishop Berkeley, the philosopher, called them.

Another difficulty is given by differentials "of higher orders than the first." Let us take up again the considerations of the fourth chapter. We saw that $v = \frac{ds}{dt}$, and found that s was got by integration: $s = \int v \,.\, dt$. This is now an immediate inference, since $\frac{ds}{dt} dt = ds$. Now, let us substitute $\frac{ds}{dt}$ for v in $\frac{dv}{dt}$. Here t is the independent variable, and all of the older mathematicians treated the elements dt as constant—the interval of the independent variable was split up into atoms, so to speak, which themselves were regarded as known, and in terms of which other differentials, ds, dx, dy, were to be determined. Thus

$$\frac{dv}{dt} = \frac{d(^{ds}/_{dt})}{dt} = \frac{^1/_{dt} \,.\, d(ds)}{dt} = \frac{d^2s}{dt^2},$$

"d^2s" being written for "$d(ds)$" and "dt^2" for "$(dt)^2$". Thus the acceleration was expressed as "the second differential of the space divided by the square of dt." If $\frac{d^2s}{dt^2}$ were constant, say, a, then $\frac{d^2s}{dt} = a \,.\, dt$; and, integrating both sides:

$$\frac{ds}{dt} = \int a \,.\, dt = a\int dt = at + b,$$

where b is a new constant. Integrating again, we have:

$$s = a\int t \,.\, dt + b\int dt = \frac{at^2}{2} + bt + c,$$

which is a more general form of Galileo's result. Many complicated problems which show how far-reaching Galileo's principles are were devised by Leibniz and his school.

Thus, the infinitesimal calculus brought about a great advance in our powers of describing nature. And this advance was mainly due to Leibniz's notation; Leibniz himself attributed all of his mathematical discoveries to his improvements in notation. Those who know something of Leibniz's work know how conscious he was of the suggestive and economical value of a good notation. And the fact that we still use and appreciate Leibniz's

"$\int$" and "d," even though our views as to the principles of the calculus are very different from those of Leibniz and his school, is perhaps the best testimony to the importance of this question of notation. This fact that Leibniz's notations have become permanent is the great reason why I have dealt with his work before the analogous and prior work of Newton.

Isaac Newton (1642–1727) undoubtedly arrived at the principles and practice of a method equivalent to the infinitesimal calculus much earlier than Leibniz, and, like Roberval, his conceptions were obtained from the dynamics of Galileo. He considered curves to be described by moving points. If we conceive a moving point as describing a curve, and the curve referred to co-ordinate axes, then the velocity of the moving point can be decomposed into two others parallel to the axes of x and y respectively; these velocities are called the "fluxions" of x and y, and the velocity of the point is the fluxion of the arc. Reciprocally the arc is the "fluent" of the velocity with which it is described. From the given equation of the curve we may seek to determine the relations between the fluxions—and this is equivalent to Leibniz's problem of differentiation;—and reciprocally we may seek the relations between the co-ordinates when we know that between their fluxions, either alone or combined with the co-ordinates themselves. This is equivalent to Leibniz's general problem of integration, and is the problem to which we saw, at the end of the fourth chapter, that theoretical astronomy reduces.

Newton denoted the fluxion of x by "$\dot{x}$," and the fluxion of the fluxion (the acceleration) of $\dot{x}$ by "$\ddot{x}$." It is obvious that this notation becomes awkward when we have to consider fluxions of higher orders; and further, Newton did not indicate by his notation the independent variable considered. Thus "$\dot{y}$" might possibly mean either $\frac{dy}{dt}$ or $\frac{dy}{dx}$. We have $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d^2x}{dt^2}$; but a dot-notation for $\frac{d^nx}{dt^n}$ would be clumsy and inconvenient. Newton's notation for the "inverse method of fluxions" was far clumsier even, and far inferior to Leibniz's "$\int$".

The relations between Newton and Leibniz were at first friendly, and each communicated his discoveries to the other with a certain frankness. Later, a long and acrimonious dispute took place between Newton and Leibniz and their respective partisans. Each accused—unjustly, it seems—the other of plagiarism, and mean suspicions gave rise to meanness of conduct, and this conduct was also helped by what is sometimes called "patriotism." Thus, for considerably more than a century, British mathematicians failed to perceive the great superiority of Leibniz's notation. And thus, while the Swiss mathematicians, James Bernoulli (1654–1705), John Bernoulli (1667–1748), and Leonhard Euler (1707–1783), the French mathematicians d'Alembert (1707–1783), Clairaut (1713–1765), Lagrange (1736–1813), Laplace (1749–1827), Legendre (1752–1833), Fourier (1768–1830), and Poisson (1781–1850), and many other Continental mathematicians, were rapidly [6] extending knowledge by using the infinitesimal calculus in all branches of pure and applied mathematics, in England comparatively little progress was made. In fact, it was not until the beginning of the nineteenth century that there was formed, at Cambridge, a Society to introduce and spread the use of Leibniz's notation among British mathematicians: to establish, as it was said, "the principles of pure *d*-ism in opposition to the *dot*-age of the university."

The difficulties met and not satisfactorily solved by Newton, Leibniz, or their immediate successors, in the principles of the infinitesimal calculus, centre about the conception of a "limit"; and a great part of the meditations of modern mathematicians, such as the Frenchman Cauchy (1789–1857), the Norwegian Abel (1802–1829), and the German Weierstrass (1815–1897), not to speak of many still living, have been devoted to the putting of this conception on a sound logical basis.

We have seen that, if $y = x^2$, $\frac{dy}{dx} = 2x$. What we do in forming $\frac{dy}{dx}$ is to form $\frac{(x + \Delta x)^2 - x^2}{\Delta x}$, which is readily found to be $2x + \Delta x$, and then consider that, as Δx approaches 0 more and more, the above quotient approaches $2x$. We express this by saying that the "limit, as h [Δx] approaches 0," is $2x$. We do not consider Δx as being a fixed "infinitesimal" or as an absolute zero (which would make the above quotient become indeterminate $\frac{0}{0}$), nor need we suppose that the quotient *reaches* its limit (the state of Δx being 0). What we need to consider is that "Δx" should represent a variable which can take values differing from 0 by as little as we please. That is to say, if we choose *any* number, however small, there is a value which Δx can take, and which differs from 0 by less than that

[6] It is difficult for a mathematician not to think that the sudden and brilliant dawn on eighteenth-century France of the magnificent and apparently all-embracing physics of Newton and the wonderfully powerful mathematical method of Leibniz inspired scientific men with the belief that the goal of all knowledge was nearly reached and a new era of knowledge quickly striding towards perfection begun; and that this optimism had indirectly much to do in preparing for the French Revolution.

number. As before, when we speak of a "variable" we mean that we are considering a certain *class*. When we speak of a "limit," we are considering a certain *infinite* class. Thus the sequence of an infinity of terms 1, ½, ¼, ⅛, 1/16, and so on, whose law of formation is easily seen, has the limit 0. In this case 0 is such that any number greater than it is greater than some term of the sequence, but 0 itself is not greater than any term of the sequence and is not a term of the sequence. A sequence like 1, $1 + \frac{1}{2}$, $1 + \frac{1}{2} + \frac{1}{4}$, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$, has an analogous *upper* limit 2. A function $f(x)$, as the independent variable x approaches a certain value, like $\frac{2x}{x}$ as x approaches 0, may have a value (in this case 2, though *at* 0, $\frac{2x}{x}$ is indeterminate). The question of the limits of a function in general is somewhat complicated, but the most important limit is $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ as Δx approaches 0; this, if $y = f(x)$, is $\frac{dy}{dx}$.

That the infinitesimal calculus, with its rather obscure "infinitesimals"—treated like finite numbers when we write $\frac{dy}{dx}\, dx = dy$ and $\frac{1}{{}^{dy}/_{dx}} = \frac{dx}{dy}$, and then, on occasion, neglected—leads so often to correct results is a most remarkable fact, and a fact of which the true explanation only appeared when Cauchy, Gauss (1777–1855), Riemann (1826–1866), and Weierstrass had developed the theory of an extensive and much used class of functions. These functions happen to have properties which make them especially easy to be worked with, and nearly all the functions we habitually use in mathematical physics are of this class. A notable thing is that the complex numbers spoken of in the second chapter *make* this theory to a great extent.

Large tracts of mathematics have, of course, not been mentioned here. Thus, there is an elaborate theory of integer numbers to be referred to in a note to the seventh chapter, and a geometry using the conceptions of the ancient Greeks and methods of modern mathematical thought; and very many men still regard space-perception as something mathematics deals with. We will return to this soon. Again, algebra has developed and branched off; the study of functions in general and in particular has grown; and soon a list of some of the many great men who have helped in all this would not be very useful. Let us now try to resume what we have seen of the development of mathematics along what seem to be its main lines.

In the earliest times men were occupied with particular questions—the properties of particular numbers and geometrical properties of particular figures, together with simple mechanical questions. With the Greeks, a more general study of classes of geometrical figures began. But traces of an earlier exception to this study of particulars are afforded by "algebra." In it and its later form symbols—like our present x and y—took the place of numbers, so that, what is a great advance in economy of thought and other labour, a part of calculation could be done with symbols instead of numbers, so that the one result stated, in a manner analogous to that of Greek geometry, a proposition valid for a whole infinite class of different numbers.

The great revolution in mathematical thought brought about by Descartes in 1637 grew out of the application of this general algebra to geometry by the very natural thought of substituting the numbers expressing the lengths of straight lines for those lines. Thus a point in a plane—for instance—is determined in position by two numbers x and y, or coordinates. Now, as the point in question varies in position, x and y both vary; to every x belongs, in general, one or more y's, and we arrive at the most beautiful idea of a single algebraical equation between x and y representing the whole of a curve—the one "equation of the curve" expressing the general law by which, given any particular x out of an infinity of them, the corresponding y or y's can be found.

The problem of drawing a tangent—the limiting position of a secant, when the two meeting points approach indefinitely close to one another—at any point of a curve came into prominence as a result of Descartes' work, and this, together with the allied conceptions of velocity and acceleration "at an instant," which appeared in Galileo's classical investigation, published in 1638, of the law according to which freely falling bodies move, gave rise at length to the powerful and convenient "infinitesimal calculus" of Leibniz and the "method of fluxions" of Newton. Mathematically, the finding of the tangent at the point of a curve, and finding the velocity of a particle describing this curve when it gets to that point, are identical problems. They are expressed as finding the "differential quotient," or the "fluxion" at the point. It is now known to be very probable that the above two methods, which are theoretically—but not practically—the same, were discovered independently; Newton discovered his first, and Leibniz published his first, in 1684. The finding of the areas of curves and of the shapes of the curves which moving particles describe under given forces showed themselves, in this calculus, as results of the inverse process to that of the direct process which serves to find tangents and the law of attraction to a given point from the datum of the path described by a particle. The direct process is called "differentiation," the inverse process "integration."

Newton's fame is chiefly owing to his application of this method to the solution, which, in its broad outlines, he gave of the problem of the motion of the bodies in the solar system, which includes his discovery of the law according to which all matter gravitates towards—is attracted by—other matter. This was given in his *Principia* of 1687; and for more than a century afterwards mathematicians were occupied in extending and applying the calculus.

Then came more modern work, more and more directed towards the putting of mathematical methods on a sound logical basis, and the separation of mathematical processes from the sense-perception of space with which so much in mathematics grew and grows up. Thus trigonometry took its place by algebra as a study of certain mathematical functions, and it began to appear that the true business of geometry is to supply beautiful and suggestive pictures of abstract—"analytical" or "algebraical" or even "arithmetical," as they are called—processes of mathematics. In the next chapter we shall be concerned with part of the work of logical examination and reconstruction.

CHAPTER VI

MODERN VIEWS OF LIMITS AND NUMBERS

LET us try to form a clear idea of the conception which showed itself to be fundamental in the principles of the infinitesimal calculus, the conception of a *limit*.

Notice that the limit of a sequence is a number which is already defined. We cannot prove that there is a limit to a sequence unless the limit sought is among the numbers already defined. Thus, in the system of "numbers" —here we must refer back to the second chapter—consisting of all fractions (or ratios), we can say that the sequence (where 1 and 2 are written for the ratios $1/1$ and $2/1$) $1, 1 + 1/2, 1 + 1/2 + 1/4, \ldots$, has a limit (2), but that the sequence

$$1, 1 + 4/10, 1 + 4/10 + 1/100, 1 + 4/10 + 1/100 + 4/1000, \ldots,$$

$$\text{or } 1 \cdot 4142 \ldots,$$

got by extracting the square root of 2 by the known process of decimal arithmetic, has not. In fact, it can be proved that there is no ratio such that it is a limit for the above sequence. If there were, and it were denoted by "x," we would have $x^2 = 2$. Here we come again to the question of incommensurables and "irrational numbers." The Greeks were quite right in distinguishing so sharply between numbers and magnitudes, and it was a tacit, natural, and unjustified—not, as it happens, incorrect—presupposition that the series of numbers, completed into the series of what are called "real numbers," exactly corresponds to the series of points on a straight line. The series of points which represents the sequence last named seems undoubtedly to possess a limit; this limiting point was assumed to represent some number, and, since it could not represent an integer or a ratio, it was said to represent an "irrational number," $\sqrt{2}$. Another irrational number is that which is represented by the incommensurable ratio of the circumference of a circle to its diameter. This number is denoted by the Greek letter "π," and its value is nearly 3·1416. . . . Of course, the process of approximation by decimals never comes to an end.

The subject of limits forced itself into a very conspicuous place in the seventeenth and eighteenth centuries owing to the use of infinite series as a means of approximate calculation. I shall distinguish what I call "sequences" and "series." A sequence is a collection—finite or infinite—of numbers; a series is a finite or infinite collection of numbers *connected by addition*. Sequences and series can be made to correspond in the following way. To the sequence 1, 2, 3, 4, . . . belongs a series of which the terms are got by subtracting, in order, the terms of the sequence from the ones immediately following them, thus:

$$(2-1)+(3-2)+(4-3)+\ .\ .\ .\ =1+1+1+\ .\ .\ .\ ;$$

and from a series the corresponding sequence can be got by making the sum of the first, the first two, the first three, . . . terms the first, second, third . . . term of the sequence respectively. Thus, to the series $1+1+1+$ . . . corresponds the sequence 1, 2, 3, . . .

Now, if a series has only a finite number of terms, it is possible to find the sum of all the terms; but if the series is unending, we evidently cannot. But in certain cases the corresponding sequence has a limit, and this limit is called by mathematicians, neither unnaturally nor accurately, "the sum to infinity of the series." Thus, the sequence 1, $1+½$, $1+½+¼$ . . . has the limit 2, and so the sum to infinity of the series $1+½+¼+⅛+$ . . . is 2. Of course, all series do not have a sum: thus $1+1+1+$ . . . to infinity has not—the terms of the corresponding sequence increase continually beyond all limits. Notice particularly that the terms of a sequence may increase continually, and yet have a limit—those of the above sequence with limit 2 so increase, but not beyond 2, though they do beyond any number less than 2; also notice that the terms of a sequence may increase beyond all limits even if the terms of the corresponding series continually diminish, remaining positive, towards 0. The series $1+½+⅓+¼+⅕+$ . . . is such a series; the terms of the sequence slowly increase beyond all limits, as we see when we reflect that the sums

$$⅓+¼,\ ⅕+⅙+⅐+⅛,\ ⅑+\ .\ .\ .\ +\tfrac{1}{16},\ .\ .\ .$$

are all greater than ½. It is very important to realise the fact illustrated by this example; for it shows that the conditions under which an infinite series has a sum are by no means as simple as they might appear at first sight.

The logical scrutiny to which, during the last century, the processes and conceptions of mathematics have been subjected, showed very plainly that it was a sheer assumption that such a process as 1·4142 . . . , though *all* its terms are less than 2, for example, has any limit at all. When we replace numbers by points on a straight line, we feel fairly sure that there is one point which behaves to the points representing the above sequence in the same sort of way as 2 to the sequence 1, $1 + \frac{1}{2}$, $1 + \frac{1}{2} + \frac{1}{4}$, . . . Now, if our system of numbers is to form a *continuum,* as a line seems to our thoughts to be; so that we can affirm that our number system is adequate, when we introduce axes in the manner of analytical geometry, to the description of all the phenomena of change of position which take place in our space,[7] *then* we must have a number $\sqrt{2}$ which is the limit of the sequence 1·4142 . . . if 2 is of the series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$, for to every point of a line must correspond a number which is subject to the same rules of calculation as the ratios or integers. Thus we must, to justify from a logical point of view our procedure in the great mathematical methods, show what irrationals are, and define them *before* we can prove that they are limits. We cannot take a series, whose law is evident, which has no ratio for sum, and yet such that the terms of the corresponding sequence all remain less than some fixed number (such as $1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \ldots$, when all the terms of the corresponding sequence are less than 3, for example), and then say that it "defines a limit." All we can prove is that *if* such a series has a limit, then, if the terms of its corresponding sequence do not decrease as we read from left to right (as in the preceding example), it cannot have more than one limit.

Some mathematicians have simply *postulated* the irrationals. At the beginning of their discussions they have, tacitly or not, said: "In what follows we will *assume* that there are such things as fill up kinds of gaps in the system of rationals (or ratios)." Such a gap is shown by this. The rationals less than ½ and those greater than ½ form two sets and ½ divides them. The rationals x such that x^2 is greater than 2 and those x's such that x^2 is less than 2 form two analogous sets, but there is only an analogue to the dividing number ½ if we postulate a number $\sqrt{2}$. Thus by

[7] The only kind of change dealt with in the science of mechanics is change of position, that is, motion. It does not seem to me to be necessary to adopt the doctrine that the complete description of any physical event is of a mechanical event; for it is possible to assign and calculate with numbers of our number-continuum to other varying characteristics (such as temperature) of the state of a body besides position.

postulation we fill up these subtle gaps in the set of rationals and get a continuous set of real numbers. But we can avoid this postulation if we define "$\sqrt{2}$" as the name of the class of rationals x such that x^2 is less than 2 and "(½)" as the name of the class of rationals x such that x is less than ½. Proceeding thus, we arrive at a set of *classes*, some of which correspond to rationals, as (½) to ½, but the rest satisfy our need of a set without gaps. There is no reason why we should not say that these classes *are* the *real numbers* which include the irrationals. But we must notice that rationals are never real numbers; ½ is not (½), though analogous to it. We have much the same state of things as in the second chapter, where 2, +2 and ²⁄₁ were distinguished and then deliberately confused because, with the mathematicians, we felt the importance of analogy in calculation. Here again we identify (½) with ½, and so on.

Thus, integers, positive and negative "numbers," ratios, and real "numbers" are all different things: real numbers are classes, ratios and positive and negative numbers are relations. Integers, as we shall see, are classes. Very possibly there is a certain arbitrariness about this, but this is unimportant compared with the fact that in modern mathematics we have reduced the definitions of all "numbers" to logical terms. Whether they are classes or relations or propositions or other logical entities is comparatively unimportant.

Integers can be defined as certain classes. Mathematicians like Weierstrass stopped before they got as far as this: they reduced the other numbers of analysis to logical developments out of the conception of integer, and thus freed analysis from any remaining trace of the sway of geometry. But it was obvious that integers had to be defined, if possible, in logical terms. It has long been recognised that two collections consist of the same number of objects if, and only if, these collections can be put in such a relation to one another that to every object of each one belongs one and only one object of the other. We must not think that this implies that we have already the idea of the *number one*. It is true that "one and only one" seems to use this idea. But "the class a has one and only one member" is simply a short way of expressing: "x is a member of a, and if y is also a member of a, then y is identical with x." It is true, also, that we use the idea of the *unity* or the *individuality* of the things considered. But this unity is a property of each individual, while the number 1 is a property of a *class*. If a class of pages of a book is itself, under the name of a "volume," a member of a class of books, the same class of pages has both a number (say 360) and a unity as being itself a member of a class.

The relation spoken of above in which two classes possessing the same number stand to one another does not involve counting. Think of the fingers on your hands. If to every finger of each hand belongs, by some process of correspondence, one and only one—remember the above meaning of this phrase—of the other, they are said to have "the same number." This is a definition of what "the same number" is to mean for us; the word "number" by itself is to have, as yet, no meaning for us; and, to avoid confusion, we had better replace the phrase "have the same number" by the words "are equivalent." Any other word would, of course, do, but this word happens to be fairly suggestive and customary. Now, if the variable *u* is any class, "the number of *u*" is defined as short for the phrase: "the class whose members are classes which are similar to *u*." Thus the number of *u* is an entity which is purely logical in its nature. Some people might urge that by "number" they mean something different from this, and that is quite possible. All that is maintained by those who agree to the process sketched above is: (1) Classes of the kind described are identical in all known arithmetical properties with the undefined things people call "integer numbers"; (2) It is futile to say: "These classes are not *numbers*," if it is not also said what *numbers* are—that is to say, if "the number of" is not defined in some more satisfactory way. There may be more satisfactory definitions, but this one is a perfectly sound foundation for all mathematics, including the theory not touched upon here of *ordinal numbers* (denoted by "first," "second," . . .) which apply to sets arranged in some order, known at present.

To illustrate (1), think of this. Acording to the above definition 2 is the general idea we call "couple." We say: "Mr. and Mrs. A. are a couple"; our definition would ask us to say in agreement with this: "The class consisting of Mr. and Mrs. A. is a member of the class 2." We define "2" as "the class of classes *u* such that, if *x* is a *u*, *u* lacking *x* is a 1"; the definition of "3" follows that of "2"; and so on. In the same way, we see that the class of fingers on your right hand and the class of fingers on your left hand are each of them members of the class 5. It follows that the classes of the fingers are equivalent in the above sense.

Out of the striving of human minds to reproduce conveniently and anticipate the results of experience of geometrical and natural events, mathematics has developed. Its development gave priceless hints to the development of logic, and then it appeared that there is no gap between the science of number and the science of the most general relations of objects of thought. As for geometry and mathematical physics, it becomes possible clearly to separate the logical parts from those parts which formulate the data of our experience.

We have seen that mathematics has often made great strides by sacrificing accuracy to analogy. Let us remember that, though mathematics and logic give the highest forms of certainty within the reach of us, the process of mathematical discovery, which is so often confused with what is discovered, has led through many doubtful analogies and errors arriving from the great help of symbolism in making the difficult easy. Fortunately symbolism can also be used for precise and subtle analysis, so that we can say that it can be made to show up the difficulties in what appears easy and even negligible—like $1 + 1 = 2$. This is what much modern fundamental work does.

CHAPTER VII

THE NATURE OF MATHEMATICS

IN the preceding chapters we have followed the development of certain branches of knowledge which are usually classed together under the name of "mathematical knowledge." These branches of knowledge were never clearly marked off from all other branches of knowledge: thus geometry was sometimes considered as a logical study and sometimes as a natural science—the study of the properties of the space we live in. Still less was there an absolutely clear idea of what it was that this knowledge was about. It had a name—"mathematics"—and few except "practical" men and some philosophers doubted that there was something about which things were known in that kind of knowledge called "mathematical." But what it was did not interest very many people, and there was and is a great tendency to think that the question as to what mathematics is could be answered if we only knew all the facts of the development of our mathematical knowledge. It seems to me that this opinion is, to a great extent, due to an ambiguity of language: one word—"mathematics"—is used both for our knowledge of a certain kind and the thing, if such a thing there be, about which this knowledge is. I have distinguished, and will now explicitly distinguish, between "Mathematics," a collection of truths of which we know something, and "mathematics," our knowledge of Mathematics. Thus, we may speak of "Euclid's mathematics" or "Newton's mathematics," and say truly that mathematics has developed and therefore had history; but Mathematics is eternal and unchanging, and therefore has no history—it does not belong, even in part, to Euclid or Newton or anybody else, but is something which is discovered, in the course of time, by human minds. An analogous distinction can be drawn between "Logic" and "logic." The small initial indicates that we are writing of a psychological process which may lead to Truth; the big initial indicates that we are writing of the entity—the part of Truth—to which this process leads us. The reason why mathematics is important is that Mathematics is not incomprehensible, though it is eternal and unchanging.

Grammatical usage makes us use a capital letter even for "mathematics" in the psychological sense when the word begins a sentence, but in this case I have guarded and will guard against ambiguity.

That particular function of history which I wish here to emphasise will now, I think, appear. In mathematics we gradually learn, by getting to know some thing about mathematics, to know that there is such a thing as Mathematics.

We have, then, glanced at the mathematics of primitive peoples, and have seen that at first discoveries were of isolated properties of abstract things like numbers or geometrical figures, and of abstract relations between concrete things like the relations between the weights and the arms of a lever in equilibrium. These properties were, at first, discovered and applied, of course, with the sole object of the satisfaction of bodily needs. With the ancient Greeks comes a change in point of view which perhaps seems to us, with our defective knowledge, as too abrupt. So far as we know, Greek geometry was, from its very beginning, deductive, general, and studied for its own interest and not for any applications to the concrete world it might have. In Egyptian geometry, if a result was stated as universally true, it was probably only held to be so as a result of induction —the conclusion from a great number of particular instances to a general proposition. Thus, if somebody sees a very large number of officials of a certain railway company, and notices that all of them wear red ties, he might conclude that *all* the officials of that company wear red ties. This might be probably true: it would not be certain: for *certainty* it would be necessary to know that there was some rule according to which all the officials were compelled to wear red ties. Of course, even then the conclusion would not be certain, since these sort of laws may be broken. Laws of *Logic*, however, cannot be broken. These laws are not, as they are sometimes said to be, laws of *thought*; for logic has nothing to do with the way people think, any more than poetry has to do with the food poets must eat to enable them to compose. Somebody might *think* that 2 and 2 make 5: we know, by a process which rests on the laws of Logic, that they make 4.

This is a more satisfactory case of induction: Fermat stated that no integral values of x, y, and z can be found such that $x^n + y^n = z^n$, if n be an integer greater than 2. This theorem has been proved to be true for $n = 3, 4, 5, 7$, and many other numbers, and there is no reason to doubt that it is true. But to this day no general proof of it has been given.[8] This, then, is an example of a mathematical proposition which has been reached and stated as probably true by induction.

Now, in Greek geometry, propositions were stated and proved by the

[8] This is an example of the "theory of numbers," the study of the properties of integers, to which the chief contributions, perhaps, have been made by Fermat and Gauss.

laws of Logic—helped, as we now know, by tacit appeals to the conclusions which common sense draws from the pictorial representation in the mind of geometrical figures—about *any* triangles, say, or *some* triangles, and thus not about one or two particular things, but about an *infinity* of them. Thus, consider any two triangles *ABC* and *DEF*. It helps the thinking of most of us to draw pictures of particular triangles, but our conclusions do not hold merely for these triangles. If the sides *BA* and *AC* are equal in length to the sides *ED* and *DF* respectively, and the angle at *A* is equal to the angle at *D*, then *BC* is equal to *EF*. This is proved rather imperfectly in the fourth proposition of the first Book of Euclid's *Elements*.

When we examine into and complete the reasonings of geometricians, we find that the conception of space vanishes, and that we are left with logic alone. Philosophers and mathematicians used to think—and some do now—that, in geometry, we had to do, not with the space of ordinary life in which our houses stand and our friends move about, and which certain quaint people say is "annihilated" by electric telegraphs or motor cars, but an abstract form of the same thing from which all that is personal or material has disappeared, and only things like *distance* and *order* and *position* have remained. Indeed, some have thought that position did not remain; that, in abstract space, a circle, for example, had no position of its own, but only with respect to other things. Obviously, we can only, in practice, give the position of a thing with respect to other things—"relatively" and not "absolutely." These "relativists" denied that position had any properties which could not be practically discovered. Relativism, in a thought-out form, seems quite tenable; in a crude form, it seems like excluding the number 2, as distinguished from classes of two things, from notice as a figment of the brain, because it is not visible or tangible like a poker or a bit of radium or a mutton-chop.

In fact, a perfected geometry reduces to a series of deductions holding not only for figures in space, but for any abstract things. Spatial figures give a striking illustration of some abstract things; and that is the secret of the interest which analytical geometry has. But it is into algebra that we must now look to discover the nature of Mathematics.

We have seen that Egyptian arithmetic was more general than Egyptian geometry: like algebra, by using letters to denote unknown numbers, it began to consider propositions about *any* numbers. In algebra and algebraical geometry this quickly grew, and then it became possible to treat branches of mathematics in a systematic way and make whole classes of problems subject to the uniform and almost mechanical working of one method. Here we must again recall the economical function of science.

At the same time as methods—algebra and analytical geometry and the infinitesimal calculus—grew up from the application of mathematics to natural science, grew up also the new conceptions which influenced the form which mathematics took in the seventeenth, eighteenth, and nineteenth centuries. The ideas of *variable* and *function* became more and more prominent. These ideas were brought in by the conception of motion, and, unaffected by the doubts of the few logicians in the ranks of the mathematicians, remained to fructify mathematics. When mathematicians woke up to the necessity of explaining mathematics logically and finding out what Mathematics is, they found that, in mathematics the striving for generality had led, from very early times, to the use of a method of deduction used but not recognised and distinguished from the method usually used by the Aristotelians. I will try to indicate the nature of these methods, and it will be seen how the ideas of variable and function, in a form which does not depend on that particular kind of variability known as motion, come in.

A *proposition* in logic is the kind of thing which is denoted by such a phrase as: "Socrates was a mortal and the husband of a scold." If—and this is the characteristic of modern logic—we notice that the notions of variable and function (correspondence, relation) which appeared first in a special form in mathematics, are fundamental in all the things which are the objects of our thought, we are led to replace the particular conceptions in a proposition by variables, and thus see more clearly the structure of the proposition. Thus: "x is a y and has the relation R to z, a member of the class u" gives the general form of a multitude of propositions, of which the above is a particular case; the above proposition may be true, but it is not a judgment of logic, but of history or experience. The proposition is false if "Kant" or "Westminster Abbey" is substituted for "Socrates": it is neither if "x," a sign for a variable, is, and then becomes what we call a "propositional function" of x and denote by "ϕx" or "ψx." If more variables are involved, we have the notation "$\phi(x,y)$," and so on.

Relations between propositional functions may be true or false. Thus x is a member of the class a, and a is contained in the class b, together imply that x is a b, is true. Here the *implication* is true, and we do not say that the *functions* are. The kind of implication we use in mathematics is of the form: "If ϕx is true, then ψx is true"; that is, any particular value of x which makes ϕx true also makes ψx true.

From the perception that, when the notions of variable and function are introduced into logic, as their fundamental character necessitates, all mathematical methods and all mathematical conceptions can be defined in purely logical terms, leads us to see that Mathematics is only a part of Logic and is the class of all propositions of the form: $\phi(x,y,z, \ldots)$ implies, for all values of the variables, $\psi(x,y,z, \ldots)$. The structure of the propositional functions involves only such ideas as are fundamental in logic, like implication, class, relation, the relation of a term to a class of which it is a member, and so on. And, of course, Mathematics depends on the notion of Truth.

When we say that "$1 + 1 = 2$," we seem to be making a mathematical statement which does not come under the above definition. But the statement is rather mistakenly written: there is, of course, only *one* whole class of unit classes, and the notation "$1 + 1$" makes it look as if there were two. Remembering that 1 is a class of certain classes, what the above proposition means is: If x and y are members of 1, and x differs from y, then x and y together make up a member of 2.

At last, then, we arrive at seeing that the nature of Mathematics is independent of us personally and of the world outside, and we can feel that our own discoveries and views do not affect the Truth itself, but only the extent to which we or others see it. Some of us discover things in science, but we do not really create anything in science any more than Columbus created America. Common sense certainly leads us astray when we try to use it for the purposes for which it is not particularly adapted, just as we may cut ourselves and not our beards if we try to shave with a carving knife; but it has the merit of finding no difficulty in agreeing with those philosophers who have succeeded in satisfying themselves of the truth and position of Mathematics. Some philosophers have reached the startling conclusion that Truth is made by men, and that Mathematics is created by mathematicians, and that Columbus created America; but common sense, it is refreshing to think, is at any rate above being flattered by philosophical persuasion that it really occupies a place sometimes reserved for an even more sacred Being.

BIBLIOGRAPHY

THE view that science is dominated by the principle of the economy of thought has been, in part,[9] very thoroughly worked out by Ernst Mach (see especially the translation of his *Science of Mechanics*, 5th ed., La Salle, Ill., 1942). On the history of mathematics, we may mention W. W. Rouse Ball's books, *A Primer of the History of Mathematics* (7th ed., 1930), and the fuller *Short Account of the History of Mathematics* (4th ed., 1908, both published in London by Macmillan), and Karl Fink's *Brief History of Mathematics* (Chicago and London, 3rd ed., 1910).

As text-books of mathematics, De Morgan's books on *Arithmetic, Algebra,* and *Trigonometry* are still unsurpassed, and his *Trigonometry and Double Algebra* contains one of the best discussions of complex numbers, for students, that there is. As De Morgan's books are not all easy to get,

[9] *Cf.* above, pp. 5, 11, 13, 15, 16, 42.

the reprints of his *Elementary Illustrations of the Differential and Integral Calculus* and his work *On the Study and Difficulties of Mathematics* (Chicago and London, 1899 and 1902) may be recommended. Where possible, it is best to read the works of the great mathematicians themselves. For elementary books, Lagrange's *Lectures on Elementary Mathematics,* of which a translation has been published at Chicago and London (2nd ed., 1901), is the most perfect specimen.

The questions dealt with in the fourth chapter are more fully discussed in Mach's *Mechanics.* An excellent collection of methods and problems in graphical arithmetic and algebra, and so on, is contained in H. E. Cobb's book on *Elements of Applied Mathematics* (Boston and London: Ginn & Co., 1911).

PART II

Historical and Biographical

COMMENTARY ON

The Great Mathematicians

AT the outset of assembling this anthology I decided that I ought to include a biographical history of the subject. This would provide a setting for the other selections, and also serve as a small reference manual for the general reader. It was not easy to find a history which was brief, authoritative, elementary and readable. W. W. R. Ball's *A Primer of the History of Mathematics* is a book of merit but rather old-fashioned. J. W. N. Sullivan's *The History of Mathematics in Europe,* an admirable outline, carries the story only as far as the end of the eighteenth century; I commend this book to your attention. Dirk Struik's *A Concise History of Mathematics* has solid virtues but is a trifle too advanced for my purposes and at times dull. A few French and German books which might have been suitable were not considered because of the labor of translating them.

Turnbull's excellent little volume, a biographical history, turned out to meet the standard in all respects. It is the story of several great mathematicians, "representatives of their day in this venerable science." "I have tried to show," says Professor Turnbull in his preface, "how a mathematician thinks, how his imagination, as well as his reason, leads him to new aspects of the truth. Occasionally it has been necessary to draw a figure or quote a formula—and in such cases the reader who dislikes them may skip, and gather up the thread undismayed a little further on. Yet I hope that he will not too readily turn aside in despair, but will, with the help of the accompanying comment, find something to admire in these elegant tools of the craft." There is overlapping between this survey and the preceding selection, but the two books are complementary, and the reader who enjoys one will derive no less pleasure from the other. Jourdain makes ideas the heroes of his account while Turnbull devotes a good deal of space to lively sketches of the men who made the ideas.

H. W. Turnbull, distinguished for his researches in algebra (determinants, matrices, theory of equations), is Regius professor of mathematics at the University of St. Andrews in Scotland, a Fellow of the Royal Society, and, as demonstrated not only in this volume but in other writings, a gifted simplifier of mathematical ideas.

We think of Euclid as of fine ice; we admire Newton as we admire the Peak of Teneriffe. Even the intensest labors, the most remote triumphs of the abstract intellect, seem to carry us into a region different from our own—to be in a terra incognita *of pure reasoning, to cast a chill on human glory.* —WALTER BAGEHOT

Many small make a great. —CHAUCER

Everything of importance has been said before by somebody who did not discover it. —ALFRED NORTH WHITEHEAD

1 The Great Mathematicians

By HERBERT WESTREN TURNBULL

PREFACE

THE usefulness of mathematics in furthering the sciences is commonly acknowledged: but outside the ranks of the experts there is little inquiry into its nature and purpose as a deliberate human activity. Doubtless this is due to the inevitable drawback that mathematical study is saturated with technicalities from beginning to end. Fully conscious of the difficulties in the undertaking, I have written this little book in the hope that it will help to reveal something of the spirit of mathematics, without unduly burdening the reader with its intricate symbolism. The story is told of several great mathematicians who are representatives of their day in this venerable science. I have tried to show how a mathematician thinks, how his imagination, as well as his reason, leads him to new aspects of the truth. Occasionally it has been necessary to draw a figure or quote a formula—and in such cases the reader who dislikes them may skip, and gather up the thread undismayed a little further on. Yet I hope that he will not too readily turn aside in despair, but will, with the help of the accompanying comment, find something to admire in these elegant tools of the craft.

Naturally in a work of this size the historical account is incomplete: a few references have accordingly been added for further reading. It is pleasant to record my deep obligation to the writers of these and other larger works, and particularly to my college tutor, the late Mr. W. W. Rouse Ball, who first woke my interest in the subject. My sincere thanks are also due to several former and present colleagues in St. Andrews who have made a considerable and illuminating study of mathematics among the Ancients: and to kind friends who have offered many valuable suggestions and criticisms.

In preparing the Second Edition I have had the benefit of suggestions which friends from time to time have submitted. I am grateful for this means of removing minor blemishes, and for making a few additions. In particular, a date list has been added.

PREFACE TO THIRD EDITION

A FEW additions have been made to the earlier chapters and to Chapter VI, which incorporate results of recent discoveries among mathematical inscriptions and manuscripts, particularly those which enlarge our knowledge of the mathematics of Ancient Babylonia and Egypt. I gratefully acknowledge the help derived from reading the *Manual of Greek Mathematics* (1931) by Sir Thomas Heath. It provides a short but masterly account of these developments, for which the scientific world is greatly indebted.

H. W. T.

December, 1940.

PREFACE TO FOURTH EDITION

AT the turn of the half-century it is appropriate to add a postscript to Chapter XI, which brought the story of mathematical development as far as the opening years of the century. What has happened since has followed very directly from the wonderful advances that opened up through the algebraical discoveries of Hamilton, the analytical theories of Weierstrass and the geometrical innovations of Von Staudt, and of their many great contemporaries. One very noteworthy development has been the rise of American mathematics to a place in the front rank, and this has come about with remarkable rapidity and principally through the study of abstract algebra such as was inspired by Peirce, a great American disciple of the Hamiltonian school. Representative of this advance in algebra is Wedderburn who built upon the foundations laid, not only by Peirce, but also by Frobenius in Germany and Cartan in France. Through abandoning the commutative law of multiplication by inventing quaternions, Hamilton had opened the door for the investigation of systems of algebra distinct from the ordinary familiar system. Algebra became algebras just as, through the discovery of non-Euclidean systems, geometry became geometries. This plurality, which had been unsuspected for so long, naturally led to the study of the classification of algebras. It was here that Wedderburn, following a hint dropped by Cartan, attained great success. The matter led to deeper and wider understanding of abstract theory, while at the same time it provided a welcome and fertile medium for the further developments in quantum mechanics. Simultaneously with this abstract approach to algebra a powerful advance was made in the technique of algebraical manipulation through the discoveries of Frobenius, Schur and A. Young in the theory of groups and of their representations and applications.

Similar trends may be seen in arithmetic and analysis where the same plurality is in evidence. Typical of this are the theory of valuation and the recognition of Banach spaces. The axiom of Archimedes (p. 99) is here in jeopardy: which is hardly surprising once the concept of regular equal steps upon a straight line had been broadened by the newer forms of geometry. Arithmetic and analysis were, so to speak, projected and made more abstract. It is remarkable that, with these trends towards generalization in each of the four great branches of pure mathematics, the branches lose something of their distinctive qualities and grow more alike. Whitehead's description of geometry as the science of cross-classification remains profoundly true. The applications of mathematics continue to extend, particularly in logic and in statistics.

H. W. T.

May, 1951.

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DATE LIST

? 18th Century B.C.	Ahmes (? 1800–).
6th " "	Thales (640–550), Pythagoras (569–500).
5th " "	Anaxagoras (500–428), Zeno (495–435), Hippocrates (470–), Democritus (? 470–).
4th " "	Archytas (? 400), Plato (429–348), Eudoxus (408–355), Menaechmus (375–325).
3rd " "	Euclid (? 330–275), Archimedes (287–212), Apollonius (? 262–200).
2nd " "	Hipparchus (? 160–).
1st " A.D.	Menelaus (? 100).
2nd " "	Ptolemy (? 100–168).
3rd " "	Hero (? 250), Pappus (? 300), Diophantus (–320 ?).
6th " "	Arya-Bhata (? 530).
7th " "	Brahmagupta (? 640).
12th " "	Leonardo of Pisa (1175–1230).
16th " "	Scipio Ferro (1465–1526), Tartaglia (1500–1557), Cardan (1501–1576), Copernicus (1473–1543), Vieta (1540–1603), Napier (1550–1617), Galileo (1564–1642), Kepler (1571–1630), Cavalieri (1598–1647).
17th " "	Desargues (1593–1662), Descartes (1596–1650), Fermat (1601–1665), Pascal (1623–1662), Wallis (1616–1703), Barrow (1630–1677), Gregory (1638–1675), Newton (1642–1727), Leibniz (1646–1716), Jacob Bernoulli (1654–1705), John Bernoulli (1667–1748).
18th " "	Euler (1707–1783), Demoivre (1667–1754), Taylor (1685–1741), Maclaurin (1698–1746), D'Alembert (1717–1783), Lagrange (1736–1813), Laplace (1749–1827), Cauchy (1759–1857).
19th " "	Gauss (1777–1855), Von Staudt (1798–1867), Abel (1802–1829), Hamilton (1805–1865), Galois (1811–1832), Riemann (1826–1866), Sylvester (1814–1897), Cayley (1821–1895), Weierstrass (1815–1897), and many others.
20th " "	Ramanujan (1887–1920), and many living mathematicians.

CHAPTER I

EARLY BEGINNINGS: THALES, PYTHAGORAS AND THE PYTHAGOREANS

TO-DAY with all our accumulated skill in exact measurements, it is a noteworthy event when lines driven through a mountain meet and make a tunnel. How much more wonderful is it that lines, starting at the corners of a perfect square, should be raised at a certain angle and successfully brought to a point, hundreds of feet aloft! For this, and more, is what is meant by the building of a pyramid: and all this was done by the Egyptians in the remote past, far earlier than the time of Abraham.

Unfortunately we have no actual record to tell us who first discovered enough mathematics to make the building possible. For it is evident that such gigantic erections needed very accurate plans and models. But many general statements of the rise of mathematics in Egypt are to be found in the writings of Herodotus and other Greek travellers. Of a certain king Sesostris, Herodotus says:

'This king divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levied yearly. But everyone from whose part the river tore anything away, had to go to him to notify what had happened; he then sent overseers who had to measure out how much the land had become smaller, in order that the owner might pay on what was left, in proportion to the entire tax imposed. In this way, it appears to me, geometry originated, which passed thence to Hellas.'

Then in the *Phaedrus* Plato remarks:

'At the Egyptian city of Naucratis there was a famous old god whose name was Theuth; the bird which is called the Ibis was sacred to him, and he was the inventor of many arts, such as arithmetic and calculation and geometry and astronomy and draughts and dice, but his great discovery was the use of letters.'

According to Aristotle, mathematics originated because the priestly class in Egypt had the leisure needful for its study; over two thousand years later exact corroboration of this remark was forthcoming, through the discovery of a papyrus, now treasured in the Rhind collection at the British Museum. This curious document, which was written by the priest Ahmes, who lived before 1700 B.C., is called 'directions for knowing all dark things'; and the work proves to be a collection of problems in geometry and arithmetic. It is much concerned with the reduction of fractions such as $2/(2n+1)$ to a sum of fractions each of whose numerators is unity. Even with our improved notation it is a complicated matter to work through such remarkable examples as:

$$2/29 = 1/24 + 1/58 + 1/174 + 1/232.$$

There is considerable evidence that the Egyptians made astonishing progress in the science of exact measurements. They had their land surveyors, who were known as *rope stretchers,* because they used ropes, with knots or marks at equal intervals, to measure their plots of land. By this simple means they were able to construct right angles; for they knew that three ropes, of lengths three, four, and five units respectively, could be formed into a right-angled triangle. This useful fact was not confined to Egypt: it was certainly known in China and elsewhere. But the Egyptian skill in practical geometry went far beyond the construction of right angles: for it included, besides the angles of a square, the angles of other regular figures such as the pentagon, the hexagon and the heptagon.

If we take a pair of compasses, it is very easy to draw a circle and then to cut the circumference into *six* equal parts. The six points of division form a regular hexagon, the figure so well known as the section of the honey cell. It is a much more difficult problem to cut the circumference into *five* equal parts, and a very much more difficult problem to cut it into *seven* equal parts. Yet those who have carefully examined the design of the ancient temples and pyramids of Egypt tell us that these particular figures and angles are there to be seen. Now there are two distinct methods of dealing with geometrical problems—the practical and the theoretical. The Egyptians were champions of the practical, and the Greeks of the theoretical method. For example, as Röber has pointed out, the Egyptians employed a practical rule to determine the angle of a regular heptagon. And although it fell short of theoretical precision, the rule was accurate enough to conceal the error, unless the figure were to be drawn on a grand scale. It would barely be apparent even on a circle of radius 50 feet.

Unquestionably the Egyptians were masters of practical geometry; but whether they knew the theory, the underlying reason for their results, is another matter. Did they know that their right-angled triangle, with sides of lengths three, four and five units, contained an *exact* right angle? Probably they did, and possibly they knew far more. For, as Professor D'Arcy Thompson has suggested, the very *shape* of the Great Pyramid indicates a considerable familiarity with that of the regular pentagon. A certain obscure passage in Herodotus can, by the slightest literal emendation, be made to yield excellent sense. It would imply that the area of each triangular face of the Pyramid is equal to the square of the vertical height; and this accords well with the actual facts. If this is so, the ratios of height, slope and base can be expressed in terms of the 'golden section', or of the radius of a circle to the side of the inscribed decagon. In short, there was already a wealth of geometrical and arithmetical results treasured by the priests of Egypt, before the early Greek travellers became acquainted with mathematics. But it was only after the keen imaginative eye of the Greek fell upon these Egyptian figures that they yielded up their wonderful secrets and disclosed their inner nature.

Among these early travellers was THALES, a rich merchant of Miletus, who lived from about 640 to 550 B.C. As a man of affairs he was highly successful: his duties as merchant took him to many countries, and his native wit enabled him to learn from the novelties which he saw. To his admiring fellow-countrymen of later generations he was known as one of the Seven Sages of Greece, many legends and anecdotes clustering round his name. It is said that Thales was once in charge of some mules, which were burdened with sacks of salt. Whilst crossing a river one of the animals slipped; and the salt consequently dissolving in the water, its load became instantly lighter. Naturally the sagacious beast deliberately submerged itself at the next ford, and continued this trick until Thales hit upon the happy expedient of filling the sack with sponges! This proved an effectual cure. On another occasion, foreseeing an unusually fine crop of olives, Thales took possession of every olive-press in the district; and having made this 'corner', became master of the market and could dictate his own terms. But now, according to one account, as he had *proved* what could be done, his purpose was achieved. Instead of victimizing his buyers, he magnanimously sold the fruit at a price reasonable enough to have horrified the financier of to-day.

Like many another merchant since his time Thales early retired from commerce, but unlike many another he spent his leisure in philosophy and mathematics. He seized on what he had learnt in his travels, particularly from his intercourse with the priests of Egypt; and he was the first to bring out something of the true significance of Egyptian scientific lore. He was both a great mathematician and a great astronomer. Indeed, much of his popular celebrity was due to his successful prediction of a solar eclipse in 585 B.C. Yet it is told of him that in contemplating the stars during an evening walk, he fell into a ditch; whereupon the old woman attending him exclaimed, 'How canst thou know what is doing in the heavens when thou seest not what is at thy feet?'

We live so far from these beginnings of a rational wonder at natural things, that we run the risk of missing the true import of results now so very familiar. There are the well-known propositions that a circle is bisected by any diameter, or that the angles at the base of an isosceles triangle are equal, or that the angle in a semicircle is a right angle, or that the sides about equal angles in similar triangles are proportional. These and other like propositions have been ascribed to Thales. Simple as they are, they mark an epoch. They elevate the endless details of Egyptian mensuration to general truths; and in like manner his astronomical results replace what was little more than the making of star catalogues by a genuine science.

It has been well remarked that in this geometry of Thales we also have the true source of algebra. For the theorem that the diameter bisects a circle is indeed a true equation; and in his experiment conducted, as Plutarch says, 'so simply, without any fuss or instrument' to determine the height of the Great Pyramid by comparing its shadow with that of a vertical stick, we have the notion of equal ratios, or proportion.

The very idea of abstracting all solidity and area from a material shape, such as a square or triangle, and pondering upon it as a pattern of lines, seems to be definitely due to Thales. He also appears to have been the first to suggest the importance of a geometrical *locus*, or curve traced out by a point moving according to a definite law. He is known as the father of Greek mathematics, astronomy and philosophy, for he combined a practical sagacity with genuine wisdom. It was no mean achievement, in his day, to break through the pagan habit of mind which concentrates on particular cults and places. Thales asserted the existence of the abstract and the more general: these, said he, were worthier of deep study than the intuitive or sensible. Here spoke the philosopher. On the other hand he gave to mankind such practical gifts as the correct number of days in the year, and a convenient means of finding by observation the distance of a ship at sea.

Thales summed up his speculations in the philosophical proposition 'All things are water'. And the fact that all things are not water is trivial compared with the importance of his outlook. He saw the field; he asked the right questions; and he initiated the search for underlying law beneath all that is ephemeral and transient.

Thales never forgot the debt that he owed to the priests of Egypt; and when he was an old man he strongly advised his pupil PYTHAGORAS to pay them a visit. Acting upon this advice, Pythagoras travelled and gained a wide experience, which stood him in good stead when at length he settled and gathered round him disciples of his own, and became even more famous than his master. Pythagoras is supposed to have been a native of Samos, belonging like Thales to the Ionian colony of Greeks planted on the western shores and islands of what we now call Asia Minor. He lived from about 584 B.C. to 495 B.C. In 529 B.C. he settled at Crotona, a town of the Dorian colony in South Italy, and there he began to lecture upon philosophy and mathematics. His lecture-room was thronged with enthusiastic hearers of all ranks. Many of the upper classes attended, and even women broke a law which forbade them to attend public meetings, and flocked to hear him. Among the most attentive was Theano, the young and beautiful daughter of his host Milo, whom he married. She wrote a biography of her husband, but unfortunately it is lost.

So remarkable was the influence of this great master that the more attentive of his pupils gradually formed themselves into a society or brotherhood. They were known as the Order of the Pythagoreans, and they were soon exercising a great influence far across the Grecian world. This influence was not so much political as religious. Members of the Society shared everything in common, holding the same philosophical beliefs, engaging in the same pursuits, and binding themselves with an oath not to reveal the secrets and teaching of the school. When, for example, Hippasus perished in a shipwreck, was not his fate due to a broken promise? For he had divulged the secret of the sphere with its twelve pentagons!

A distinctive badge of the brotherhood was the beautiful star pentagram—a fit symbol of the mathematics which the school discovered. It was also the symbol of health. Indeed, the Pythagoreans were specially interested in the study of medicine. Gradually, as the Society spread, teachings once treasured orally were committed to writing. Thereby a copy of a treatise by Philolaus, we are told, ultimately came into the possession of Plato; probably a highly significant event in the history of mathematics.

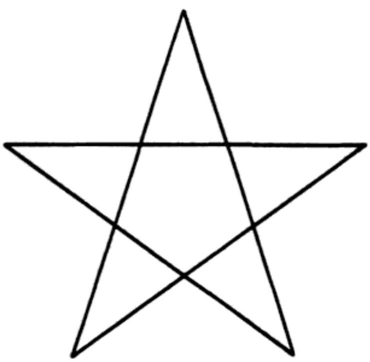


FIGURE 1

In mathematics the Pythagoreans made very great progress, particularly in the theory of numbers and in the geometry of areas and solids. As it was the generous practice among members of the brotherhood to attribute all credit for each new discovery to Pythagoras himself, we cannot be quite certain about the authorship of each particular theorem. But at any rate in the mathematics which are now to be described, his was the dominating influence.

In thinking of these early philosophers we must remember that open air and sunlight and starry nights formed their surroundings—not our grey mists and fettered sunshine. As Pythagoras was learning his mensuration from the priests of Egypt, he would constantly see the keen lines cast by the shadows of the pillars across the pavements. He trod chequered floors with their arrays of alternately coloured squares. His mind was stirred by interesting geometrical truths learnt from his master Thales, his interest in number would lead him to count the squares, and the sight of the long straight shadow falling obliquely across them would suggest sequences of special squares. It falls maybe across the centre of the first, the fourth, the seventh; the arithmetical progression is suggested. Then again the square is interesting for its *size.* A fragment of more diverse pattern would demonstrate a larger square enclosing one exactly half its size. A

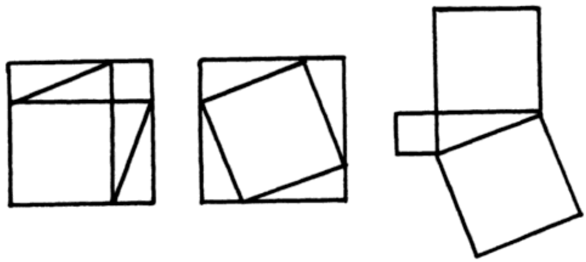
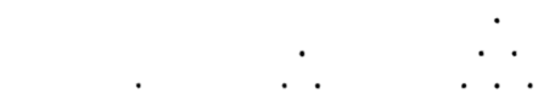


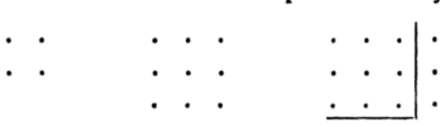
FIGURE 2

little imaginative thought would reveal, within the larger, a smaller square placed unsymmetrically, and so would lead to the great theorem which somehow or other was early reached by the brotherhood (and some say by Pythagoras himself), that the square on one side of a right-angled triangle is equal to the sum of the squares on the remaining sides. The above figures (Figure 2) actually suggest the proof, but it is quite possible that several different proofs were found, one being by the use of similar triangles. According to one story, when Pythagoras first discovered this fine result, in his exultation he sacrificed an ox!

Influenced no doubt by these same orderly patterns, he pictured numbers as having characteristic designs. There were the triangular numbers, one, three, six, ten, and so on, ten being the *holy tetractys,* another sym-



bol highly revered by the brotherhood. Also there were the square numbers, each of which could be derived from its predecessor by adding an

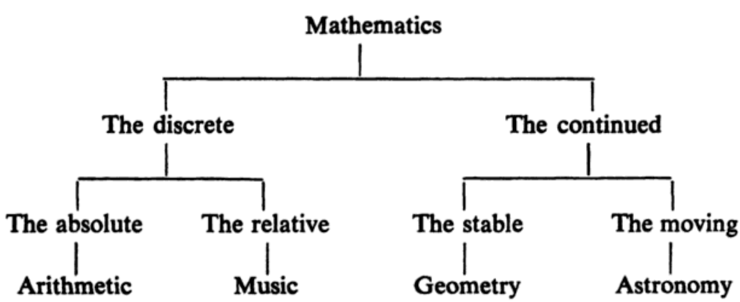


L-shaped border. Great importance was attached to this border: it was called a gnomon (γνώμων, carpenter's rule). Then it was recognized that each odd number, three, five, seven, etc., was a gnomon of a square number. For example, seven is the gnomon of the square of three to make the square of four. Pythagoras was also interested in the more abstract natural objects, and he is said to have discovered the wonderful harmonic progressions in the notes of the musical scale, by finding the relation between the length of a string and the pitch of its vibrating note. Thrilled by his discovery, he saw in numbers the element of all things. To him numbers were no mere attributes: three was not that which is common to three cats or three books or three Graces: but numbers were themselves the stuff out of which all objects we see or handle are made—the rational reality. Let us not judge the doctrine too harshly; it was a great advance on the cruder water philosophy of Thales.

So, in geometry, *one* came to be identified with the point; *two* with the line, *three* with the surface, and *four* with the solid. This is a noteworthy disposition that really is more fruitful than the usual allocation in which the line is said to have one, the surface two, and the solid three, dimensions.

More whimsical was the attachment of *seven* to the maiden goddess Athene 'because seven alone within the decade has neither factors nor product'. *Five* suggested marriage, the union of the first even with the first genuine odd number. *One* was further identified with reason; *two* with opinion—a wavering fellow is Two; he does not know his own mind: *four* with justice, steadfast and square. Very fanciful no doubt: but has not Ramanujan, one of the greatest arithmeticians of our own days, been thought to treat the positive integers as his personal friends? In spite of this exuberance the fact remains, as Aristotle sums it up: 'The Pythagoreans first applied themselves to mathematics, a science which they improved; and, penetrated with it, they fancied that the principles of mathematics were the principles of all things'. And a younger contemporary, Eudemus, shrewdly remarked that 'they changed geometry into a liberal science; they diverted arithmetic from the service of commerce'.

To Pythagoras we owe the very word mathematics and its doubly twofold branches:



This classification is the origin of the famous Quadrivium of knowledge.

In geometry Pythagoras or his followers developed the theory of space-filling figures. The more obvious of these must have been very well known. If we think of each piece in such a figure as a unit, the question arises, can we fill a flat surface with repetitions of these units? It is very likely that this type of inquiry was what first led to the theorem that the three angles of a triangle are together equal to two right angles. The same train of thought also extends naturally to solid geometry, including the conception of regular solids. One of the diagrams (Figure 3) shows six equal triangles filling flat space round their central point. But five such equilateral triangles can likewise be fitted together, to form a blunted bell-tent-shaped figure, spreading from a central vertex: and now their bases form a regular pentagon. Such a figure is no longer flat; it makes a solid angle, the corner, in fact, of a regular icosahedron. The process could be repeated by surrounding each vertex of the original triangles with five triangles. Exactly twenty triangles would be needed, no more and no less, and the

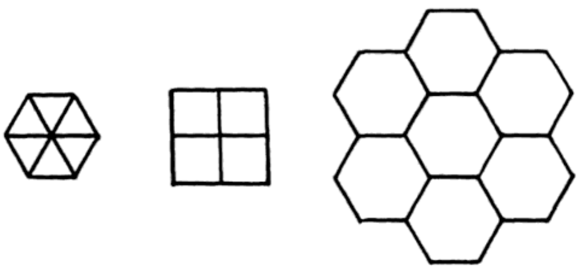


FIGURE 3

result would be the beautiful figure of the icosahedron of twenty triangles surrounding its twelve vertices in circuits of five.

It is remarkable that in solid geometry there are only five such regular figures, and that in the plane there is a very limited number of associations of regular space-filling figures. The three simplest regular solids, including the cube, were known to the Egyptians. But it was given to Pythagoras to discover the remaining two—the dodecahedron with its twelve pentagonal faces, and the icosahedron. Nowadays we so often become acquainted with these regular solids and plane figures only after a long excursion through the intricacies of mensuration and plane geometry that we fail to see their full simplicity and beauty.

Another kind of problem that interested Pythagoras was called the *method of application of areas*. His solution is noteworthy because it provided the geometrical equivalent of solving a quadratic equation in algebra. The main problem consisted in drawing, upon a given straight line, a figure that should be the size of one and the shape of another given figure. In the course of the solution one of three things was bound to happen. The base of the constructed figure would either fit, fall short of, or exceed the length of the given straight line. Pythagoras thought it proper to draw attention to these three possibilities; accordingly he introduced the terms *parabole, ellipsis* and *hyperbole.* Many years later his nomenclature was adopted by Apollonius, the great student of the conic section, because the same threefold characteristics presented themselves in the construction of the curve. And we who follow Apollonius still call the curve the parabola, the ellipse, or the hyperbola, as the case may be. The same threefold classification underlies the signs $=$, $<$, $>$ in arithmetic.

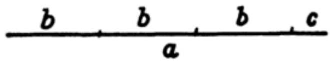


FIGURE 4

Many a time throughout the history of mathematics this classification has proved to be the key to further discoveries.

For example, it is closely connected with the theory of irrational numbers; and this brings us to the greatest achievement of Pythagoras, who is credited with discovering the (ἄλογον) irrational. In other words, he proved that it was not always possible to find a common measure for two given lengths a and b. The practice of measuring one line against another must have been very primitive. Here is a long line a, into which the shorter line b fits three times, with a still shorter piece c left over (Figure 4). To-day we express this by the equation $a = 3b + c$, or more generally by $a = nb + c$. If there is no such remainder c, the line b measures a; and a is called a multiple of b. If, however, there is a remainder c, further subdivision might perhaps account for each length a, b, c without remainder: experiment might show, for instance, that in tenths of inches, $a = 17$, $b = 5$, $c = 2$. At one time it was thought that it was *always* possible to reduce lengths a and b to such multiples of a smaller length. It appeared to be simply a question of patient subdivision, and sooner or later the desired measure would be found. So the required subdivision, in the present example, is found by measuring b with c. For c fits twice into b with a remainder d; and d fits exactly twice into c *without* remainder. Consequently d measures c, and also measures b and also a. This is how the numbers 17, 5 and 2 come to be attached to a, b, and c: namely a contains d 17 times.

Incidentally this shows how naturally the arithmetical progression arises. For although the original subdivisions, and extremity, of the line a occur at distances 5, 10, 15, 17, measured from the left in quarter inches, they occur at distances 2, 7, 12, 17, from the right. These numbers form a typical arithmetical progression, with a rhythmical law of succession that alone would be incentive for a Pythagorean to study them further.

This reduction of the comparison of a line a with a line b to that of the number 17 with 5, or, speaking more technically, this reduction of the ratio $a : b$ to $17 : 5$ would have been agreeable to the Pythagorean. It exactly fitted in with his philosophy; for it helped to reduce space and geometry to pure number. Then came the awkward discovery, by Pythagoras himself, that the reduction was not always possible; that something in geometry eluded whole numbers. We do not know exactly how this discovery of the *irrational* took place, although two early examples can be cited. First when a is the diagonal and b is the side of a square, no common measure can be found; nor can it be found in a second example, when a line a is divided in *golden section* into parts b and c. By this is meant that the ratio of a, the whole line, to the part b is equal to the ratio of b to the other part c. Here c may be fitted once into b with remainder d: and then d may be fitted once into c with a remainder e: and so on. It is not hard to prove that such lengths $a, b, c, d, \ldots$ form a geometrical progression without end; and the desired common measure is never to be found.

If we prefer algebra to geometry we can verify this as follows. Since it is given that $a = b + c$ and also $a : b :: b : c$, it follows that $a(a - b) = b^2$. This is a quadratic equation for the ratio $a : b$, whose solution gives the result

$$a : b = \sqrt{5} + 1 : 2.$$

The presence of the surd $\sqrt{5}$ indicates the irrational.

The underlying reason why such a problem came to be studied is to be found in the star badge of the brotherhood (p. 83); for every line therein is divided in this golden section. The star has five lines, each cut into three

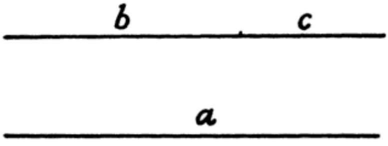


FIGURE 5

parts, the lengths of which can be taken as a, b, a. As for the ratio of the diagonal to the side of a square, Aristotle suggests that the Pythagorean proof of its irrationality was substantially as follows:

If the ratio of diagonal to side is commensurable, let it be $p : q$, where

p and q are whole numbers prime to one another. Then p and q denote the number of equal subdivisions in the diagonal and the side of a square respectively. But since the square on the diagonal is double that on the side, it follows that $p^2 = 2q^2$. Hence p^2 is an even number, and p itself must be even. Therefore p may be taken to be $2r$, p^2 to be $4r^2$, and consequently q^2 to be $2r^2$. This requires q to be even; which is impossible because two numbers p, q, prime to each other cannot both be even. So the original supposition is untenable: no common measure can exist; and the ratio is therefore irrational.

This is an interesting early example of an indirect proof or *reductio ad absurdum*; and as such it is a very important step in the logic of mathematics.

We can now sum up the mathematical accomplishments of these early Greek philosophers. They advanced in geometry far enough to cover roughly our own familiar school course in the subject. They made substantial progress in the theoretical side of arithmetic and algebra. They had a geometrical equivalent for our method of solving quadratic equations; they studied various types of progressions, arithmetical, geometrical and harmonical. In Babylon, Pythagoras is said to have learnt the 'perfect proportion'

$$a : \frac{a+b}{2} :: \frac{2ab}{a+b} : b$$

which involves the arithmetical and harmonical means of two numbers. Indeed, to the Babylonians the Greeks owed many astronomical facts, and the sexagesimal method of counting by sixties in arithmetic. But they lacked our arithmetical notation and such useful abbreviations as are found in the theory of indices. From a present-day standpoint these results may be regarded as elementary: it is otherwise with their discovery of irrational numbers. That will ever rank as a piece of essentially advanced mathematics. As it upset many of the accepted geometrical proofs it came as a 'veritable logical scandal'. Much of the mathematical work in the succeeding era was coloured by the attempt to retrieve the position, and in the end this was triumphantly regained by Eudoxus.

Recent investigations of the Rhind Papyrus, the Moscow Papyrus of the Twelfth Egyptian Dynasty, and the Strassburg Cuneiform texts have greatly added to the prestige of Egyptian and Babylonian mathematics. While no general proof has yet been found among these sources, many remarkable *ad hoc* formulae have come to light, such as the Babylonian solution of complicated quadratic equations dating from 2000 B.C., which O. Neugebauer published in 1929, and an Egyptian approximation to the area of a sphere (equivalent to reckoning $\pi = 256/81$).

CHAPTER II

EUDOXUS AND THE ATHENIAN SCHOOL

A SECOND stage in the history of mathematics occupied the fifth and fourth centuries B.C., and is associated with Athens. For after the wonderful victories at Marathon and Salamis early in the fifth century, when the Greeks defeated the Persians, Athens rose to a position of pre-eminence. The city became not only the political and commercial, but the intellectual centre of the Grecian world. Her philosophers congregated from East and West, many of whom were remarkable mathematicians and astronomers. Perhaps the greatest among these were Hippocrates, Plato, Eudoxus and Menaechmus; and contemporary with the three latter was Archytas the Pythagorean, who lived at Tarentum.

Thales and Pythagoras had laid the foundations of geometry and arithmetic. The Athenian school concentrated upon special aspects of the superstructure; and, whether by accident or design, found themselves embarking upon three great problems: (i) the *duplication of the cube*, or the attempt to find the edge of a cube whose volume is double that of a given cube; (ii) the *trisection of a given angle*; and (iii) the *squaring of a circle*, or, the attempt to find a square whose area is equal to that of a given circle. These problems would naturally present themselves in a systematic study of geometry; while, as years passed and no solutions were forthcoming they would attract increasing attention. Such is their inherent stubbornness that not until the nineteenth century were satisfactory answers to these problems found. Their innocent enunciations are at once an invitation and a paradox. Early attempts to solve them led indirectly to results that at first sight seem to involve greater difficulties than the problems themselves. For example, in trying to square the circle Hippocrates discovered that two moon-shaped figures could be drawn whose areas were together equal to that of a right-angled triangle. This diagram (Figure 6) with its three semicircles described on the respective sides of the triangle illustrates his theorem. One might readily suppose that it would be easier to determine the area of a single circle than that of these lunes, or lunules, as they are called, bounded by pairs of circular arcs. Yet such is not the case.

In this by-product of the main problem Hippocrates gave the first example of a solution in *quadratures*. By this is meant the problem of constructing a rectilinear area equal to an area bounded by one or more curves. The sequel to attempts of this kind was the invention of the integral calculus by Archimedes, who lived in the next century. But his first success in the method was not concerned with the area of a circle, but

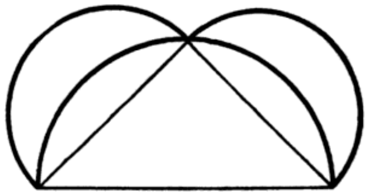


FIGURE 6

with that of a parabola, a curve that had been discovered by Menaechmus in an attempt to duplicate the cube. This shows how very interdependent mathematics had now become with its interplay between branch and branch. All this activity led to the discovery of many other new curves, including the ellipse, the hyperbola, the quadratrix, the conchoid (the shell), the cissoid (the ivy leaf), various spirals, and other curves classed as *loci on surfaces.*

The Greeks now found it useful to adopt a special classification for their problems, calling them *plane, solid* and *linear*. Problems were *plane* if their solution depended only on the use of straight lines and circles. These were of distinctly the Pythagorean type. They were *solid* if they depended upon conic sections: and they were *linear* if in addition they depended upon still more complicated curves. This early classification shows true mathematical insight, because later experience has revealed close algebraic and analytic parallels. For example, the *plane* problem corresponds in algebra to the problem soluble by quadratic equations. The Greeks quite naturally but vainly supposed that the three famous problems above were soluble by plane methods. It is here that they were wrong: for by solid or linear methods the problems were not necessarily insoluble.

One of the first philosophers to bring the new learning from Ionia to Athens was ANAXAGORAS (? 500–428 B.C.), who came from near Smyrna. He is said to have neglected his possessions, which were considerable, in order to devote himself to science, and in reply to the question, what was the object of being born, he remarked: 'The investigation of the sun, moon and heaven.' In Athens he shared the varying fortunes of his friend Pericles, the great statesman, and at one time was imprisoned for impiety. This we know from an ancient record which adds that 'while in prison he wrote (or drew) the squaring of the circle', a brief but interesting allusion to the famous problem. Nor has the geometry of the circle suffered unduly from the captivity of its devotees. Centuries later another great chapter was opened, when the Russians flung Poncelet, an officer serving under Napoleon, into prison, where he discovered the circular points at infinity. Anaxagoras, however, was famous chiefly for his work in astronomy.

HIPPOCRATES[1] was his younger contemporary, who came from Chios to Athens about the middle of the fifth century. A lawsuit originally lured him to the city: for he had lost considerable property in an attack by Athenian pirates near Byzantium. Indeed, the tastes of Athenian citizens were varied: they were not all artists, sculptors, statesmen, dramatists, philosophers, or honest seamen, in spite of the wealth there and then assembled. After enduring their ridicule first at being cheated and then for hoping to recover his money, the simple-minded Hippocrates gave up the quest, and found his solace in mathematics and philosophy.

He made several notable advances. He was the first author who is known to have written an account of elementary mathematics; in particular he devoted his attention to properties of the circle. To-day his actual work survives among the theorems of Euclid, although his original book is lost. His chief result is the proof of the statement that circles are to one another in the ratio of the squares on their diameters. This is equivalent to the discovery of the formula πr^2 for the area of a circle in terms of its radius. It means that a certain number π exists, and is the same for all circles, although his method does not give the actual numerical value of π. It is thought that he reached his conclusions by looking upon a circle as the limiting form of a regular polygon, either inscribed or circumscribed. This was an early instance of the *method of exhaustions*—a particular use of approximation from below and above to a desired limit.

The introduction of the method of exhaustions was an important link in the chain of thought culminating in the work of Eudoxus and Archimedes. It brought the prospect of unravelling the mystery of irrational numbers, that had sorely puzzled the early Pythagoreans, one stage nearer. A second important but perhaps simpler work of Hippocrates was an example of the useful device of reducing one theorem to another. The Pythagoreans already had shown how to find the geometric mean between two magnitudes by a geometrical construction. They merely drew a square equal to a given rectangle. Hippocrates now showed that to duplicate a cube was tantamount to finding *two* such geometric means. Put into more familiar algebraic language, if

$$a : x = x : b, \text{ then } x^2 = ab,$$

and if

$$a : x = x : y = y : 2a$$

then $x^3 = 2a^3$. Consequently if a was the length of the edge of the given cube, x would be that of a cube twice its size. But the statement also shows that x is the first of two geometric means between a and $2a$.

We must, of course, bear in mind that the Greeks had no such convenient algebraic notation as the above. Although they went through the

[1] Not the great physician.

same reasoning and reached the same conclusions as we can, their statements were prolix, and afforded none of the help which we find in these concise symbols of algebra.

It is supposed that the study of the properties of two such means, x and y, between given lengths a and b, led to the discovery of the parabola and hyperbola. As we should say, nowadays, the above continued proportions indicate the equations $x^2 = ay$, and $xy = 2a^2$. These equations represent a parabola and a hyperbola: taken together they determine a point of intersection which is the key to the problem. This is an instance of a *solid* solution for the duplication of the cube. It represents the ripe experience of the Athenian school; for MENAECHMUS (? 375–325 B.C.), to whom it is credited, lived a century later than Hippocrates.

Where two lines, straight or curved, cross, is a point: where three surfaces meet is a point. The two walls and the ceiling meeting at the corner of a room give a convenient example. But two curved walls, meeting a curved ceiling would also make a corner, and in fact illustrate a truly ingenious method of dealing with this same problem of the cube. The author of this geometrical novelty was ARCHYTAS (? 400 B.C.), a contemporary of Menaechmus. This time the problem was reduced to finding the position of a certain point in space: and the point was located as the meeting-place of three surfaces. For one surface Archytas chose that generated by a circle revolving about a fixed tangent as axis. Such a surface can be thought of as a ring, although the hole through the ring is completely stopped up. His other surfaces were more commonplace, a cylinder and a cone. With this unusual choice of surfaces he succeeded in solving the problem. When we bear in mind how little was known, in his day, about solid geometry, this achievement must rank as a gem among mathematical antiquities. Archytas, too, was one of the first to write upon mechanics, and he is said to have been very skilful in making toys and models—a wooden dove which could fly, and a rattle which, as Aristotle says, 'was useful to give to children to occupy them from breaking things about the house (for the young are incapable of keeping still)'.

Unlike the majority of mathematicians who lived in this Athenian era, Archytas lived at Tarentum in South Italy. He found time to take a considerable part in the public life of his city, and is known for his enlightened attitude in his treatment of slaves and in the education of children. He was a Pythagorean, and was also in touch with the philosophers of Athens, numbering Plato among his friends. He is said upon one occasion to have used his influence in high quarters to save the life of Plato.

Between Crotona and Tarentum upon the shore of the gulf of Southern Italy was the city of Elea: and with each of these places we may associate a great philosopher or mathematician. At Crotona Pythagoras had instituted his lecture-room; nearly two centuries later Archytas made his mechanical models at Tarentum. But about midway through the intervening period there lived at Elea the philosopher ZENO. This acutely original thinker played the part of philosophical critic to the mathematicians, and some of his objections to current ideas about motion and the infinitesimal were very subtle indeed. For example, he criticized the infinite geometrical progression by proposing the well-known puzzle of Achilles and the Tortoise. How, asked Zeno, can the swift Achilles overtake the Tortoise if he concedes a handicap? For if Achilles starts at A and the tortoise at B, then when Achilles reaches B the tortoise is at C, and when Achilles reaches C the tortoise is at D. As this description can go on and on, apparently Achilles never overtakes the tortoise. But actually he may do so; and this is a paradox. The point of the inquiry is not *when,* but *how* does Achilles overtake the tortoise.

Somewhat similar questions were asked by DEMOCRITUS, the great philosopher of Thrace, who was a contemporary of Archytas and Plato. Democritus has long been famous as the originator of the atomic theory, a speculation that was immediately developed by Epicurus, and later provided the great theme for the Latin poet Lucretius. It is, however, only quite recently that any mathematical work of Democritus has come to light. This happened in 1906, when Heiberg discovered a lost book of Archimedes entitled the *Method.* We learn from it that Archimedes re-

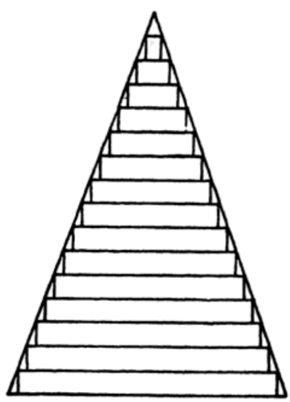


FIGURE 7

garded Democritus as the first mathematician to state correctly the formula for the volume of a cone or a pyramid. Each of these volumes was one-third part of a circumscribing cylinder, or prism, standing on the same base. To reach his conclusions, Democritus thought of these solids as built up of innumerable parallel layers. There would be no difficulty in the case of the cylinder, for each layer would be equal. But for the cone or pyramid the sizes of layer upon layer would taper off to a point. The appended diagram (Figure 7), showing the elevation of a cone or pyramid, illustrates this tapering of the layers, although the picture that Democritus had in mind consisted of very much thinner layers. He was puzzled by their diminishing sizes. 'Are they equal or unequal?' he asked, 'for if they are unequal, they will make the cone irregular as having many indentations, like steps, and unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder and to be made up of equal, not unequal circles, which is very absurd.'

This quotation is striking; for it foreshadows the great constructive work of Archimedes, and, centuries later, that of Cavalieri and Newton. It exhibits the infinitesimal calculus in its infancy. The notion of stratification—that a solid could be thought of as layer upon layer—would occur quite naturally to Democritus, because he was a physicist; it would *not* so readily have occurred to Pythagoras or Plato with their more algebraic turn of mind which attracted them to the pattern or arrangement of things. But here the acute Greek thought is once more restless. No mere rough and ready approximation will satisfy Democritus: there is discrepancy between stratified pyramid and smoothly finished whole. The deep question of the theory of limits is at issue; but how far he foresaw any solution, we do not know.

This brings us to the great arithmetical work at Athens, associated with the names of Plato (429–348 B.C.) and Eudoxus (408–355 B.C.)

Among the philosophers of Athens only two were native to the place, Socrates and Plato, master and disciple, both of whom were well read mathematicians. Plato was perhaps an original investigator; but whether this is so or not, he exerted an immense influence on the course that mathematics was to take, by founding and conducting in Athens his famous Academy. Over the entrance of his lecture room his students read the telling inscription, 'Let no one destitute of geometry enter my doors'; and it was his earnest wish to give his pupils the finest possible education. A man, said he, should acquire no mere bundle of knowledge, but be trained to see below the surface of things, seeking rather for the eternal reality and the Good behind it all. For this high endeavour the study of mathematics is essential; and numbers, in particular, must be studied, simply as numbers and not as embodied in anything. They impart a character to nature; for instance, the periods of the heavenly bodies can only be characterized by invoking the use of irrationals.

Originally the Greek word ἀριθμοί, from which we derive 'arithmetic,' meant the natural numbers, although it was at first questioned whether unity was a number; for 'how can unity, the measure, be a number, the thing measured?' But by including irrationals as numbers Plato made a great advance: he was in fact dealing with what we nowadays call the positive real numbers. Zero and negative numbers were proposed at a far later date.

There is grandeur here in the importance which Plato ascribes to arithmetic for forming the mind: and this is matched by his views on geometry, 'the subject which has very ludicrously been called mensuration' (γεωμετρία = land measuring) but which is really an art, a more than human miracle in the eyes of those who can appreciate it. In his book, the *Timaeus,* where he dramatically expounds the views of his hero Timaeus, the Pythagorean, reference is made to the five regular solids and to their supposed significance in nature. The speaker tells how that the four elements earth, air, fire and water have characteristic shapes: the cube is appropriated to earth, the octahedron to air, the sharp pyramid or tetrahedron to fire, and the blunter icosahedron to water, while the Creator used the fifth, the dodecahedron, for the Universe itself. Is it sophistry, or else a brilliant foretaste of the molecular theory of our own day? According to Proclus, the late Greek commentator, 'Plato caused mathematics in general and geometry in particular to make a very great advance, by reason of his enthusiasm for them, which of course is obvious from the way he filled his books with mathematical illustrations, and everywhere tries to kindle admiration for these subjects, in those who make a pursuit of philosophy.' It is related that to the question, What does God do? Plato replied, 'God always geometrizes.'

Among his pupils was a young man of Cnidus, named EUDOXUS, who came to Athens in great poverty, and, like many another poor student, had a struggle to maintain himself. To relieve his pocket he lodged down by the sea at the Piraeus, and every day used to trudge the dusty miles to Athens. But his genius for astronomy and mathematics attracted attention and finally brought him to a position of eminence. He travelled and studied in Egypt, Italy and Sicily, meeting Archytas, the geometer, and other men of renown. About 368 B.C., at the age of forty, Eudoxus returned to Athens in company with a considerable following of pupils, about the time when Aristotle, then a lad of seventeen, first crossed the seas to study at the Academy.

In astronomy the great work of Eudoxus was his theory of concentric spheres explaining the strange wanderings of the planets; an admirable surmise that went far to fit the observed facts. Like his successor Ptolemy, who lived many centuries later, and all other astronomers until Kepler, he found in circular motion a satisfactory basis for a complete planetary theory. This was great work; yet it was surpassed by his pure mathematics which touched the zenith of Greek brilliance. For Eudoxus placed the doctrine of irrationals upon a thoroughly sound basis, and so well was his task done that it still continues, fresh as ever, after the great arithmetical reconstructions of Dedekind and Weierstrass during the nineteenth century. An immediate effect of the work was to restore confidence in the geometrical methods of proportion and to complete the proofs of several important theorems. The *method of exhaustions* vaguely underlay the results of Democritus upon the volume of a cone and of Hippocrates on the area of a circle. Thanks to Eudoxus this method was fully explained.

An endeavour will now be made to indicate in a simple way how this great object was achieved. This study of higher arithmetic at Athens was stimulated by the Pythagorean, Theodorus of Cyrene, who is said to have been Plato's teacher. For Theodorus discovered many irrationals, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, $\sqrt{15}$, $\sqrt{17}$, 'at which point,' says Plato, 'for some reason he stopped'. The omissions in the list are obvious: $\sqrt{2}$ had been discovered by Pythagoras through the ratio of diagonal to side of a square, while $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$ are of course irrelevant. Now it is one thing to discover the *existence* of an irrational such as $\sqrt{2}$; it is quite another matter to find a way of *approach* to the number. It was this second problem that came prominently to the fore: it provided the arithmetical aspect of the *method of exhaustions* already applied to the circle: and it revealed a wonderful example of ancient arithmetic. We learn the details from a later commentator, Theon of Smyrna.

Unhampered by a decimal notation (which here is a positive hindrance, useful as it is in countless other examples), the Greeks set about their task in the following engaging fashion. To approximate to $\sqrt{2}$ they built a ladder of whole numbers. A brief scrutiny of the ladder shows how the rungs are devised: $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, $2 + 5 = 7$, $5 + 7 = 12$, and so on. Each rung of the ladder consists of two numbers x and y, whose ratio approaches nearer and nearer to the ratio $1 : \sqrt{2}$, the further down the ladder it is situated. Again, these numbers x and y, at each rung, satisfy the equation

1	1
2	3
5	7
12	17
29	41
etc.	

$$y^2 - 2x^2 = \pm 1.$$

The positive and negative signs are taken at alternate rungs, starting with a negative. For example, at the third rung $7^2 - 2 \cdot 5^2 = -1$.

As these successive ratios are alternately less than and greater than all that follow, they nip the elusive limiting ratio $1 : \sqrt{2}$ between two extremes, like the ends of a closing pair of pincers. They approximate *from both sides* to the desired irrational: $5/7$ is a little too large, but $12/17$ is a little too small. Like pendulum swings of an exhausted clock they die down—but they never actually come to rest. Here again, is the Pythagorean notion of *hyperbole* and *ellipsis*; it was regarded as very significant, and was called by the Greeks the 'dyad' of the 'great and small'.

Such a ladder could be constructed for any irrational; and another very pretty instance, which has been pointed out by Professor D'Arcy Thompson, is closely connected with the problem of the Golden Section.

1	1	Here the right member of each rung is the sum of the pair on
1	2	the preceding rung, so that the ladder may be extended with
2	3	the greatest ease. In this case the ratios approximate, again
3	5	by the little more and the little less, to the limit $\sqrt{5} + 1 : 2$.
5	8	It is found that they provide the arithmetical approach to the
etc.		golden section of a line AB, namely when C divides AB so that

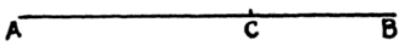


FIGURE 8

CB : AC = AC : AB. In fact, AC is roughly $3/5$ of the length AB, but more nearly $5/8$ of AB; and so on. It is only fair to say that this simplest of all such ladders has not yet been found in the ancient literature, but owing to its intimate connexion with the pentagon, it is difficult to resist the conclusion that the later Pythagoreans were familiar with it. The series 1, 2, 3, 5, 8, . . . was known in mediaeval times to Leonardo of Pisa, surnamed Fibonacci, after whom it is nowadays named.

Let us now combine this ladder-arithmetic with the geometry of a divided line. For example, let a line AB be divided at random by C, into lengths a and b, where AC $= a$, CB $= b$. Then the question still remains, what is the *exact* arithmetical meaning of the ratio $a : b$, whether or not this is irrational? The wonderful answer to this question is what has made Eudoxus so famous. Before considering it, let us take as an illustration the strides of two walkers. A tall man A takes a regular stride of length a, and his short friend B takes a stride b. Now suppose that eight strides of A cover the same ground as thirteen of B: in this case the *single* strides of A and B are in the ratio 13 : 8. The repetition of strides, to make them cover a considerable distance, acts as a magnifying glass and helps in the measurement of the *single* strides a and b, one against the other. Here we have the point of view adopted by Eudoxus. Let us, says he in effect, multiply our magnitudes a and b, whose ratio is required, and see what happens.

Let us, he continues, be able to recognize if a and b are equal, and if not, which is greater. Then if a is the greater, let us secondly be able to find multiples $2b$, $3b$, . . . , nb, of the smaller magnitude b; and thirdly, let us always be able to find a multiple nb of b which exceeds a. (The tall man may have seven-league boots and the short man may be Tom Thumb. Sooner or later the dwarf will be able to overtake one stride of his friend!) Few will gainsay the propriety of these mild assumptions: yet their mathematical implications have proved to be very subtle. This third supposition of Eudoxus has been variously credited, but to-day it is known as the *Axiom of Archimedes.*

A definition of equal ratios can now be stated. Let a, b, c, d be four given magnitudes, then the ratio $a : b$ is equal to that of $c : d$, if, *whatever* equimultiples ma, mc are chosen and whatever equimultiples nb, nd are chosen,

$$\begin{aligned} \textit{either } ma>nb,\ mc>nd, \quad &\text{(i)} \\ \textit{or } ma=nb,\ mc=nd, \quad &\text{(ii)} \\ \textit{or } ma<nb,\ mc<nd. \quad &\text{(iii)} \end{aligned}$$

On this strange threefold statement the whole theory of proportion for geometry and algebra was reared. It is impossible to develop the matter here in any convincing way, but the simplicity of the ingredients in this definition is remarkable enough to merit attention. It has the characteristic threefold pattern already noticed by Pythagoras. As far as ordinary commensurable ratios go, the statement (ii) would suffice; m and n are whole numbers and the ratios $a : b$, $c : d$ are each equal to the ratio $n : m$. But the essence of the new theory lies in (i) and (iii), because (ii) *never* holds for incommensurables—the geometrical equivalent of irrationals in arithmetic. But it is extraordinary that out of these inequalities *equal* ratios emerge.

Lastly it was a stroke of genius when Eudoxus put on record the above Axiom of Archimedes. To continue our illustration, marking time is not striding, and Eudoxus excluded marking time. However small the stride b might be, it had a genuine length. Eudoxus simply ruled out the case of a ratio $a : b$ when either a or b was zero. Thereby he avoided a trap that Zeno had already set, and into which many a later victim was to fall. So the axiom was a notice-board to warn the unwary. It also had another use; it automatically required a and b to be magnitudes of the same kind. For if a denoted length and b weight, no number of ounces could be said to exceed the length of a foot.

The logical triumphs of this great period in Grecian mathematics overshadow important but less spectacular advances which were made in numerical notation and in music. From the earliest times the significance of the numbers five and ten for counting had been recognised in Babylonia, China and Egypt: and in Homer πεμπάζειν 'to five' means to count. Eventually the Greeks systematised their written notation by using the letters of the alphabet to denote definite numbers ($\alpha = 1$, $\beta = 2$, $\gamma = 3$ and so on). A Halicarnassus inscription (circa 450 B.C.) provides perhaps the earliest attested use of this alphabetical numeration.

In music Archytas gave the numerical ratios for the intervals of the tetrachord on three scales, the enharmonic, the chromatic and the diatonic. He held that sound was due to impact, and that higher tones correspond to quicker, and lower tones to slower, motion communicated to the air.

CHAPTER III

ALEXANDRIA: EUCLID, ARCHIMEDES AND APOLLONIUS

TOWARDS the end of the fourth century B.C., the scene of mathematical learning shifted from Europe to Africa. By an extraordinary sequence of brilliant victories the young soldier-prince, Alexander of Macedonia, conquered the Grecian world, and conceived the idea of forming a great empire. But he died at the age of thirty-three (323 B.C.), only two years after founding the city of Alexandria. He had planned this stronghold near the mouth of the Nile on a magnificent scale, and the sequel largely fulfilled his hopes. Geographically it was a convenient meeting-place for Greek and Jew and Arab. There, what was finest in Greek philosophy was treasured in great libraries: the mathematics of the ancients was perfected: the intellectual genius of the Greek came into living touch with the moral and spiritual genius of the Jew: the Septuagint translation of Old Testament Scriptures was produced: and in due time it was there that the great philosophers of the early Christian Church taught and prospered. In spite of ups and downs the city endured for about six hundred years, but suffered grievous losses in the wilder times that followed. The end came in A.D. 642, when a great flood of Arab invasion surged westward, and Alexandria fell into the hands of the Calif Omar.

A great library, reputed to hold 700,000 volumes, was lost or destroyed in this series of disasters. But happily a remnant of its untold wealth filtered through to later days when the Arabs, who followed the original warriors, came to appreciate the spoils upon which they had fallen. Ptolemy, the successor of Alexander in his African dominions, had founded this library about 300 B.C. He had in effect established a University; and among the earliest of the teachers was EUCLID. We know little of his life and character, but he most probably passed his years of tuition at Athens before accepting the invitation of Ptolemy to settle in Alexandria. For twenty or thirty years he taught, writing his well-known *Elements* and many other works of importance. This teaching bore notable fruit in the achievements of Archimedes and Apollonius, two of the greatest members of the University.

The picture has been handed down of a genial man of learning, modest and scrupulously fair, always ready to acknowledge the original work of others, and conspicuously kind and patient. Some one who had begun to read geometry with Euclid, on learning the first theorem asked, 'What shall I get by learning these things?' Euclid called his slave and said, 'Give him threepence, since he must make gain out of what he learns'. Apparently Euclid made much the same impression as he does to-day. The schoolboy, for whom the base angles of an isosceles triangle 'are forced to be equal, without any nasty proof', is but re-echoing the ancient critic who remarked that two sides of a triangle were greater than the third, as was evident to an ass. And no doubt they told Euclid so.

In the *Elements* Euclid set about writing an exhaustive account of mathematics—a colossal task even in his day. The Work consisted of thirteen books, and the subjects of several books are extremely well known. Books I, II, IV, VI on lines, areas and simple regular plane figures are mostly Pythagorean, while Book III on circles expounds Hippocrates. The lesser known Book V elaborates the work of Eudoxus on proportion, which was needed to justify the properties of similar figures discussed in Book VI. Books VII, VIII and IX are arithmetical, giving an interesting account of the theory of numbers; and again much here is probably Pythagorean. Prime and composite numbers are introduced—a relatively late distinction; so are the earlier G.C.M. and L.C.M. of numbers, the theory of geometrical progressions, and in effect the theorem $a^{m+n} = a^m a^n$, together with a method of summing the progression by a beautiful use of equal ratios. Incidentally Euclid utilized this method to give his *perfect* numbers, such as 6, 28, 496, each of which is equal to the sum of its factors. The collection of perfect numbers still interests the curious; they are far harder to find than the rarest postage stamps. The ninth specimen alone has thirty-seven digits, while a still larger one is $2^{126}(2^{127}-1)$.

Book X of Euclid places the writer in the forefront among analysts. It is largely concerned with the doctrine of irrational numbers, particularly of the type $\sqrt{(\sqrt{a} \pm \sqrt{b})}$, where a and b are positive integers. Here Euclid elaborates the arithmetical side of the work of Eudoxus, having already settled the geometrical aspect in Books V and VI, and here we duly find the method of exhaustions carefully handled. After Book XI on elementary solid geometry comes the great Book XII, which illustrates the same method of exhaustions by formally proving Hippocrates' theorem for πr^2, the area of a circle. Finally in Book XIII we have the climax to which all this stately procession has been leading. The Greeks were never in a hurry; and it is soothing, in these days of bustle, to contemplate the working of their minds. This very fine book gives and proves the constructions for the five regular solids of Pythagoras, extolled by Plato; and it ends with the dodecahedron, the symbol of the Universe itself.

By this great work Euclid has won the admiration and helped to form the minds of all his successors. To be sure a few logical blemishes occur in his pages, the gleanings of centuries of incessant criticism; but the wonder is that so much has survived unchanged. In point of form he left nothing to be desired, for he first laid down his careful definitions, then his common assumptions or axioms, and then his postulates, before proceeding with the orderly arrangement of their consequences. There were, however, certain gaps and tautologies among these preliminaries of his work: they occur in the geometrical, not in the Eudoxian parts of his books; and it has been one of the objects of latter-day criticism to supply what Euclid left unsaid.

But on one point Euclid was triumphant; in his dealing with parallel lines. For he made no attempt to hide, by a plausible axiom, his inability to prove a certain property of coplanar lines. Most of his other assumptions, or necessary bases of his arguments, were such as would reasonably command general assent. But in the case of parallel lines he started with the following elaborate supposition, called the *Parallel Postulate*:

If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

By leaving this unproved, and by actually proving its converse, Euclid laid himself open to ridicule and attack. Surely, said the critics, this is no proper assumption; it must be capable of proof. Hundreds of attempts were vainly made to remove this postulate by proving its equivalent; but each so-called proof carried a lurking fallacy. The vindication of Euclid came with the discovery in the nineteenth century of non-Euclidean geometry, when fundamental reasons were found for some such postulate. There is dignity in the way that Euclid left this curious rugged excrescence, like a natural outcrop of rock in the plot of ground that otherwise had been so beautifully smoothed.

Many of his writings have come down to us, dealing with astronomy, music and optics, besides numerous other ways of treating geometry in his *Data* and *Division of Figures*. But his *Book of Fallacies* with its intriguing title, and the *Porisms* are lost; and we only learn of them indirectly through Pappus, another great commentator. It is one of the historical puzzles of mathematics to discover what porisms were, and many geometers, notably Simson in Scotland and Chasles in France, have tried to do so. Very likely they were properties relating to the organic description of figures—a type of geometry that appealed to Newton, Maclaurin, and to workers in projective geometry of our own days. Geometry at Alexandria was in fact a wide subject, and it has even been thought by some that the *porisms* consisted of an analytical method, foreshadowing the co-ordinate geometry of Descartes.

Euclid was followed by ARCHIMEDES of Samos, and APOLLONIUS of Perga. After the incomparable discoveries of Eudoxus, so well consolidated by Euclid, it was now the time for great constructive work to be launched; and here were the giants to do it. Archimedes, one of the greatest of all mathematicians, was the practical man of common sense, the Newton of his day, who brought imaginative skill and insight to bear upon metrical geometry and mechanics, and even invented the integral calculus. Apollonius, one of the greatest of geometers, endowed with an eye to see form and design, followed the lead of Menaechmus, and perfected the geometry of conic sections. They sowed in rich handfuls the seeds of pure mathematics, and in due time the harvest was ingathered by Kepler and Newton.

Little is known of the outward facts in the life of ARCHIMEDES. His father was Phidias the astronomer, and he was possibly related to Hiero II, King of Syracuse, who certainly was his friend. As a youth Archimedes spent some time in Egypt, presumably at Alexandria with the immediate successors of Euclid—perhaps studying with Euclid himself. Then on returning home he settled in Syracuse, where he made his great reputation. In 212 B.C., at the age of seventy-five, he lost his life in the tumult that followed the capture of Syracuse by the Romans. Rome and Carthage were then at grips in the deadly Punic wars, and Sicily with its capital Syracuse lay as a 'No man's land' between them. During the siege of Syracuse by the Romans, Archimedes directed his skill towards the discomfiture of the enemy, so that they learnt to fear the machines and contrivances of this intrepid old Greek. The story is vividly told by Plutarch, how at last Marcellus, the Roman leader, cried out to his men, 'Shall we not make an end of fighting against this geometrical Briareus who uses our ships like cups to ladle water from the sea, drives off our sambuca ignominiously with cudgel-blows, and by the multitude of missiles that he hurls at us all at once, outdoes the hundred-handed giants of mythology!' But all to no purpose, for if the soldiers did but see a piece of rope or wood projecting above the wall, they would cry, 'There it is,' declaring that Archimedes was setting some engine in motion against them, and would turn their backs and run away. Of course the geometrical Briareus attached no importance to these toys; they were but the diversions of geometry at play. Ignoble and sordid, unworthy of written record, was the business of mechanics and every sort of art which was directed to use and profit. Such was the outlook of Archimedes.

He held these views to the end; for even after the fall of the city he was still pondering over mathematics. He had drawn a diagram in the sand on the ground and stood lost in thought, when a soldier struck him down. As Whitehead has remarked:

'The death of Archimedes at the hands of a Roman soldier is symbolical of a world change of the first magnitude. The Romans were a great race, but they were cursed by the sterility which waits upon practicality. They were not dreamers enough to arrive at new points of view, which could give more fundamental control over the forces of nature. No Roman lost his life because he was absorbed in the contemplation of a mathematical diagram.'

Many, but not all, of the wonderful writings of Archimedes still survive. They cover a remarkable mathematical range, and bear the incisive marks of genius. It has already been said that he invented the integral calculus. By this is meant that he gave strict proofs for finding the areas, volumes and centres of gravity of curves and surfaces, circles, spheres, conics, and spirals. By his method of finding a tangent to a spiral he even embarked on what is nowadays called differential geometry. In this work he had to invoke algebraic and trigonometric formulae; here, for example, are typical results:

$$1^2 + 2^2 + 3^2 + \; . \; . \; . + n^2 = \tfrac{1}{6}n(n+1)(2n+1),$$
$$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \; . \; . \; . + \sin\,(2n-1)\frac{\pi}{2n} = \cot\frac{\pi}{4n}.$$

This last is the concise present-day statement of a geometrical theorem, arising in his investigation of the value of π, which he gave approximately in various ways, such as

$$3\tfrac{1}{7} > \pi > 3\tfrac{10}{71}.$$

Elsewhere he casually states approximations to $\sqrt{3}$ in the form

$$\tfrac{265}{153} < \sqrt{3} < \tfrac{1351}{780},$$

which is an example of the ladder-arithmetic of the Pythagoreans (p. 97). As these two fractions are respectively equal to

$$\tfrac{1}{3}\left(5 + \frac{1}{5 + \tfrac{1}{10}}\right) \text{ and } \tfrac{1}{3}\left(5 + \cfrac{1}{5 + \cfrac{1}{10 + \tfrac{1}{5}}}\right),$$

it is natural to suppose that Archimedes was familiar with continued fractions, or else with some virtually equivalent device; especially as $\sqrt{3}$ itself is given by further continuation of this last fraction, with denominators 10, 5, 10, 5 in endless succession. The same type of arithmetic occurs elsewhere in his writings, as well as in those of his contemporary, Aristarchus of Samos, a great astronomer who surmised that the earth travels round the sun.

Allusion has already been made to the recent discovery of the *Method* of Archimedes, a book that throws light on the mathematical powers of Democritus. It also reveals Archimedes in a confidential mood, for in it he lifts the veil and tells us how some of his results were reached. He *weighed* his parabola to ascertain the area of a segment, and this experiment suggested the theorem that the parabolic area is two-thirds of the area of a circumscribing parallelogram (Figure 9). He admits the value of such experimental methods for arriving at mathematical truths, which afterwards of course must be rigidly proved.

Indeterminate equations, with more unknowns than given equations, have attracted great interest from the earliest days. For example, there may be *one* equation for *two* unknowns:

$$3x - 2y = 5.$$

Many whole numbers x and y satisfy this equation, but it is often interest-

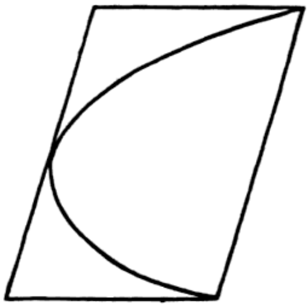


FIGURE 9

ing to discover the simplest numbers to do so. Such problems are closely connected with continued fractions, and perhaps Archimedes was beginning to recognize this. At all events we are told that he set the *Cattle Problem* to his friends in Alexandria.

The problem dealt with eight herds, four of bulls and four of cows, according to the colours, white, black, yellow and dappled. Certain facts were stated; for example, that the dappled bulls exceeded the yellow bulls in multitude by ($\frac{1}{6} + \frac{1}{7}$) of the number of white bulls: and the problem required, for its solution, the exact size of each herd. In other words there were eight unknown numbers to be found, but unfortunately, when turned into algebra, the data of the problem provided only seven equations. One of these equations, typical of all seven, can readily be formed from the facts already cited. If x denotes the number of dappled bulls, y that of the white bulls, and z that of the yellow bulls, it follows that

$$x = (\tfrac{1}{6} + \tfrac{1}{7})y + z.$$

From seven such equations for eight unknowns, of which only three x, y, z occur in this particular equation, all the unknowns have to be found.

There are, of course, an infinite number of solutions to seven equations for eight unknowns. The simplest solution of the above equation, taken apart from its context, is $x = 14$, $y = 42$, $z = 1$. As this does not fit in with the other six equations, a more complicated set of numbers for x, y, z must be found. Who would guess that the smallest value of x satisfying all seven such innocent looking equations is a number exceeding 3½ million? In our decimal notation this is a number seven figures long. But Archimedes improved on the problem by stating that 'when the white bulls joined in number with the black, they stood firm, with depth and breadth of equal measurement; and the plains of Thrinakia, far stretching all ways, were filled with their multitude'. Taking this to mean that the total number of black and white bulls was square, an enterprising investigator, fifty years ago, showed that the smallest such herd amounted to a number 200,000 figures long. The plains of Thrinakia would have to be replaced by the Milky Way.

The so-called *Axiom of Archimedes* bears his name probably because of its application on a grand scale, when he showed that the amount of sand in the world was finite. This appears in the *Sand-Reckoner*, a work full of quaint interest, and important for its influence on the arithmeticians of the last century. The opening sentences run as follows:

> 'There are some, King Gelon, who think that the number of the sand is infinite in multitude: and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. And again, there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its multitude.'

So far from weeping to see such quantities of sand, Archimedes cheerfully fancies the whole Universe to be stuffed with sandgrains and then proceeds to count them. After a tilt at the astronomer Aristarchus for talking of the ratio of the centre of a sphere to the surface—'it being easy to see that this is impossible, the centre having no magnitude'—he gently puts Aristarchus right and then turns to the problem. First he settles the question, how many grains of sand placed side by side would measure the diameter of a poppy seed. Then, how many poppy seeds would measure a finger breadth. From poppy seed to finger breadth, from finger breadth to stadium, and so on to a span of 10,000 million stadia, he serenely carries out his arithmetical reductions. Mathematically he is developing something more elaborate than the theory of indices: his arithmetic might be called *the theory of indices of indices*, in which he classifies his gigantic numbers by orders and periods. The first order consists of all numbers from 1 to $100{,}000{,}000 = 10^8$, and the first period ends with the number $10^{800{,}000{,}000}$. This number can be expressed more compactly as $(10^8)^{10^8}$, but in the ordinary decimal notation consists of eight hundred million and one figures. Archimedes advances through further periods of this enormous size, never pausing in his task until the hundred-millionth period is reached.

In conclusion:

'I conceive that these things, King Gelon, will appear incredible to the great majority of people who have not studied mathematics, but that to those who are conversant therewith and have given thought to the question of the distances and sizes of the earth, the sun and moon, and the whole universe, the proof will carry conviction. And it was for this reason that I thought the subject would be not inappropriate for your consideration.'

One cannot pass from the story of Archimedes without reference to his work on statics and hydrostatics, in which he created a new application for mathematics. Like the rest of his writings this was masterly. Finally, in a book now lost, he discussed the semiregular solids, which generalize on the Pythagorean group of five regular solids. When each face of the solid is to be a regular polygon, exactly thirteen and no more forms are possible, as Kepler was one of the first to verify.

The third great mathematician of this period was APOLLONIUS of Perga in Pamphilia (? 262–200 B.C.), who earned the title 'the great geometer'. Little is known of him but that he came as a young man to Alexandria, stayed long, travelled elsewhere, and visited Pergamum, where he met Eudemus, one of the early historians of our science. Apollonius wrote extensively, and many of his books are extant. His prefaces are admirable, showing how perfect was the style of the great mathematicans when they were free from the trammels of technical terminology. He speaks with evident pleasure of some results: 'the most and prettiest of these theorems are new.'

What Euclid did for elementary geometry, Apollonius did for conic sections. He defined these curves as sections of a cone standing on a circular base; but the cone may be *oblique*. He noticed that not only were all sections parallel to the base, circular, but that there was also a secondary set of circular sections.

Although a circle is much easier to study than an ellipse, yet every property of a circle gives rise to a corresponding property of an ellipse. For example, if a circle and tangent are looked at obliquely, what the eye sees is an ellipse and its tangent. This matter of perspective leads on to projective geometry; and in this manner Apollonius simplified his problems. By pure geometry he arrived at the properties of conics which we nowadays express by equations such as

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

and $ax^2 + bxy + cy^2 = 1$, and even $\sqrt{ax} + \sqrt{by} = 1$. In the second equation *a, b, c* denote given multiples of certain squares and a rectangle, the total area being constant. From our analytical geometry of conics he had clearly very little to learn except the notation, which improves on his own. He solved the difficult problem of finding the shortest and longest distances from a given point P to a conic. Such lines cut the curve at right angles and are called *normals.* He found that as many as four normals could be drawn from favourable positions of P, and less from others. This led him to consider a still more complicated curve called the *evolute*, which he fully investigated. He worked with what is virtually an equation of the sixth degree in x and y, or its geometrical equivalent—in its day a wonderful feat. His general problem, [*locus*] *ad tres et quattuor lineas,* will be considered when we turn to the work of Pappus.

Another achievement of Apollonius was his complete solution of a problem about a circle satisfying three conditions. When a circle passes through a given point, *or* touches a given line, *or* touches a given circle, it is said to satisfy one condition. So the problem of Apollonius really involved nine cases, ranging from the description of a circle through three given points to that of a circle touching three given circles. The simplest of these cases were probably quite well known: in fact, one of them occurs in the *Elements* of Euclid.

Apollonius was also a competent arithmetician and astronomer. It is reported that he wrote on *Unordered Irrationals,* and invented a 'quick delivery' method of approximating to the number π. Here, to judge from his title, it looks as if he had begun the theory of uniform convergence.

It may now be wondered what was left for their successors to discover after Archimedes and Apollonius had combed the field? So complete was their work that only a few trivial gaps needed to be filled, such as the addition of a focus to a parabola or a directrix to a conic, properties which Apollonius seems to have overlooked. The next great step could not be taken until algebra was abreast of geometry, and until men like Kepler, Cavalieri and Descartes were endowed with both types of technique.

CHAPTER IV

THE SECOND ALEXANDRIAN SCHOOL: PAPPUS AND DIOPHANTUS

WITH the death of Apollonius the golden age of Greek mathematics came to an end. From the time of Thales there had been almost a continuous chain of outstanding mathematicians. But until about the third century A.D., when Hero, Pappus and Diophantus once more brought fame to Alexandria, there seems to have been no mathematician of pre-eminence. During this interval of about half a millennium the pressure of Roman culture had discouraged Greek mathematics, although a certain interest in mechanics and astronomy was maintained; and the age produced the great astronomer HIPPARCHUS, and two noteworthy commentators, MENELAUS and PTOLEMY. Menelaus lived about the year A.D. 100, and Ptolemy was perhaps fifty years his junior. There is a strange monotony in trying to detail any facts whatsoever about these men—so little is known for certain, beyond their actual writings. The same uncertainty hangs over Hero, Pappus and Diophantus, whose names may be associated together as forming the Second Alexandrian School, because they each appear to have been active about the year A.D. 300. Yet Pappus and Diophantus are surrounded by mystery. Each seems to be a solitary echo of bygone days, in closer touch with Pythagoras and Archimedes than with their contemporaries, or even with each other.

MENELAUS is interesting, more particularly to geometers, because he made a considerable contribution to spherical trigonometry. Many new theorems occur in his writings—new in the sense that no earlier records are known to exist. But it is commonly supposed that most of the results originated with Hipparchus, Apollonius and Euclid. A well-known theorem, dealing with the points in which a straight line drawn across a triangle meets the sides, still bears his name. For some reason, hard to fathom, it is often classed to-day as 'modern geometry', a description which scarcely does justice to its hoary antiquity. The occasion of its appearance in the work of Menelaus is the more significant because he used it to prove a similar theorem for a triangle drawn on a sphere. Menelaus gave several theorems which hold equally well for triangles and other figures, whether they are drawn on a sphere, or on a flat plane. They include a very fundamental theorem known as the *cross ratio* property of a transversal drawn across a pencil of lines. This too is 'modern geometry'. He also gave the celebrated theorem that the angles of a spherical triangle are together greater than two right angles.

PTOLEMY (? 100–168 A.D.), who was a good geometer, will always be remembered for his work in astronomy. He treated this subject with a completeness comparable to that which Euclid achieved in geometry. His compilation is known as the Almagest—a name which is thought to be an Arabic abbreviation of the original Greek title.[2] His work made a strong appeal to the Arabs, who were attracted by the less abstract branches of mathematics; and through the Arabs it ultimately found a footing in mediaeval Europe. In this way a certain planetary theory called the Ptolemaic system became commonly accepted, holding sway for many centuries until it was superseded by the Copernican system. Ptolemy, following the lead

[2] Meaning 'The Great Compilation.'

of Hipparchus, chose one of several competing explanations of planetary motion, and interpreted the facts by an ingenious combination of circular orbits, or epicycles. Fundamental to his theory was the supposition that the Earth is fixed in space: and, if this is granted, his argument follows very adequately. But there were other explanations, such as that of Aristarchus, the friend of Archimedes, who supposed that the Earth travels round the Sun. When, therefore, Copernicus superseded the Ptolemaic theory by his own well-known system, centred on the Sun, he was restoring a far older theory to its rightful place.

HERO of Alexandria was a very practical genius with considerable mathematical powers. It is generally assumed that all the great mathematicians of the Hellenic world were Greek; but it is supposed that Hero was not. He was probably an Egyptian. At any rate there is in his work a strong bias towards the applications and away from the abstractions of mathematics, which is quite in keeping with the national characteristics of Egypt. Yet Hero proved to be a shrewd follower of Archimedes, bringing his mathematics to bear on engineering and surveying. He not only made discoveries in geometry and physics, but is also reputed to have invented a steam engine. One of his most interesting theorems proves that, when light from an object is broken by reflection on mirrors, the path of the ray between object and eye is a minimum. This is an instance of a *principle of least action*, which was formally adopted for optics and dynamics by Hamilton in the last century, and which has recently been incorporated in the work of Einstein. We may, therefore, regard Hero as the pioneer of Relativity (*c.* 250 A.D.).

At the beginning of the fourth century there was a revival of pure mathematics, when something of the Pythagorean enthusiasm for geometry and algebra existed once again in Alexandria under the influence of Pappus and Diophantus. PAPPUS wrote a great commentary called the Collection (*συναγωγή*); and happily many of his books are preserved. They form a valuable link with still more ancient sources, and particularly with the lost work of Euclid and Apollonius. As an expositor, Pappus rivals Euclid himself, both in completeness of design and wealth of outlook. To discover what Euclid and his followers were about, from reading the Collection, is like trying to follow a masterly game of chess by listening to the comments of an intelligent onlooker who is in full sight of the board.

Pappus was somewhat vain and occasionally unscrupulous, but he had enough sympathy to enter into the spirit of each great epoch. The space-filling figures of Pythagorean geometry made him brood over the marvels of bee-geometry; for God has endowed these sagacious little creatures with a power to construct their honey cells with the *smallest* enclosing surface. How far the bee knows this is not for the mathematician to say, but the fact is perfectly true. Triangular, or square, cells could be crowded together, each holding the same amount of honey as the hexagonal cell; but the hexagonal cell requires least wax. Like the mirrors of Hero, this again suggests *least action* in nature: and Pappus was disclosing another important line of inquiry. He put the question, What is the maximum volume enclosed by a given superficial area? This was perhaps the earliest suggestion of a branch of mathematics called the *calculus of variations.*

Most striking, and in true Archimedean style, is his famous theorem which determines the volume of a surface of revolution. His leading idea may be grasped by first noting that the volume of a straight tube is known if its cross-section A and its length *l* are given. For the volume is the product A.*l*. Pappus generalized this elementary result by considering such a tube to be no longer straight but circular. The cross-section A was taken to be the same at every place; but the length of the tube would need further definition. For example, the length of an inflated bicycle tyre is least if measured round the inner circle in contact with the rim, and is greatest round the outermost circle. This illustration suggests that an *average* or mean length *l* may exist for which the formula A.*l* still gives the volume. Pappus found that for such a circular tube this was so, and he located his average length as that of the circle passing through the centroid of each cross-section A. By centroid is meant that special point of a plane area often called the centre of gravity. As the shape of the section A is immaterial to his result, the theorem is one of the most general conclusions in ancient mathematics. In later years P. Guldin (1577–1643) without even the excuse of an unconscious re-discovery, calmly annexed this theorem, and it has become unjustly associated with his name.

As two further examples of important geometrical work by Pappus, the properties of the two following diagrams may be given. There are no hidden subtleties about the drawing of either figure. In the first (Figure 10),

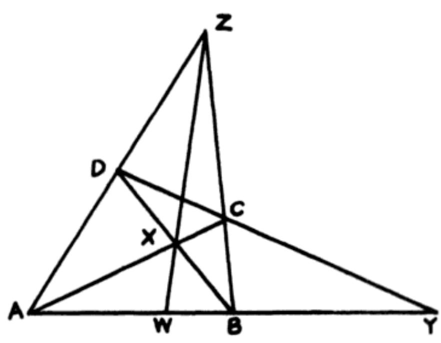


FIGURE 10

A, B, C, D are four points through which various straight lines have been drawn; and these intersect as shown at X, Y, Z. The line joining ZX is produced and cuts AB at W. The interest of this construction lies in the fact that, no matter what the shape of the quadrilateral ABCD may be, the lines AW, AB, AY are in harmonical progression. In the second figure (Figure 11), ABC and DEF are any two straight lines. These trios of points are joined crosswise by the three pairs of lines meeting at X, Y, Z. Then it follows that X, Y, Z are themselves in line. Here the interest lies in the *symmetry* of the result. It has nine lines meeting by threes in nine points: but it also has nine points lying by threes on the nine lines, as the reader may verify. This nice balance between points and lines of a figure is an early instance of reciprocation, or the *principle of duality*, in geometry.

In the parts of geometry which deal with such figures of points and lines, Pappus excelled. He gave a surprisingly full account of kindred properties connected with the quadrilateral, and particularly with a grouping of six points upon a line into three pairs. This so-called *involution* of six points would be effected by erasing the line ZXW in the first of the above figures, and re-drawing it so as to cross the other six lines at random in six distinct points.

In a significant passage of commentary on Apollonius, Pappus throws

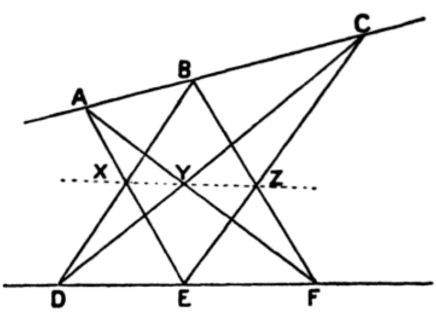


FIGURE 11

light upon what was evidently a very famous problem—the *locus ad tres et quattuor lineas*. It sums up so well the best Greek thought upon conics and it so very nearly inaugurates analytical geometry that it deserves special mention. Apollonius, says Pappus, considered the locus or trace of a roving point P in relation to three or four fixed lines. Suppose P were at a distance x from the first line, y from the second, z from the third, and t from the fourth. Suppose further that these distances were measured in specified directions, but not necessarily at right angles to their several lines. Then, as P moves, the values of x, y, z, t would vary; although it would always be possible to construct a rectangle of area xy, or a solid rectangular block of volume xyz. But as space is three-dimensional there is apparently nothing in geometry corresponding to the product $xyzt$ derived from four lines. On the other hand the ratio $x : y$ of two lines is a *number*, and there is nothing to prevent us from multiplying together as many such ratios as we like. So from the four lines x, y, z, t we can form two ratios $x : y$ and $z : t$, and then multiply them together. This gives $xz : yt$. Now if the resulting ratio is given as a constant, and equal to c, we can write

$$xz/yt = c, \text{ or } xz = cyt.$$

This is a way of stating the Apollonian problem about four lines: it indicates that the rectangle of the distances x, z from P to two of the lines is proportional to that of its distances y, t from the other two. When this happens, as Apollonius proved, P describes a conic. By a slight modification the same scrutiny may be applied to the problem, if three and not four lines are given. Pappus continues his commentary by generalizing the result with any number of lines: but it will be clearest if we confine ourselves to six lines.

If the distances of the point P from six given lines are x, y, z, u, v, w, then we can form them into three ratios $x : y$, $z : u$, $v : w$. If, further, it is given that the product of these three ratios is fixed, then we can write

$$\frac{x}{y} \cdot \frac{z}{u} \cdot \frac{v}{w} = c.$$

Pappus draws the correct conclusion that, when this happens, the point P is constrained to lie upon a certain locus or curve. But after a few more remarks he turns aside as if ashamed of having said something obvious. He had nevertheless again made one of the most general statements in all ancient geometry. He had begun the theory of Higher Plane Curves. For the number of ratios involved in such a constant product defines what is called the *order* of the locus. So a conic is a curve of order two, because it involves two ratios, as is shown in the Apollonian case above. In the simpler case, when only one ratio $x : y$ is utilized, the locus is a straight line. For this reason a straight line is sometimes called a curve of the first order. But Pappus had suggested curves of order higher than the second. These are now called cubics, quartics, quintics, and so on. To be sure, particular cases of cubic and other curves had already been discovered. The ancients had invented them for *ad hoc* purposes of trisecting an angle, and the like: but mathematicians had to wait for Descartes to clinch the matter.

The other great mathematician who brought fame to Alexandria was DIOPHANTUS. He is celebrated for his writings on algebra, and lived at the time of Pappus, or perhaps a little earlier. This we gather from a letter of Psellus, who records that Anatolius, Bishop of Laodicea about A.D. 280, dedicated to Diophantus a concise treatise on the Egyptian method of reckoning. Diophantus was devoted to algebra, as the wording of a Greek epigram indicates, which tells us the scanty record of his life. His boyhood lasted ⅙th of his life; his beard grew after $\frac{1}{12}$th more; he married after ⅐th more, and his son was born five years later; the son lived to half his father's age, and the father died four years after his son.

If x was the age when he died, then,

$$\tfrac{1}{6}x + \tfrac{1}{12}x + \tfrac{1}{7}x + 5 + \tfrac{1}{2}x + 4 = x;$$

and Diophantus must have lived to be eighty-four years old.

The chief surviving writings of Diophantus are six of the thirteen books forming the *Arithmetica*, and fragments of his *Polygonal Numbers* and *Porisms*. Twelve hundred years after they were written these books began to attract the attention of scholars in Europe. As Regiomontanus observed in 1463: 'In these old books the very flower of the whole of arithmetic lies hid, the *ars rei et census* which to-day we call by the Arabic name of Algebra.' This work of Diophantus has a twofold importance: he made an essential improvement in mathematical notation, while at the same time he added large instalments to the scope of algebra as it then existed. The full significance of his services to mathematics only became evident with the rise of the early French school in the fifteenth and sixteenth centuries.

The study of notation is interesting, and covers a wider sphere than at first sight may be supposed. For it is the study of symbols; and as words are symbols of thought, it embraces literature itself. Now we may concentrate our attention on the literal symbol as it appears to the eye in a mathematical formula and in a printed sentence; or else on the thing signified, on the sense of the passage, and on the thought behind the symbol. A good notation is a valuable tool; it brings its own fitness and suggestiveness, it is easy to recognize and is comfortable to use. Given this tool and the material to work upon, advance may be expected. In their own language and in their geometrical notation the Greeks were well favoured: and a due succession of triumphs followed. But their arithmetic and algebra only advanced in spite of an unfortunate notation. For the Greeks were hampered by their use of letters α, β, γ for the numbers 1, 2, 3, and this concealed from them the flexibility of ordinary arithmetical calculations. On the other hand, the very excellence of our *decimal notation* has made these operations wellnigh trivial. Before the notation was widely known, even simple addition, without the help of a ball frame, was a task of some skill. The chief merits of this notation are the sign 0 for zero, and the use of one symbol, its meaning being decided by its context, to denote several distinct things, as, for example, the writing of 11 to denote *ten* and *one*. The history of this usage has been traced to a source in Southern India, dating shortly after the time of Diophantus. Thence it spread to the Moslem world and so to mediaeval Europe.

In the previous chapters many algebraic formulæ have occurred. They are, of course, not a literal transcription of the Greek, but are concise symbolic statements of Greek theorems originally given in verbal sentences or in geometrical form. For instance, a^2 has been used instead of 'the square on AB'. The earliest examples of this symbolic algebra occur in the work of Vieta, who lived in the sixteenth century, though it only came into general use about the year A.D. 1650. Until that time the notation of Diophantus had been universally adopted.

An old classification speaks of

Rhetorical Algebra,
Syncopated Algebra,
Symbolic Algebra,

and these names serve to indicate broad lines of development. By the rhetorical is meant algebra expressed in ordinary language. Then syncopations, or abbreviations, similar to our use of H.M.S. for His Majesty's Ship, and the like, became common among the ancients. To Diophantus more than to any other we owe this essential improvement. The third, symbolic algebra, became finally established, once Vieta had invented it, through the influence of Napier, Descartes and Wallis.

A typical expression of symbolic algebra is

$$(250x^2 + 2520) \div (x^4 + 900 - 60x^2):$$

and this serves to indicate the type of complication which Diophantus successfully faced. His syncopations enabled him to write down, and deal with, equations involving this or similar expressions. For $250x^2$ he wrote $\Delta^{\Upsilon}\sigma\nu$: here the letter ν meant 50 and σ, 200, according to the ordinary Greek practice. But the Δ^{Υ} was short for the Greek word meaning *power* (it is our word, dynamic); and power meant the square of the unknown number. Diophantus used the letter ϛ for the first power of the unknown, and the abbreviation of the word *cube* for the third power. He used no sign for *plus*, but a sort of inverted ψ for *minus*, the letter ι for *equals*, and a special phrase to denote the *division* of one expression by another. It is interesting that his idea of addition and subtraction was 'forthcoming' and 'wanting', and that the Greek word for wanting is related to the Pythagorean term *ellipse*.

Those who have solved quadratic equations will remember the little refrain—'the square of half the co-efficient of x'. It is a quotation from Diophantus, who dealt with such equations very thoroughly. He even ventured on the easier cases of cubic equations. Yet he speaks of 'the impossible solution of the absurd equation $4 = 4x + 20$': such an equation requires a negative solution; and it was not until much later that negative numbers as things in themselves were contemplated. But fractions and alternative roots of quadratic equations presented to him no difficulties.

We need not go far into the puzzles of 'problems leading to simple equations' to convince ourselves of the value of using several letters x, y, z for the unknown quantities. Each different symbol comes like a friendly hand to help in disentangling the skein. As Diophantus attempted such problems with the sole use of his symbol ς he was, so to speak, tying one hand behind his back and successfully working single-handed. This was clearly the chief drawback of his notation. Nevertheless he cleverly solved simultaneous equations such as

$$yz = m\,(y + z),\ zx = n\,(z + x),\ xy = p\,(x + y);$$

and it is evident from this instance that he saw the value of *symmetry* in algebra.

All this is valuable for its general influence upon mathematical manipulation: and had the genius of Diophantus taken him no farther, he would still be respected as a competent algebraist. But he attained far greater heights, and his abiding work lies in the Theory of Numbers and of Indeterminate Equations. Examples of these last occurred in the *Cattle Problem* of Archimedes (p. 105) and in the relation $2x^2 - y^2 = 1$ (of p. 97). His name is still attached to simple equations, such as enter the Cattle Problem, although he never appears to have interested himself in them. Instead he was concerned with the more difficult quadratic and higher types, as, for example, the equation

$$x^4 + y^4 + z^4 = u^2.$$

He discovered four whole numbers x, y, z, u for which this statement was true. Centuries later his pages were eagerly read by Fermat, who proved to be a belated but brilliant disciple. 'Why', says Fermat, 'did not Diophantus seek *two* fourth powers such that their sum is square? This problem is, in fact, impossible, as by my method I am able to prove with all rigour.' No doubt Diophantus had experimented far enough with the easier looking equation $x^4 + y^4 = u^2$ to prove that no solution was available.

This brings us to the close of the Hellenic period; and we are now in a position to appreciate the contribution which the Greeks made to mathematics. They virtually sketched the whole design that was to give incessant opportunities for the mathematicians and physicists of later centuries. In some parts of geometry and in the theory of the irrational the picture had been actually completed.

Within a glittering heap of numerical and geometrical puzzles and trifles —the accumulation in Egypt or the East of bygone ages—the Greeks had found order. Their genius had made mathematics and music out of the discord. And now in turn their own work was to appear as a wealth of scattered problems whose interrelations would be seen as parts of a still grander whole. New instruments were to be invented—the decimal notation, the logarithm, the analytical geometry of Descartes, and the magnifying glass. Each in its way has profoundly modified and enriched the mathematics handed down by the Greeks. Such profound changes have been wrought that we have been in some danger of losing a proper perspective in mathematics as a whole. So ingrained to-day is our habit of microscopic scrutiny that we are apt to think that all accuracy is effected by examining the infinitesimal under a glass or by reducing everything to decimals. It is well to remember that, even in the scientific world, this is but a partial method of arriving at exact results. Speaking numerically, multiplication, and not division, was the guiding process of the Greeks. The spacious definition of equal ratios which the astronomer Eudoxus bequeathed was not the work of a man with one eye glued to a micrometer.

CHAPTER V

THE RENAISSANCE: NAPIER AND KEPLER; THE RISE OF ANALYSIS

AFTER the death of Pappus, Greek mathematics and indeed European mathematics lay dormant for about a thousand years. The history of the science passed almost entirely to India and Arabia; and by far the most important event of this long period was the introduction of the Indian decimal notation into Europe. The credit for this innovation is due to Leonardo of Pisa, who was mentioned on p. 98, and certainly ranks as a remarkable mathematician in these barren centuries. From time to time there were others of merit and even of genius; but, judged by the lofty standard of past achievements and of what the future held in store, no one rose supreme. The broad fact remains: Pappus died in the middle of the fourth century, and the next great forward step for Western mathematics was taken in the sixteenth century.

It is still an obscure historical problem to determine whether Indian mathematics is independent of Greek influence. When Alexander conquered Eastern lands he certainly reached India, so that at any rate there was contact between East and West. This took place about 300 B.C., whereas the early mathematical work of India is chiefly attributed to the far later period A.D. 450–650. So in the present state of our knowledge it is safest to assume that considerable independent work was done in India. An unnamed genius invented the decimal notation; he was followed by ĀRYABHATA and BRAHMAGUPTA, who made substantial progress in algebra and trigonometry. Their work brings us to the seventh century, an era marked by the fall of Alexandria and the rise of the Moslem civilization.

The very word *algebra* is part of an Arabic phrase for 'the science of reduction and cancellation', and the digits we habitually use are often called the Arabic notation. These survivals remind us that mathematical knowledge was mediated to western Europe through the Arabs. But it will be clear from what has already been said that the Arabs were in no sense the originators of either algebra or the number notation. The Arabs rendered homage to mathematics; they valued the ancient learning whether it came from Greece or India. They proved apt scholars; and soon they were industriously translating into Arabic such valuable old manuscripts as their forerunners had not destroyed. In practical computation and the making of tables they showed their skill, but they lacked the originality and genius of Greece and India. Great tracts of Diophantine algebra and of geometry left them quite unmoved. For long centuries they were the safe custodians of mathematical science.

Then came the next chapter in the story, when northern Italy and the nations beyond the Alps began to feel their wakening strength. Heart and mind alike were stirred by the great intellectual and spiritual movements of the Renaissance and the Reformation. Once again mathematics was investigated with something of the ancient keenness, and its study was greatly stimulated by the invention of printing. There were centres of learning, in touch with the thriving city life of Venice and Bologna and other celebrated towns of mediaeval Europe. Italy led the way; France, Scotland, Germany and England were soon to follow. The first essential advance beyond Greek and Oriental mathematics was made by SCIPIO FERRO (1465–1526), who picked up the threads where Diophantus left them. Ferro discovered a solution to the cubic equation.

$$x^3 + mx = n;$$

and, as this solved a problem that had baffled the Greeks, it was a remarkable achievement.

Scipio was the son of a paper-maker in Bologna whose house can still be precisely located. He became Reader in mathematics at the University in 1496 and continued in office, except for a few years' interval at Venice, till his death in the year 1526. In those days mathematical discoveries were treasured as family secrets, only to be divulged to a few intimate disciples. So for thirty years this solution was carefully guarded, and it only finally came to light owing to a scientific dispute. Such wranglings were very fashionable: they were the jousts and tournaments of the intellectual world, and mathematical devices, often double-edged, were the weapons. Some protagonists preferred to spar with slighter blades—only drawing their mightiest swords as a last resort. Among them were Tartaglia and Cardan, both very celebrated, and ranking with Scipio as leading figures in this drama of the Cubic Equation. Scipio himself was dragged rather unwillingly into the fray: others relished it.

NICCOLO FONTANA (1500–1557) received the nickname TARTAGLIA because he stammered. When he was quite a little lad he had been almost killed by a wound on the head, which permanently affected his speech. This had occurred in the butchery that followed the capture of Brescia, his native town, by the French. His father, a postal messenger, was amongst the slain, but his mother escaped, and rescued the boy. Although they lived in great poverty, Tartaglia was determined to learn. Lacking the ordinary writing materials, he even used tombstones as slates, and eventually rose to a position of eminence for his undoubted mathematical ability. He emulated Ferro by solving a new type of cubic equation, $x^3 + mx^2 = n$; and when he heard of the original problem, he was led to re-discover Ferro's solution. This is an interesting example of what frequently happens,—the mere knowledge that a certain step had been taken being inducement enough for another to take the same step. Tartaglia was the first to apply mathematics to military problems in artillery.

GIROLAMO CARDAN (1501–1576) was a turbulent man of genius, very unscrupulous, very indiscreet, but of commanding mathematical ability. With strange versatility he was astrologer and philosopher, gambler and algebraist, physician yet father and defender of a murderer, heretic yet receiver of a pension from the Pope. He occupied the Chair of Mathematics at Milan and also practised medicine. In 1552 he visited Scotland at the invitation of John Hamilton, Archbishop of St. Andrews, whom he cured of asthma. He was interested one day to find that Tartaglia held a solution of the cubic equation. Cardan begged to be told the details, and eventually under a pledge of secrecy obtained what he wanted. Then he calmly proceeded to publish it as his own unaided work in the *Ars Magna*, which appeared in 1545. Such a blot on his pages is deplorable because of the admittedly original algebra to be found in the book. He seems to have been equally ungenerous in the treatment of his pupil Ferrari, who was the first to solve a quartic equation. Yet Cardan combined piracy with a measure of honest toil; and he had enough mathematical genius in him to profit by these spoils. He opened up the general theory of the cubic and quartic equations, by discussing how many roots an equation may have. He surmised the need not only for negative but for complex (or imaginary) numbers to effect complete solutions. He also found out the more important relations between the roots.

By these mathematical achievements, so variously conducted, Italy made a substantial advance. It was now possible to state, in an algebraic formula, the solution of the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$

The matter had proceeded step by step from the simple to the quadratic, the cubic and the quartic equation. Naturally the question of the quintic and higher equations arose, but centuries passed before further light was thrown upon them. About a hundred years ago a young Scandinavian mathematician named Abel found out the truth about these equations. They proved to be insoluble by finite algebraic formulae such as these Italians had used. Cardan, it would seem, had unwittingly brought the algebraic theory of equations to a violent full stop!

Now what was going on at this time elsewhere in Europe? Something very significant in Germany, and a steady preparation for the new learning in France, Flanders and England. Contemporary with Scipio Ferro were three German pioneers, DÜRER, STIFEL and COPERNICUS. Dürer is renowned for his art; Stifel was a considerable writer on algebra; and Copernicus revolutionized astronomy by postulating that the Earth and all the planets revolve around the Sun as centre. About this time, in 1522, the first book on Arithmetic was published in England: it was a fine scholarly production by TONSTALL, who became Bishop of London. In the preface the author explains the reason for his belated interest in arithmetic. Having forgotten what he had learnt as a boy, he realized his disadvantage when certain gold- and silver-smiths tried to cheat him, and he wished to check their transactions.

Half a century later another branch of mathematics came into prominence, when STEVINUS left his mark in work on Statics and Hydrostatics. He was born at Bruges in 1548, and lived in the Low Countries. Then once more the scene shifts to Italy, where GALILEO of Pisa (1564–1642) invented dynamics, by rebuilding the scanty and ill-conceived system which had come down from the time of Aristotle. Galileo showed the importance of experimental evidence as an essential prelude to a theoretical account of moving objects. This was the beginning of physical science—which really lies outside our present scope—and by taking this step Galileo considerably enlarged the possible applications of mathematics. In such applications it was no longer possible for the mathematician to make his discoveries merely by sitting in his study or by taking a walk. He had to face stubborn facts, often very baffling to common sense, but always the outcome of systematic experiments. Two of the first to do this were Galileo and his contemporary, Kepler. Galileo found out the facts of dynamics for himself by dropping pebbles from a leaning tower at Pisa. Kepler took, for the basis of his astronomical speculations, the results of patient observations made by Tycho Brahe, of whom more anon.

The latter half of the sixteenth century also saw the rise of mathematics in France and Scotland. France produced VIETA, and Scotland NAPIER. The work of these two great men reminds us how deep was the influence of Ancient Greece upon the leaders of this mathematical Renaissance. Allusion has already been made to the share which Vieta took in improving the notation of algebra: he also attacked several outstanding problems that had baffled the Greeks, and he made excellent progress. He showed, for example, that the famous problem of trisecting an angle really depended on the solution of a cubic equation. Also he reduced the problem of squaring a circle to that of evaluating the elegant expression:

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}} \times \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})} \times \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})})} \times \ldots$$

Here was a considerable novelty—the first actual formula for the time-honoured number π, which Archimedes had located to lie somewhere between $3\tfrac{1}{7}$ and $3\tfrac{10}{71}$. Vieta was also the first to make explicit use of that wonderful principle of duality, or reciprocation, which was hinted at by Pappus. We had an instance in the figure 11 of p. 112. For Vieta pointed out the importance of a polar triangle, obtained from a spherical triangle ABC. He drew three great circular arcs whose poles were respectively A, B, C; and then he formed a second triangle from these arcs. The study of the two triangles jointly turned out to be easier than that of the original triangle by itself.

Perhaps the most remarkable of all these eminent mathematicians was John Napier, Baron of Merchiston, who discovered the logarithm. This achievement broke entirely new ground, and it had great consequences, both practical and theoretical. It gave not only a wonderful labour-saving device for arithmetical computation, but it also suggested several leading principles in higher analysis.

John Napier was born in 1550 and died in 1617: he belonged to a noble Scottish family notable for several famous soldiers. His mother was sister of Adam Bothwell, first reformed Bishop of Orkney, who assisted at the marriage of his notorious kinsman, the Earl of Bothwell, to Queen Mary, and who also anointed and crowned the infant King James VI. Scotland was a country where barbarous hospitality, hunting, the military art and keen religious controversy occupied the time and attention of Napier's contemporaries: a country of baronial leaders whose knowledge of arithmetic went little farther than counting on the fingers of their mail-clad hands. It was a strange place for the nurture of this fair spirit who seemed to belong to another world. The boy lost his mother when he was thirteen, and in the same year was sent to the University of St. Andrews, where he matriculated in 'the triumphant college of St. Salvator'. In those days St. Andrews was no home of quiet academic studies: accordingly the Bishop, who always took a kindly interest in the lad, advised a change. 'I pray you, schir,' he wrote to John's father, 'to send your son Jhone to the schuyllis; oyer to France or Flanderis; for he can leyr na guid at hame, nor get na proffeit in this maist perullous worlde.' So abroad he went; but it is probable that he soon returned to Merchiston, his home near Edinburgh, where he was to spend so many years of his serene life.

During the year at St. Andrews his interest was aroused in both arithmetic and theology. The preface to his *Plain Discovery of the Whole Revelation of St. John,* which was published in 1593, contains a reference to his 'tender yeares and barneage in Sanct Androis' where he first was led to devote his talents to the study of the Apocalypse. His book is full of profound but, it is to be feared, fruitless speculations; yet in form it follows the finest examples of Greek mathematical argument, of which he was master, while in sober manner of interpretation it was far ahead of its time. Unlike Cardan, before him, and Kepler, after him, he was innocent of magic and astrology.

Napier acquired a great reputation as an inventor; for with his intellectual gifts he combined a fertile nimbleness in making machines. His constant efforts to fashion easier modes of arithmetical calculation led him to produce a variety of devices. One was a sort of chess-arithmetic where digits moved like rooks and bishops on a board: another survives under the name of Napier's Bones. But what impressed his friends was a piece of artillery of such appalling efficiency that it was able to kill all cattle within the radius of a mile. Napier, horrified, refused to develop this terrifying invention, and it was forgotten.

During his sojourn abroad he eagerly studied the history of the Arabic notation, which he traced to its Indian source. He brooded over the mysteries of arithmetic and in particular over the principle which underlies the number notation. He was interested in reckoning not only, as is customary, in *tens,* but also in *twos.* If the number eleven is written 11, the notation indicates *one* ten and *one*. In the common scale of ten each number is denoted by so many *ones,* so many *tens,* so many *hundreds,* and so on. But Napier also saw the value of a binary scale—in which a number is broken up into parts 1, 2, 4, 8, etc. Thus he speaks with interest of the fact that any number of pounds can be weighed by loading the other scale pan with one or more from among the weights 1 lb., 2 lb., 4 lb., 8 lb., and so on.

When Napier returned to Scotland he wrote down his thoughts on arithmetic and algebra, and many of his writings remain. They are very systematic, showing a curious mixture of theory and practice: the main business is the theory, but now and then comes an illustration that 'would please the mechanicians more than the mathematicians'. Somewhere on his pages the following table appears:

	I	II	III	IIII	V	VI	VII	. . .
1	2	4	8	16	32	64	128	. . .

Perhaps the reader thinks that it is simple and obvious; yet in the light of the sequel, it is highly significant. Men were still feeling for a notation of indices, and the full implications of the Arabic decimal notation had hardly yet been grasped. Napier was looking with the eyes of a Greek-trained mathematician upon this notation as upon a new plaything. He saw in the above parallel series of numbers the matching of an arithmetical with a geometrical progression. A happy inspiration made him think of these two progressions as *growing continuously from term to term.* The above table then became to him a sort of slow kinematograph record, implying that things are happening *between* the recorded terms. By the year 1590, or perhaps earlier, he discovered logarithms—the device which replaces multiplication by addition in arithmetic: and his treatment of the matter shows intimate knowledge of the correspondence between arithmetical and geometrical progressions. So clearly did he foresee the practical benefit of logarithms in astronomy and trigonometry, that he deliberately turned aside from his speculations in algebra, and quietly set himself the lifelong task of producing the requisite tables. Twenty-five years later they were published.

Long before the tables appeared, they created a stir abroad. There dwelt on an island of Denmark the famous Tycho Brahe, who reigned in great pomp over his sea-girt domain. It was called Uraniburg—the Castle of the Heavens—and had been given to him by a beneficent monarch, King Frederick II, for the sole purpose of studying astronomy. Here prolonged gazings and much accurate star chronicling proceeded; but the stars in their courses were getting too much for Tycho. Like a voice from another world word came of a portentous arithmetical discovery in Scotland, the *terra incognita.* The Danish astronomer looked for an early publication of the logarithmic tables; but it was long before they were completed. Napier, in fact, was slow but sure. 'Nothing', said he, 'is perfect at birth. I await the judgment and criticism of the learned on this, before unadvisedly publishing the others and exposing them to the detraction of the envious.' The first tables appeared in 1614, and immediately attracted the attention of mathematicians in England and on the Continent—notably BRIGGS and KEPLER. The friendship between Napier and Briggs rapidly grew, but was very soon to be cut short: for in 1617, worn out by his incessant toil, Napier died. One of his last writings records how 'owing to our bodily weakness we leave the actual computation of the new canon to others skilled in this kind of work, more particularly to that very learned scholar, my dear friend, Henry Briggs, public Professor of Geometry in London'.

A picturesque account of their first meeting has been handed down. The original publication had so delighted Briggs that

'he could have no quietness in himself, until he had seen that noble person whose only invention they were. . . . Mr. Briggs appoints a certain day when to meet in Edinburgh; but failing thereof, Merchiston was fearful he would not come. It happened one day as John Marr and the Lord

Napier were speaking of Mr. Briggs: "Ah, John," saith Merchiston, "Mr. Briggs will not now come": at the very instant one knocks at the gate; John Marr hasted down and it proved to be Mr. Briggs, to his great contentment. He brings Mr. Briggs up into My Lord's chamber, where almost one quarter of an hour was spent, each beholding other with admiration before one word was spoken: at last Mr. Briggs began. "My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto Astronomy, viz. the Logarithms: but My Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy." '

Exactly: and perhaps this was the highest praise. It is pleasant to record the excellent harmony existing between Napier, Briggs and Kepler. Kepler the same year discovered his third great planetary *canon* which he published in the *Ephemerides* of 1620, a work inscribed to Napier; and there for frontispiece was a telescope of Galileo, the elliptic orbit of a planet, the system of Copernicus, and a female figure with the Napierian logarithm of half the radius of a circle arranged as a glory round her head!

And what was a logarithm? Put into unofficial language it can be explained somewhat as follows. A point G may be conceived as describing a straight line TS with diminishing speed, slowing towards its destination S, in such wise that the speed is always proportional to the distance it has to go. When the point G is at the place *d* its speed is proportional to the distance *d*S. What a problem in dynamics to launch on the world,

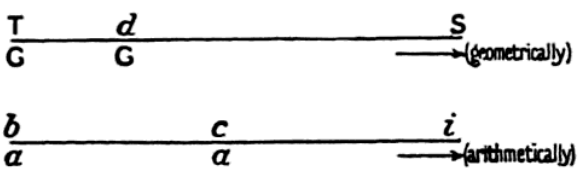


FIGURE 12

before dynamics were even invented! This motion Napier called *decreasing geometrically*. Alongside this, and upon a parallel line *bi*, a point *a* moves off uniformly from its starting position *b*. This Napier called *increasing arithmetically*. The race between the moving points G and *a* is supposed to begin at T and *b*, both starting off at the same speed; and then at any subsequent instant the places reached by G and *a* are recorded. When G has reached *d* let *a* have reached *c*. Then the number measuring the length *bc* is called by Napier the *logarithm* of the number measuring *d*S. In short, the distance *a* has gone is the logarithm of the distance G has to go.

Beginning with this as his definition Napier built up not only the theoretical properties of logarithms, but also his seven-figure tables. The definition is in effect the statement of a differential equation; and his superstructure provides the complete solution. It even suggests a theory of functions on a genuinely arithmetical basis. As this was done before either the theory of indices or the differential calculus had been invented, it was a wonderful performance.

Napier was also a geometer of some imagination. He devised new methods in spherical trigonometry. Particularly beautiful is his treatment of a right-angled spherical triangle as part of a fivefold figure, reminiscent of the Pythagorean symbol.

The story of Napier shows how the time was ripe for logarithms to be invented, and it is scarcely surprising that another should also have discovered them. This was his contemporary BÜRGI, a Swiss watchmaker, who reached his conclusions through the idea of indices, and published his results in 1620. Great credit must also be given to Briggs for the rapid progress he made in fashioning logarithmic tables of all kinds. None but an expert mathematician of considerable originality could have done the work so quickly.

The rapid spread of Napier's logarithms on the Continent was due to the enthusiasm of KEPLER, an astronomer, who was born in 1571 of humble parents near Stuttgart in Würtemberg, and died at Ratisbon in 1630. He was a man of affectionate disposition, abundant energy and methodical habits, with the intuition of true genius and the readiness to look for new relations between familiar things. He combined a love of general principles with the habit of attending to details. To his knowledge of ancient and mediaeval lore which included, in one comprehensive grasp, the finest Greek-geometry and the extravagances of astrology, he added the new learning of Copernicus and Napier. He learnt of the former in his student days at Tübingen whence at the age of twenty-two he migrated to Gratz in Austria, where he was appointed Professor. There he imprudently married a wealthy widow—a step which brought him no happiness. Within three years of his appointment he became famous through the publication of his *Mysterium*, a work full of fancies and strange theories of the heavens.

Kepler's interest in the stars and planets developed as he corresponded with the great Tycho Brahe at Uraniburg, who held even kings spellbound by his discoveries. When in course of time Brahe lost royal favour and began to wander, he accepted a post at the new observatory near Prague. He even persuaded Kepler, who also was rather unsettled, to become his assistant. This arrangement was made in 1599 at the instigation of Rudolph II, a taciturn monarch much addicted to astrology, who hoped that these two astrological adepts would bring distinction to his kingdom. In this he was disappointed: for collaboration was not a success between these two strong personalities, with their widely different upbringing. Yet the experience was good for Kepler, especially as he also came under the influence of Galileo. It helped to stabilize his wayward genius. When Tycho died in 1601, Kepler succeeded him as astronomer; but his career was dogged by bad luck. He was often unpaid; his wife died;—nor did a second matrimonial venture prove more successful, although he acted with the greatest deliberation: for he carefully analysed and weighed the virtues and defects of several young ladies until he found his desire. It is a warning to all scientists that there *are* matters in life which elude weights and measures. The axiom of Archimedes has its limitations!

Kepler brimmed over with new ideas. Possessed with a feeling for number and music, and imbued through and through with the notions of Pythagoras, he sought for the underlying harmony in the cosmos. Temperamentally he was as ready to listen as to look for a clue to these secrets. Nor was there any current scientific reason to suppose that light would yield more significant results than sound. So he brought all his genius to bear on the problem of the starry universe: and he dreamt of a harmony in arithmetic, geometry and music that would solve its deepest mysteries. Eventually he was able to disclose his great laws of planetary motion, two in 1609, and the third and finest in the *Harmonices Mundi* of 1619.

These laws, which mark an epoch in the history of mathematical science, are as follows:

1. The orbit of each planet is an ellipse, with the sun at a focus.
2. The line joining the planet to the sun sweeps out equal areas in equal times.
3. The square of the period of the planet is proportional to the cube of its mean distance from the sun.

The period in the case of the earth is, of course, a year. So this third law states that a planet situated twice as far from the sun would take nearly three years to perform its orbit, since the cube of two is only a little less than the square of three. This first law itself made a profound change in the scientific outlook upon nature. From ancient times until the days of Copernicus and Tycho Brahe, circular motion had reigned supreme. But the circle was now replaced by the ellipse: and with the discovery that the ellipse was a path actually performed in the heavens and by the earth itself, a beautiful chapter in ancient geometry had unexpectedly become the centre of a practical natural philosophy. In reaching this spectacular result Kepler inevitably pointed out details in the abstract theory that Apollonius had somehow missed—such as the importance of the focus of a conic, and even the existence of a focus for a parabola. Then by a shrewd combination of his new ideas with the original conical properties, Kepler began to see ellipses, parabolas, hyperbolas, circles, and pairs of lines as so many phases of *one* type of curve. To Kepler, starlight, radiating from points unnumbered leagues away, suggested that in geometry parallel lines have a common point at infinity. Kepler therefore not only found out something to interest the astronomer; he made essential progress in geometry. An enthusiastic geometer once lamented that here was a genius spoilt for mathematics by his interest in astronomy!

The second law of Kepler is remarkable as an early example of the infinitesimal calculus. It belongs to the same order of mathematics as the definition that Napier gave for a logarithm. Again we must remember that this calculus, as a formal branch of mathematics, still lay hidden in the future. Yet Kepler made further important contributions by his accurate methods of calculating the size of areas within curved boundaries. His interest in these matters arose partly through reading the ancient work of Archimedes and partly through a wish to improve on the current method of measuring wine casks. Kepler recorded his results in a curious document, which incidentally contained an ingenious number notation based on the Roman system, where subtraction as well as addition is involved. Kepler used symbols analogous to I, V, X, L, but instead of the numbers one, five, ten and fifty he selected one, three, nine, twenty-seven, and so on. In this way he expressed any whole number very economically; for instance,

$$20 = 27 - 9 + 3 - 1.$$

As an algebraist he also touched upon the theory of recurring series and difference relations. He performed prodigies of calculation from the sheer love of handling numbers. The third of his planetary laws, which followed ten years after the other two, was no easy flight of genius: it represented prolonged hard work.

Something may be quoted of the contents in the *Harmonices Mundi* which enshrines this great planetary law. It is typical of the work of this extraordinary man. In it he makes a systematic search into the theory of musical intervals, and their relations to the distances between the planets and the sun: he discusses the significance of the five Platonic regular solids for interplanetary space: he elaborates the properties of the thirteen semi-regular solids of Archimedes: he philosophizes on the place of harmonic and other algebraic progressions in civil life, drawing his illustrations from the dress of Cyrus as a small boy, and the equity of Roman marriage laws. Few indeed are the great discoverers in science who can rival Kepler in richness of imagery! For Kepler, every planet sang its tune: Venus a monotone, the Earth (in the sol-fa notation) the notes *m*, *f*, *m*, signifying that in this world man may expect but misery and hunger. This gave Kepler an opportunity for a Latin pun—'in hoc nostro domicilio *mi*seriam et *fa*men obtinere'. The italics are his, and in fact the whole book was written in solemn mediaeval Latin. The song of Mercury, in his arpeggio-like orbit, is

d r m f s l t d' r' m' d' s m d

—stated originally of course in the staff notation. As for the comets, surely they must be live things, darting about with will and purpose 'like fishes in the sea'! This frisky skirl of Mercury amid the sober hummings of the other planets, is no idle fancy: it duly records a curious fact, that the orbit of Mercury is more strongly elliptical, and less like a circle, than that of any other planet. It was this very peculiarity of Mercury which provided Einstein with one of his clues leading to the hypothesis of Relativity.

Carlyle, in his *Frederick the Great* (Book III, Chapter XIV) has preserved a delightful picture of John Kepler as he appeared to a contemporary, Sir Henry Wotton, Ambassador to the King of Bohemia.

" 'He hath a little black Tent . . .,' says the Ambassador, 'which he can suddenly set up where he will in a Field; and it is convertible (like a windmill) to all quarters at pleasure; capable of not much more than one man, as I conceive, and perhaps at no great ease; exactly close and dark, —save at one hole, about an inch and a half in the diameter to which he applies a long perspective Trunk, with the convex glass fitted to the said hole, and the concave taken out at the other end . . .' . . . An ingenious person, truly, if there ever was one among Adam's Posterity. Just turned fifty, and ill-off for cash. This glimpse of him, in his little black tent with perspective glasses, while the Thirty-Years War blazes out, is welcome as a date."

CHAPTER VI

DESCARTES AND PASCAL: THE EARLY FRENCH GEOMETERS AND THEIR CONTEMPORARIES

HITHERTO the mathematicians of outstanding ability, whose names have survived, have been comparatively few; but from the beginning of the seventeenth century the number increased so rapidly that it is quite impossible in a short survey to do justice to all. In France alone there were as many mathematicians of genius as Europe had produced during the preceding millennium. Three names will therefore be singled out to be representatives of their time, Descartes and Pascal from among the French, and Newton from among the English. In this heroic age that followed the performances of Napier and Kepler, mathematics attained a remarkable prestige. The age was mathematical; the habits of mind were mathematical; and its methods were deemed necessary for an exact philosophy, or an exact anything else. It was the era when what is called modern philosophy began; and the pioneers among its philosophers, like the Greek philosophers of old, were expert mathematicians. They were Descartes and Leibniz.

DESCARTES was born of Breton parents near Tours in 1596 and died at Stockholm in 1650. In his youth he was delicate, and until the age of twenty his friends despaired of his life. After receiving the traditional scholastic education of mathematics, physics, logic, rhetoric and ancient languages, at which he was an apt pupil, he declared that he had derived no other benefit from his studies than the conviction of his utter ignorance and profound contempt for the systems of philosophy then in vogue.

> 'And this is why, as soon as my age permitted me to quit my preceptors,' he says, 'I entirely gave up the study of letters; and resolving to seek no other science than that which I could find in myself or else in the great book of the world, I employed the remainder of my youth in travel, in seeing courts and camps, in frequenting people of diverse humours and conditions, . . . and above all in endeavouring to draw profitable reflection from what I saw. For it seemed to me that I should meet with more truth in the reasonings which each man makes in his own affairs, and which if wrong would be speedily punished by failure, than in those reasonings which the philosopher makes in his study.'

In this frame of mind he led a roving, unsettled life; sometimes serving in the army, sometimes remaining in solitude. At the age of three and twenty, when residing in his winter quarters at Neuberg on the Danube, he conceived the idea of a reformation in philosophy. Thereupon he began his travels, and ten years later retired to Holland to arrange his thoughts into a considered whole. In 1638 he published his *Discourse on Method* and his *Meditations.* An immense sensation was produced by the *Discourse*, which contained important mathematical work. The name of Descartes became known throughout Europe; Princes sought him; and it was only the outbreak of the civil war in England which prevented him from accepting a liberal appointment from Charles I. Instead, he went to Sweden at the invitation of Queen Christina, arriving at Stockholm in 1649, where it was hoped that he would found an Academy of Sciences. Such a replica of the Platonic School in Athens already existed in Paris. But his health gave way under the severity of the climate, and shortly after his arrival he died.

The work of Descartes changed the face of mathematics: it gave geometry a universality hitherto unattained; and it consolidated a position which made the differential calculus the inevitable discovery of Newton and Leibniz. For Descartes founded *analytical geometry*, and by so doing provided mathematicians with occupation lasting over two hundred years.

Descartes was led to his analytical geometry by systematically fitting algebraic symbols to the still fashionable rhetorical geometry. Examples of this procedure have already been given on p. 14 and elsewhere. Those examples were stated in algebraic formulae in order to convey the sense of the propositions more readily to the reader. Strictly speaking, they were an anachronism before the time of Descartes. His next step concerned the famous Apollonian problem (p. 112) [*locus*] *ad tres et quattuor lineas*, disclosed by Pappus. It will be recalled that a point moves so that the product of its oblique distances from certain given lines is proportional to that of its distances from certain others. Descartes took a step that from one point

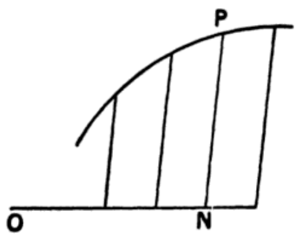


FIGURE 13

of view was simplicity itself—he enlisted the fact that plane geometry is *two*-dimensional. So he expressed everything in the figure in terms of two variable lengths, x and y, together with fixed quantities. This at once gave an algebraic statement for the results of Pappus: it put them into a form now typified by $f(x, y) = 0$, an equation where x and y alone are variable. The fundamental importance of this result lies in the further consequence that such an equation can be looked on as the definition of y in terms of x. It defined y as a function of x: it did geometrically very much what Napier's definition of a logarithm did dynamically. It also gave a new significance to the method of Archimedes for discussing the area of a curve, using an abscissa ON and an ordinate NP: in the notation of Descartes ON became x and NP, y. But, besides this, it linked the wealth of Apollonian geometry with what Archimedes had found; by forging this link Descartes rendered his most valuable service to mathematics.

Although Descartes deserves full credit for this, because he took considerable pains to indicate its significance, he was not alone in the discovery. Among others to reach the same conclusion was FERMAT—another of the great French mathematicians, a man of deeper mathematical imagination than Descartes. But Fermat had a way of hiding his discoveries.

Before indicating some of the principal consequences of this new method in geometry, there are other aspects of the notation which should be mentioned. The letter x has become world-famous: and it was the methodical Descartes who first set the fashion of denoting variables by *x*, *y*, *z* and constants by *a*, *b*, *c*. He also introduced indices to denote continued products of the same factor, a step which completed the improvements in notation originating with Diophantus. The fruitful suggestion of negative and fractional indices followed soon afterwards: it was due to WALLIS, one of our first great English mathematicians. A profound step in classification also was taken when Descartes distinguished between two classes of curves, *geometrical* and *mechanical*, or, as LEIBNIZ preferred to call them, *algebraic* and *transcendental*. By the latter is meant a curve, such as the spiral of Archimedes, whose Cartesian equation has no finite degree.

Apollonius had solved the problem of finding the shortest distance from a given point to a given ellipse, or other conic. Following this lead Des-

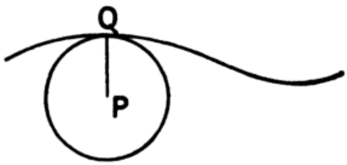


FIGURE 14

cartes addressed himself to the same problem in general: he devised a method of determining the shortest line PQ from a given point P to a given curve. Such a line meets the curve at right angles in the point Q, and is often called the *normal* at Q to the curve. Descartes took a circle with centre P, and arranged that the radius should be just large enough for the circle to reach the curve. The point where it reached the curve gave him Q, the required foot of the normal. His way of getting the proper radius was interesting; it depended on solving a certain equation, two of whose roots were equal. It is hardly appropriate to go into further details

FIGURE 15

here; but the reader who has some familiarity with analytical geometry, and has found the tangent to a circle or conic by the method of equal roots, has really employed the same general principle. Had Descartes been so inclined he could also have used his method for finding a tangent to a curve, i.e. a line PQ touching a given curve at a point Q (Figure 15). This is one of the first problems of the differential calculus; and one of the earliest solutions was found by Fermat and not by Descartes.

Fermat had discovered how to draw the tangent at certain points of a curve, namely at points Q which were, so to speak, at a crest or in the trough of a wave of the curve. They were points at a maximum or minimum distance from a certain standard base line called the axis of x. By so

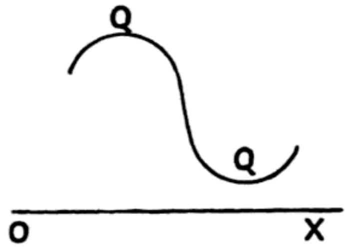


FIGURE 16

doing, Fermat had followed up a fertile hint, which Kepler had let fall, concerning the behaviour of a variable quantity near its maximum or minimum values.

An interesting curve, still called the Cartesian oval, was discovered by Descartes, and has led to far-reaching research in geometry and analysis. It was found in an endeavour to improve the shape of a lens, so as to condense a pencil of light to an accurate focus. Although a lens of this shape would successfully focus a wide-angled pencil of light, if it issued from a certain particular position, the lens would be otherwise useless. But it has a physical, besides a mathematical, interest: for the principle underlying its construction is identical with that which Hero of Alexandria first noticed in the case of plane mirrors. It is the principle of Least Action, which was ultimately exhibited in a general form by Hamilton.

All this mathematical work was but part of a comprehensive philosophical programme culminating in a theory of vortices, by which Descartes sought to account for the planetary motions. Just as Kepler had thought of comets as live fishes darting through a celestial sea, Descartes imagined the planets as objects swirling in vast eddies. It remained for Newton not only to point out that this theory was incompatible with Kepler's planetary laws, but to propose a truer solution.

In philosophy Descartes made a serious attempt to build up a system in the only way which would appeal to a mathematician—by first framing his axioms and postulates. In doing this he was the true symbol of an age, filled with self-confidence after the triumphs of Copernicus, Napier and Kepler. We cannot but admire the intellectual force of a man who undertook to revise philosophy and achieved so much. Nevertheless he lacked certain gifts that might be thought essential to success in the venture. He was cold, prudent and selfish, and offered a great contrast to his younger contemporary, the mathematician and philosopher, Blaise Pascal.

The analytical geometry of Descartes is a kind of machine: and 'the clatter of the co-ordinate mill', as Study has remarked, may be too insistent. The phenomenal success of this machine in the hands of Newton, Euler and Lagrange almost completely diverted thought from pure geometry. The great geometrical work in France, contemporary with that of Descartes, actually sank into oblivion for about two centuries, until it came into prominence once more, a hundred years ago. Two of the early French geometers were PASCAL and DESARGUES, and their work was the natural continuation of what Kepler had begun in projective geometry. Desargues, who was an engineer and architect residing at Lyons, gave to the ancient geometry of Apollonius its proper geometrical setting. He showed, for example, with grand economy, how to cut conics of different shapes from a single cone, and that a right circular cone. He won the admiration of Chasles, the great French geometer of the nineteenth century, who speaks of Desargues as an artist, but goes on to say that his work

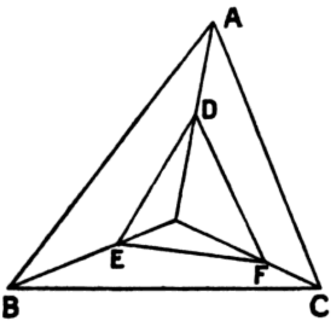


FIGURE 17

bears the stamp of universality uncommon in that of an artist. Desargues had the distinction of finding out one of the most important theorems of geometry, which takes its place, with a theorem of Pappus already quoted, as a fundamental element in the subject. It runs as follows: if two triangles ABC and DEF are such that AD, BE, CF meet in a point, then BC, EF; CA, FD; AB, DE, taken in pairs, meet in three points which are in line (Figure 17). The theorem is remarkable because it is easier to prove if the triangles are not in the same plane. As a rule, solid geometry is more difficult to handle than plane geometry—but not invariably. The perspective outline drawing of a cube on a sheet of paper is a more complicated figure than the actual outline of the solid cube. Desargues began the method of disentangling plane figures by raising them out of the flat into three dimensions. This is a choice method that has only lately borne its finest fruit in the *many-dimensional* geometry of Segre and the Italian school.

The work of Desargues is intimately linked with that of Pascal. Even in the grand century which produced Descartes, Fermat and Desargues, the fourth great French mathematician, BLAISE PASCAL, stands out for the brilliancy of his genius and for his astonishing gifts. He was born at Clermont-Ferrand in Auvergne on 19th June, 1623; and was educated with the greatest care by his father, who was a lawyer and president of the Court of Aids. As it was thought unwise to begin mathematics too early, the boy was put to the study of languages. But his mathematical curiosity was aroused, when he was twelve years old, on being told in reply to a question as to the nature of geometry, that it consisted in constructing exact figures and in studying the relations between the parts. Pascal was doubtless stimulated by the injunction against reading it, for he gave up his playtime to the new study, and before long had actually deduced several leading properties of the triangle. He found out for himself the fact that the angles of a triangle are together equal to two right angles. When his father knew of it, he was so overcome with wonder that he wept for joy, repented, and gave him a copy of Euclid. This, eagerly read and soon mastered, was followed by the conics of Apollonius, and within four years Pascal had written and published an original essay on conic sections, which astounded Descartes. Everything turned on a miracle of a theorem that Pascal called 'L'hexagramme mystique', commonly acknowledged to be the greatest theorem of mediaeval geometry. It states that, if a hexagon is inscribed in a conic, the three points of intersection of pairs of opposite sides always lie on a straight line: and from this proposition he is said to have deduced hundreds of corollaries, the whole being infused with the method of projection. The theorem has had a remarkably rich history, after the two hundred year eclipse, culminating in the enchantments of Segre when he presents it as a cubic locus in space of four dimensions, transfigured yet in its simplest and most inevitable form!

During these years Pascal was fortunate in enjoying the society in Paris of Roberval, Mersenne and other mathematicians of renown, whose regular weekly meetings finally grew into the French Academy. Such a stimulating atmosphere bore fruit after the family removed to Rouen, where at the age of eighteen Pascal amused himself by making his first calculating machine, and six years later he published his *Nouvelles Expériences sur le vide*, containing important experimental results which verified the work of Torricelli upon the barometer. Pascal was, in fact, as capable and original in the practical and experimental sciences as in pure geometry. At Rouen his father was greatly influenced by the Jansenists, a newly formed religious sect who denied certain tenets of Catholic doctrine, and in this atmosphere occurred his son's first conversion. A second conversion took place seven years later, arising from a narrow escape in a carriage accident. Henceforth Pascal led a life of self-denial and charity, rarely equalled and still more rarely surpassed. When one of his friends was condemned for heresy, Pascal undertook a vigorous defence in *A Letter written to a Provincial*, full of scathing irony against the Jesuits. Then the idea came to him to write an Apologia of the Christian Faith, but in 1658 his health, always feeble, gave way; and after some years of suffering borne with noble patience he died at the age of thirty-nine. The notes in which he jotted down his thoughts in preparation for this great project, have been treasured up and published in his *Pensées*, a literary classic.

In Pascal the simplest faith graced the holder of the highest intellectual gifts: and for him mathematics was something to be taken up or laid aside at the will of God. So when in the years of his retirement, as he lay awake suffering, certain mathematical thoughts came to him and the pain disappeared, he took this as a divine token to proceed. The problem which occurred to him concerned a curve called the cycloid, and in eight days he found out its chief properties by a brilliant geometrical argument. This curve may be described by the rotation of a wheel: if the axle is fixed, like that of a flywheel in a machine, a point on the rim describes a circle; but if the wheel rolls along a line, a point on the rim describes a cycloid. Galileo, Descartes and others were interested in the cycloid, but Pascal surpassed them all. To do so he made use of a new tool, the *method of indivisibles* recently invented by the Italian CAVALIERI. Though Pascal threw out a challenge, no one could compete with him: and his work may be regarded as the second chapter in the integral calculus, to which Archimedes had contributed the first.

An account of Pascal, the mathematician, would be incomplete without reference to his algebra, which, in the present-day sense of the word, he practically founded. It arose out of a game of chance that had formed a topic of discussion between Pascal and Fermat. From the debate the notion of mathematical *probability* emerged; this in turn Pascal looked upon as a problem in arrangements or combinations of given things and in counting those arrangements. With characteristic insight he lit upon the proper mechanism for handling the subject. It was the *Arithmetic Triangle*, a device already used by Napier for another purpose, and dating from still earlier times.

1	1	1	1	1	1
1	2	3	4	5	
1	3	6	10		
1	4	10			
1	5				
1					

Certain numbers are written down in a triangular table, as shown by the diagram. The table can at any stage be enlarged by affixing further numbers, one each at the right-hand extremities of the rows, with a single 1 added at the bottom of the first column to start a new row. For example, underneath the 5 of the second row, and alongside the 10 of the third row a new number can be placed. This number is 15, the sum of the 5 and the 10. According to this rule of simple addition each new number is entered in the table. The diagram exhibits a 1 in the top left-hand corner followed by five parallel diagonals, the fifth and last being (1, 5, 10, 10, 5, 1). A sixth, which has not been filled in, would consist of 1, 6, 15, 20, 15, 6, 1, according to the addition rule. Instead of locating an entry, 10 for example, as standing in the fourth row and third column, it is more important to locate it by the *fifth diagonal* and *third* column. Pascal discovered that this gave the number of combinations of *five* things taken *two* at a time; and he found a formula for the general case, when the number stood in the mth diagonal and the $(n + 1)$th column. He stated this correctly to be $(n + 1)(n + 2)(n + 3) \ldots (m)/1.2.3. \ldots (m - n)$. He also utilized the diagonals for working out the binomial expansion of $(a + b)^m$. For example,

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Numbers and quantities are not always so important for their size or bulk as for their patterns and arrangements. What Pascal did was to bring this notion of pattern, common enough in geometry, to bear upon number itself—a highly significant step in the history of mathematics. By so doing he created higher algebra and prepared the way for Bernoulli, Euler and Cayley. 'Let no one say that I have said nothing new', writes Pascal in his *Pensées*; 'the arrangement of the subject is new. When we play tennis, we both play with the same ball, but one of us places it better.'

FERMAT, who shared with Pascal the beginning of this algebra, is most famous for his theory of numbers. In the margin of a copy of Diophantus he made a habit of scribbling notes of ideas which came into his mind as he read. These notes are unique in their interest and profundity: he seemed to grasp properties of whole numbers by intuition rather than reason. The most celebrated note, which is often called *Fermat's Last Theorem*, has baffled the wit of all his analytical successors: for no one has yet been able to say whether Fermat was right or wrong. The theorem asserts that it is impossible to find whole numbers x, y, z which satisfy the equation

$$x^n + y^n = z^n$$

when n is an integer greater than 2. He adds: 'I have found for this a truly wonderful proof, but the margin is too small to hold it.' The problem has led to a wealth of new methods and new ideas about number; valuable prizes have been offered for a solution; but to-day its quiet challenge still remains unanswered.

Great things were also going on in Italy and England during this early seventeenth century. CAVALIERI of Bologna will always rank as a remarkable geometer who went far in advancing the integral calculus by his *Method of Indivisibles,* following up Kepler's wine-cask geometry. One of his theorems is a gem: upon concentric circles equally spaced apart he drew a spiral of Archimedes whose starting-point was the centre. Then in order to discover its area he re-drew the figure with all the circles straightened out into parallel lines the same distances apart as before. As a result the spiral became a parabola: and 'Unless I am mistaken', he adds, 'this is a new and very beautiful way of describing a parabola.' This is an early example of a transcendental mathematical transformation that not only preserves the area of a sector of the original curve but also the length of its arc.

Another very fine piece of work was done in 1695 by PIETRO MENGOLI, who gave an entirely new setting to the celebrated logarithm, by showing that it was intimately linked with a harmonical progression. His definition and treatment was on true Eudoxian lines and rigorous enough to satisfy the strictest arithmetical disciple of Weierstrass.

It is natural that, in these years succeeding Napier's death, a great deal of attention was bestowed upon the logarithm. Besides the practical business of constructing tables there was the still more interesting theory of logarithms to consider. The stimulus of analytical geometry encouraged several mathematicians to treat the logarithm by the method of co-ordinates. This led to a beautiful result that connected the area between a hyperbola and its asymptote with the logarithm. It was found in 1647 by GREGOIRE DE SAINT VINCENT, of Flanders: but several others turned their attention to the matter, reaching the same general conclusions more or less independently; notably Mercator, Mersennes, Brouncker, Wallis, James Gregory, Newton and Leibniz. (This Mercator was not the maker of geographical maps: he was a mathematician who had lived in the previous century.)

It is not difficult to suggest how this result was attained. A start was made with the geometrical progression whose sum is $1/(1-x)$; namely,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots,$$

and a curve was determined whose co-ordinate equation is $y = 1/(1-x)$. This curve is a hyperbola. Next, its area was determined, by following much the same course that Archimedes had taken for the case of the parabola. There was no difficulty in finding a requisite formula, thanks to Napier's original definition of the logarithm. It led to the result

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots,$$

which is called the logarithmic series. As may be seen, it is a union of the geometrical and harmonical progression.

Among the names which have just been given we find one Scot, one Irishman, and two Englishmen. For at last England produced mathematicians of the first rank, and in Gregory Scotland possessed a worthy successor to Napier. It is interesting to give, as typical specimens from the work of these our fellow-countrymen, the following formulae, which may be compared and contrasted with the logarithmic series:

$$\frac{4}{\pi}=\cfrac{1}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\dots}}}},$$

$$\frac{\pi}{4}=\frac{2\times4\times4\times6\times6\times8\times\dots}{3\times3\times5\times5\times7\times7\times\dots},$$

$$\frac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\dots$$

The first is due to LORD BROUNCKER, an Irish peer; the second to WALLIS, who was educated in Cambridge and later became Savilian Professor of Mathematics in Oxford. The third was given by LEIBNIZ, but is really a special case of a formula discovered by JAMES GREGORY. Two of these formulae have been slightly altered from their original statements. The reader is not asked to prove, but merely to accept the results! After all, as they stand, they are readily grasped. The row of dots, with which each concludes, signifies that the formula can be carried farther; in fact, they each have something in common with the ladder-arithmetic of Athens (p. 97). They have this in common also with Vieta's formula for π (p. 121); but they improve on it, not only for their greater simplicity, but because each converges, as Plato would have it, by 'the great and small'—each step slightly overshooting the mark. This is not always done when such sequences are used, as in the more ordinary formula

$$\frac{\pi}{4} = \tfrac{1}{4} \text{ of } 3\cdot1415926\dots = \cdot785398\dots,$$

which approximates from one side only, like the putts of a timid golfer who *never* gives the ball a chance, or like the race of Achilles and the tortoise. Such series need careful handling, as Zeno had broadly hinted;

and Gregory (by framing the notions of convergency and divergency) was the first to provide this.

In the last of these four formulae for $\frac{\pi}{4}$, the digits occur at random, and for this reason the statement is of little interest except to the practical mathematician. It is far otherwise with the other three: the *arrangement* of their parts has the inevitability of the highest works of art. It would be a pleasure to hear Pythagoras commenting upon them.

The Gregory family has long been associated with the county of Aberdeen. It had not been distinguished intellectually until John Gregory of Drumoak married Janet Anderson, herself a mathematician and a relative of the Professor of Mathematics in Paris. Many of their descendants have been eminent either as mathematicians or physicians. Chief among them all was their son James, who learnt mathematics from his mother. Unhappily, like Pascal, he died in his prime; but he lived long enough to exhibit his powers. After spending several years in Italy he occupied the Chair of Mathematics in St. Andrews for six years, followed by one year in Edinburgh. Shortly before his death he became blind.

Gregory was a great mathematical analyst, and many of his incidental results are striking. From the study of the logarithm he discovered the binomial theorem, generally and rightly attributed to Newton, who had probably found it out a few years earlier without publishing the result. It was but another case of independent discovery, as were also their invention of the reflecting telescope, and their attainments in the differential and integral calculus. The work of Gregory opened out a broad region of higher trigonometry, algebra and analysis. It is important not merely in detailed theorems but for its general aim, which was to prove that no finite algebraic formula could be found to express the functions that arise in trigonometry and logarithms. In other words, he held that circle-squarers were pursuing a vainer phantom than those who endeavour with rule and compass to trisect an angle. His project was lofty, even if it inevitably failed: it was a brilliant failure in an attempt to disentangle parts of pure mathematics which were only satisfactorily resolved during the nineteenth century.

Some of his greatest work remained in manuscript until the Gregory tercentenary (1938) gave an opportunity to publish it. This included an important general theorem which was later discovered by Brook Taylor (1715). Paper was scarce in 1670 when Gregory used the blank spaces of old letters to record his work. This was the year when BARROW produced his masterpiece, the *Lectiones Geometricae*, in which the foundations of the differential and the integral calculus were truly but geometrically laid.

If it is asked what is the peculiar national contribution made by our country to mathematics, the reply is: the mathematics of interpolation—the mathematical art of reading between the lines. As an illustration let us consider the arithmetical triangle of Pascal, supposing it to be a fragment of an Admiralty chart. The numbers indicate the depth in fathoms at various points on the surface of the sea. Such a chart with these particular readings obviously indicates a submarine valley trending downwards south-east. What the chart does not show is the actual depth at positions intermediate between the readings. Mathematical interpolation is concerned with discovering a formula for the most probable depth consistent with these measured soundings. Certain isolated points are given: what is happening between? Napier, Briggs, Wallis, Gregory and Newton, each in his way gave an answer.

From gap to gap
One hangs up a huge curtain so,
Grandly, nor seeks to have it go
Foldless and flat against the wall.

Indeed, some faith was needed to believe that there *was* a curtain, and some imagination to see its pattern. For Napier it was the pattern of the logarithm; Wallis wrought a continuous chain out of the isolated exponents x^1, x^2, x^3, . . . , by filling in fractional indices. Newton found out the pattern which fills in the triangle of Pascal; and from this he discovered the binomial theorem in its general form. Briggs suggested and Gregory found an interpolation formula of very wide application, while Newton supplemented it with several other alternatives which have usually been attributed to Stirling, Bessel and Gauss.

CHAPTER VII

ISAAC NEWTON

IN the country near Grantham during a great storm, which occurred about the time of Oliver Cromwell's death, a boy might have been seen amusing himself in a curious fashion. Turning his back to the wind he took a jump, which of course was a long jump. Then he turned his face to the wind and again took a jump, which was not nearly so long as his first. These distances he carefully measured, for this was his way of ascertaining the force of the wind. The boy was Isaac Newton, and he was one day to measure the force, if force it be, that carries a planet in its orbit.

From school at Grantham his friends took him to tend sheep and go regularly to the Grantham market. But as he *would* read mathematics instead of minding his business, it was at last agreed that he should go back to school, and from school to college. At school he lodged with Mr. Clark, apothecary, and in his lodgings spent much time, hammering and knocking. In the room were picture-frames and pictures of his own making, portraits, and drawings of birds and beasts and ships. Somewhere in the house might be seen a clock that was worked by water, and a mill which had a mouse as its miller. The boy made a carriage which could be propelled by the passenger, and a sundial that stood in the yard. To the little ladies of the house he was a very good friend, making tables and chairs for their dolls. His schoolfellows looked up to him as a skilful mechanic. As for his studies, when he first came to school he was somewhat lazy, but a fight that he had one day woke him up, and thereafter he made good progress. This quiet boy had great powers which were yet to be brought out.

In his twentieth year he went to Cambridge, where for more than thirty years he lived at Trinity College. He entered the college as a sizar, that is to say, being too poor to live in the style of other undergraduates he received help from the college. His tutor invited him to join a class reading Kepler's *Optics*. So Newton procured a copy of the book, and soon surprised the tutor by mastering it. Then followed a book on astrology; but this contained something which puzzled him. It was a diagram of the heavens. He found that, in order to understand the diagram, he must first understand geometry. So he bought Euclid's *Elements*, but was disappointed to find it too simple. He called it a 'trifling book' and threw it aside (an act of which he lived to repent). But turning to the work of Descartes he found his match, and by fighting patiently and steadily he won the battle.

After taking his degree Isaac Newton still went on learning all the mathematics and natural philosophy that Cambridge could teach him, and finding out new things for himself, until the Lucasian Professor of Mathematics in the University had become so convinced of the genius of this young man that, incredible as it may seem, he gave up to him his professorship. Isaac Barrow, the master of Newton's college, who thus resigned, was at no time a man to prefer self-interest before honour. He was possessed of great personal courage, and is reputed to have fought with a savage dog in an early morning's walk, and to have defended a ship from pirates. He was a mathematician of no mean powers; and as a divine he gained a lasting reputation.

Newton made three famous discoveries: one was in light, one was in mathematics, and one in astronomy. We are not to suppose that these flashed upon him all at once. They were prepared for by long pondering. 'I keep', said he, 'the subject of my inquiry constantly before me, and wait till the first dawning opens gradually, by little and little, into a full and clear light.' Early in his career he discovered that white light was composed of coloured lights, by breaking up a sunbeam and making the separate beams paint a rainbow ribbon of colours upon a screen. This discovery was occasioned by the imperfection of the lenses in telescopes as they were then made. Newton chose to cure the defect by inventing a reflecting telescope with a mirror to take the place of the principal lens, because he found that mirrors do not suffer from this awkwardness of lenses. It is one of his distinctions, shared with Archimedes and a few other intellectual giants, that his own handiwork was so excellent. In the chapel of his college there is a statue, holding a prism:

—Newton, with his prism and silent face;
The marble index of a mind for ever
Voyaging through strange seas of thought, alone.

In mathematics his most famous discovery was the differential and integral calculus—which he called the method of *Fluxions*: and in astronomy it was the conception and elaboration of universal gravitation. It would be a mistake to suppose that he dealt with these subjects one by one: rather they were linked together, and reinforced each other. Already at the age of twenty-three, when for parts of the years 1665 and 1666 the college was shut down owing to the plague, Newton had thought out, in his quiet country home, the principles of gravitation and, for the better handling of the intense mathematical difficulties which the principles involved, he had worked at the fluxional calculus. In the space of three years after first reading geometry, he had so completely mastered the range of mathematics from Archimedes to Barrow, that he had fitted their wonderful infinitesimal geometry into a systematic discipline. Newton gave to analysis the same universality that Descartes had already given to geometry.

Newton may be said to have fused the points of view adopted by Napier and Descartes into a single whole. Napier thought of points M and N racing along parallel tracks OX and OY, N moving steadily and M at a variable speed. The co-ordinates of Descartes provide a chart of the race in the following way: the lines OX and OY can be placed, no longer parallel, but at right angles to each other, and a curve can be plotted, traced by a point P which is simultaneously abreast of the points N and M. In this way two figures can be drawn, one the Napierian and the other the Cartesian. The figures are symbols of two lines of thought—the kinematical and the geometrical. Newton may never actually have drawn such figures side by side, but he certainly had the two trains of thought. 'I fell by degrees on the method of fluxions,' he remarks: and by *fluxions* he simply meant what we call the simultaneous speeds of the points N and M. Then by seeking to compare the speed of M with that of N, he devised the method which the geometrical figure suggests. 'Fluxions' was his name

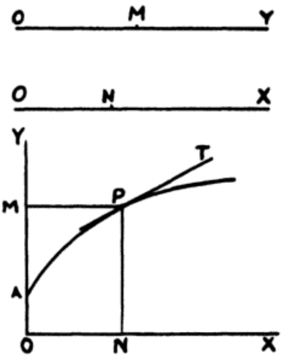


FIGURE 18

for what we call the differential and integral calculus, but he kept the discovery to himself.

In after years LEIBNIZ announced that *he* had found this new mathematical method. Then a quarrel arose between the followers of Newton and the followers of Leibniz, and unhappily it grew into a quarrel between the great men themselves. It is enough to say that the time was ripe for such a discovery: and both Newton and the German philosopher were sufficiently gifted to effect it. Newton was the first to do so, and only brought the trouble unwittingly on his head by refraining from publishing his results. It is also probable that Leibniz was influenced more by Pascal and Barrow than by Newton: and in turn we owe to Leibniz the record of parts of Pascal's work which would otherwise have been lost.

About this time the Royal Society of London was founded by King Charles II. It corresponded with the Academy of Paris, and provided a rendezvous for the leading mathematicians and natural philosophers in the country. Two of the Fellows of this Society were Gregory and Newton, who had become friends through their common interest in the reflecting telescope. Besides maintaining a correspondence, they may actually have met. They were certainly brought into contact with other leading mathematicians and astronomers. Among those who have not already been mentioned in the last chapter were Wren, Hooke, and Halley.

Christopher Wren is now so famous as the builder of St. Paul's Cathedral that we never hear of his scientific fame, though a man of science he was. Hooke, who was in appearance a puny little man, was a hard student, often working till long after midnight, but caring too excessively for his own reputation. When Newton found out anything, Hooke would commonly remark, 'That is just what I found out before.' But he was a great inventor, whose eager speculations stirred people up to think about the questions which Newton was to solve. Halley was an astronomer—a very active man, always travelling about the world to make some addition to his science. Every one has heard of Halley's comet, and to Halley is due the credit of bringing Newton before the world as the discoverer of gravitation.

One day these three friends were talking earnestly together: the subject of their conversation was the whirlpool theory of Descartes, which they felt to be hardly a satisfactory explanation of planetary motion. It did not seem to give a proper explanation of the focal position of the Sun within the elliptic orbit. Instead of imagining the planets to be propelled by a whirling current, they preferred to think that each planet was forcibly attracted by the Sun. 'Supposing', said they, 'the Sun pulls a planet with such and such a force, how ought the planet to go? We want to see clearly that the planet will go in an ellipse. If we can see that, we shall be pretty sure that the Sun *does* pull the planet in the way we supposed.' 'I can answer that,' said Hooke: upon which Wren offered him forty shillings on condition of his producing the answer within a certain time. However, nothing more was heard of Hooke's solution. So at last after several months Halley went to Cambridge, to consult Newton; and, without mentioning the discussion which had taken place in London, he put the question: If a planet were pulled by the Sun with a force which varies inversely as the square of the distance between them, in what sort of a curve ought the planet to go? Newton, to Halley's astonishment and delight answered, 'An ellipse.' 'How do you know that?' 'Why, I have calculated it.' 'Where's the calculation?' Oh, it was somewhere among his papers; he would look for it and send it to Halley. It appeared that Newton had worked all this out long before; and only now in this casual way was the matter made known to the world. Then Halley did a wise thing: he persuaded his retiring friend to develop the entire problem, explaining the whole complicated system of planetary motion. This Newton did; it was a tremendous task, taking two or three years; at the end of which appeared the famous book called *The Mathematical Principles of Natural Philosophy*, or more shortly the *Principia*, one of the supreme achievements of the human mind.

It is impossible to exaggerate the importance of the book, which at once attracted the keenest attention not only in England but throughout Europe. It was a masterpiece alike of mathematics and of natural philosophy. Perhaps the strangest part of the work was not so much the conception that the Sun pulls the planet, but that the planet pulls the Sun—and pulls equally hard! And that the whole Universe is full of falling bodies: and everything pulls everything else—literally everything, down to the minutest speck of dust. When Newton's friends had discussed the effect of the solar attraction upon a planet, they had correctly surmised the requisite force: it was determined by what is called the *law of the inverse square*. Newton had already adopted this law of force in his early conjectures, during the long vacation of 1666, over twenty years before the publication of the *Principia* (1687). That early occasion is also the date to which the well-known apple story may be referred. It is said that the sight of a falling apple set in motion the train of thought, leading Newton to his discovery of universal gravitation. But after working out the mathematical consequences of his theory and finding them to disagree with the observed facts he had tossed his pages aside. Only after many years he became aware of later and more careful calculations of the observations. This time, to his delight, they fitted his mathematical theory, and so Newton was ready with his answer, when Halley paid him the memorable visit.

In the *Principia* Newton demonstrated that, if his rule of gravitation is universally granted, it becomes the key to all celestial motions. Newton could not *prove* that it was the right key, for not all the celestial motions were known at the time, but very nearly all that have since been discovered help to prove that he was right. Even so, there was enough already known to give Newton plenty of trouble. The moon, for instance, that refuses to go round the Earth in an exact ellipse, but has all sorts of fanciful little excursions of her own—the moon was very trying to Isaac Newton.

Newton's great book was written in Latin, and, in order to make it intelligible to current habits of mind, it was couched in the style of Greek geometry. Newton had of course worked the mathematics out by fluxions, but he preferred to launch the main gravitational discovery alone, without further perplexing his readers by the use of a novel method. Outside his Cambridge lecture-room little was known of his other mathematical performances until a much later date. His *Arithmetica Universalis* was published in 1707, and two more important works, on algebra and geometry, appeared about the same time. Newton left his mark on every branch of mathematics which he touched; indeed, there are few parts of the subject which escaped his attention. Allusion has already been made to his work in interpolation and algebra. The power of his methods may be judged from one celebrated theorem which he gave without proof for determining the positions of the roots of an equation. A hundred and fifty years elapsed before Sylvester discovered how to prove his theorem.

The publication of the *Principia* forced Newton to abandon his sheltered life. In 1689 he became a Member of Parliament, and a few years later was appointed Master of the Mint. In 1705 he was knighted by Queen Anne. He died in 1727 at an advanced age, and was buried in Westminster Abbey. Voltaire has recorded his pride at having lived for a time 'in a land where a Professor of Mathematics, only because he was great in his vocation, was buried like a king who had done good to his subjects'. The world at large is often more generous in showing appreciation and gratitude than are mathematicians themselves, who feel, but are slow to exhibit their feelings. It is therefore the more noteworthy that, two hundred years later, in 1927, the English mathematical world made a pilgrimage to Grantham, to signal their respect for the genius of Newton. This alone is enough to indicate that the immense reputation which he always enjoyed was fully deserved.

It is proper to associate, with Newton, the great Dutch natural philosopher HUYGENS (1629–1693), who was in close touch with scientists of England, and did much to stimulate their wonderful advances. His own work in physics is so grand that his mathematics are apt to be overlooked. He contributed many elegant results in the infinitesimal calculus, particularly in its bearings upon mechanical phenomena, the oscillations of a pendulum, the shape of a hanging string, and the like. But he is best known for his undulatory theory of light.

As a mathematical concept this has proved to be a landmark in the history, and it is particularly interesting because it has thrown Newton's universal gravitation into intense relief. To Newton light seemed to be so many tiny particles streaming in luminous lines: to Huygens, on the contrary, light was propagated by waves. The sequel has shown that, of these rival theories, the latter is the more valuable. Not only has it provided a better key to optical puzzles, but it has also answered many purposes in the theory of electricity and magnetism. One by one, the natural phenomena were absorbed in this all-enveloping wave-theory, and gravitation alone remained untouched—a single physical exception. This unwavelike behaviour of gravitation, this action at a distance, sorely perplexed Newton himself, long before these further instances of natural behaviour had made wave motion the correct deportment. As the mystery of gravitation deepened, it became more and more the conscious aim of scientists to explain the contrast: and the matter has only lately been settled by Einstein, who solves the problem by drastically embedding gravitation in the very texture of space and time.

But it would be wrong to suppose that this left the field clear for the wave-theory. Quietly and unobtrusively other interruptions have been congregating, and reasons have once more been urged in favour of Newton's corpuscular theory of light. At the present time there is no clear-cut decision one way or the other: the work of both Newton and Huygens appears to be fulfilled in the Quantum Theory and the Wave Mechanics.

CHAPTER VIII

THE BERNOULLIS AND EULER

THE story of mathematics during the eighteenth century is centred upon Euler, and the scene of action is chiefly laid in Switzerland and Russia. About the time when Napier was experiencing the turmoil of the Reformation, violent persecution of Protestants took place in Antwerp. One of the many refugees, whom Belgium could ill afford to lose, was a certain Jacques Bernoulli, who fled to Frankfort. In 1622 his grandson settled at Basel, and there, on the frontiers of Switzerland, the BERNOULLI family were destined to bring fame to the country of their adoption. As evidence of the power of heredity, or of early home influence, their mathematical record is unparalleled. No less than nine members of the family attained eminence in mathematics or physics, four of whom received signal honours from the Paris Academy of Sciences. Of these nine the two greatest were the brothers JACOB and JOHN, great-grandsons of the fugitive from Antwerp. Jacob was fifth child in the large family, and John, thirteen years his junior, was tenth. Each in turn became Professor of Mathematics at Basel.

The elder brother settled to his distinguished career, as a mathematical analyst, only after considerable experiment and travel. At one time his father had forbidden him to study either mathematics or astronomy, hoping that he would devote himself to theology. But an inborn talent urged his son to spend his life in perfecting what Pascal and Newton had begun. Among his many discoveries, and perhaps the finest of them all, is the equiangular spiral. It is a curve to be found in the tracery of the spider's web, in the shells upon the shore and in the convolutions of the far-away nebulae. Mathematically it is related in geometry to the circle and in analysis to the logarithm. A circle threads its way over the radii by crossing them always at right angles; this spiral also crosses its radii at a constant angle—but the angle is not a right angle. Wonderful are the phœnix-like properties of the curve: let all the mathematical equivalents of burning it and tearing it in pieces be performed—it will but reappear unscathed! To Bernoulli in his old age the curve seemed to be no unworthy symbol of his life and faith; and in accordance with his wishes the spiral was engraved upon his tombstone, and with it the words *Eadem mutata resurgo.*

His younger brother JOHN (1667–1748) followed in his footsteps, continually adding fresh material to the store of analysis, which now included differential equations. His works exhibits a bolder use of negative and imaginary numbers, thereby realizing 'the great emolument' which Napier himself had hoped to bestow on mathematics by 'this ghost of a quantity,' had not his own attention been absorbed by logarithms. His sons Daniel and Nicolas Bernoulli were also very able mathematicians, and it was under their influence at college that Euler discovered his vocation.

LEONARD EULER (1707–1783) was the son of a clergyman who lived in the neighbourhood of Basel. His natural aptitude for mathematics was soon apparent from the eagerness and facility with which he mastered the elements under the tuition of his father. At an early age he was sent to the University of Basel, where he attracted the attention of John Bernoulli. Inspired by such a teacher he rapidly matured, and at the age of seventeen, when he received the degree of Master of Arts, he provoked high applause for a probationary discourse, the subject of which was a Comparison between the Cartesian and Newtonian Systems.

His father earnestly wished him to enter the ministry and directed his son to study theology. But unlike the father of Bernoulli, he abandoned his views when he saw that his son's talents lay in another direction. Leonard was allowed to resume his favourite pursuits and, at the age of nineteen, he transmitted two dissertations to the Paris Academy, one upon the masting of ships, and the other on the philosophy of sound. These essays mark the beginning of his splendid career.

About this time, in consequence of the keen disappointment at failing to attain a vacant professorship in Basel, he resolved to leave his native country. So in 1727, the year when Newton died, Euler set off for St. Petersburg to join his friends, the younger Bernoullis, who had preceded him thither a few years earlier. On the way to Russia, he learnt that Nicolas Bernoulli had fallen a victim to the stern northern climate; and the very day upon which he set foot on Russian soil the Empress Catherine I died—an event which at first threatened the dissolution of the Academy, of which she had laid the foundation. Euler, in dismay, was ready to give up all hope of an intellectual career and to join the Russian navy. But, happily for mathematics, when a change took place in the aspect of public affairs in 1730, Euler obtained the Chair of Natural Philosophy. In 1733 he succeeded his friend Daniel Bernoulli, who wished to retire; and the same year he married Mademoiselle Gsell, a Swiss lady, the daughter of a painter who had been brought to Russia by Peter the Great.

Two years later, Euler gave a signal example of his powers, when in three days he effected the solution of a problem urgently needed by members of the Academy, though deemed insoluble in less than several months' toil. But the strain of the work told upon him, and he lost the sight of an eye. In spite of this calamity he prospered in his studies and discoveries, each step seeming only to invigorate his future exertions. At about the age of thirty he was honoured by the Paris Academy when he received recognition, as also did Daniel Bernoulli and our own countryman Colin Maclaurin, for dissertations upon the flux and reflux of the sea. The work of Maclaurin contained a celebrated theorem upon the equilibrium of elliptical spheroids; that of Euler brought the hope considerably nearer of solving outstanding problems on the motions of the heavenly bodies.

In the summer of 1741 King Frederick the Great invited Euler to reside in Berlin. This invitation was accepted, and until 1766 Euler lived in Germany. On first arriving he received a royal letter written from the camp at Reichenbach, and he was soon after presented to the queen-mother, who always took a great interest in conversing with illustrious men. Though she tried to put Euler at his ease, she never succeeded in drawing him into any conversation but that of monosyllables. One day when she asked the reason for this, Euler replied, 'Madam, it is because I have just come from a country where every person who speaks is hanged.' It was during his residence in Berlin that Euler wrote a remarkable set of letters, or lessons, on natural philosophy, for the Princess of Anhalt Dessau, who was eager for instruction from so great a teacher. These letters are a model of perspicuous and interesting teaching, and it is noteworthy that Euler should have found time for such detailed elementary work, amid all his other literary interests.

For eleven years his widowed mother lived in Berlin also, receiving assiduous attention from her son, and enjoying the pleasure of seeing him universally esteemed and admired. Euler became intimate in Berlin with M. de Maupertuis, President of the Academy, a Frenchman from Brittany who strongly favoured Newtonian philosophy in preference to Cartesian. His influence was important, as it was exerted at a time when Continental opinion was still reluctant to accept the views of Newton. Maupertuis much impressed Euler with his favourite principle of least action, which Euler used with great effect in his mechanical problems.

It speaks highly for the esteem in which Euler was held that, when in 1760 a Russian army invaded Germany and pillaged a farm belonging to Euler, and the act became known to the general, the loss was immediately made good, and a gift of four thousand florins was added by the Empress Elizabeth when she learnt of the circumstance. In 1766 Euler returned to Petersburg, to spend the remainder of his days, but shortly after his arrival he lost the sight of his other eye. For some time he had been forced to use a slate, upon which in large characters he would make his calculations. Now, however, his pupils and children copied his work, writing the memoirs exactly as Euler dictated them. Magnificent work it was too, astonishing at once for its labour and its originality. He developed an amazing facility for figures, and that rare gift of mentally carrying out far-reaching calculations. It is recorded that on one occasion when two of his pupils, working the sum of a series to seventeen terms, disagreed in their results by one unit at the fiftieth significant figure, an appeal was made to Euler. He went over the calculation in his own mind, and his decision was found to be correct.

In 1771, when a great fire broke out in the town and reached Euler's house, a fellow-countryman from Basel, Peter Grimm, dashed into the flames, discovered the blind man and carried him off on his shoulders into safety. Although books and furniture were all lost, his precious writings were saved. For twelve years more Euler continued his excessive labours, until the day of his death, in the seventy-sixth year of his age.

Like Newton and many others, Euler was a man of parts, who had studied anatomy, chemistry and botany. As is reported of Leibniz, he could repeat the *Aeneid* from beginning to end, and could even remember the first and last lines in every page of the edition which he had been accustomed to use. The power seems to have been the result of his most wonderful concentration, that great constituent of inventive power, to which Newton himself has borne witness, when the senses are locked up in intense meditation, and no external idea can intrude.

Sweetness of disposition, moderation and simplicity of manner were his characteristics. His home was his joy, and he was fond of children. In spite of his affliction he was lively and cheerful, possessed of abundant energy; as his pupil M. Fuss has testified, 'his piety was rational and sincere; his devotion was fervent.'

In an untechnical account it is impossible to do justice to the mathematics of Euler: but while Newton is a national hero, surely Euler is a hero for mathematicians. Newton was the Archimedes and Euler was the Pythagoras. Great was the work of Euler in the problems of physics—but only because their mathematical pattern caught and retained his attention. His delight was to speculate in the realms of pure intellect, and here he reigns a prince of analysts. Not even geometry, not even the study of lines and figures, diverted him: his ultimate and constant aim was the perfection of the calculus and analysis. His ideas ran so naturally in this train, that even in Virgil's poetry he found images which suggested philosophic inquiry, leading on to new mathematical adventures. Adventures they were, which his more wary followers sometimes hailed with delight and occasionally condemned. The full splendour of the early Greek beginnings and the later works of Napier, Newton and Leibniz, was now displayed. Let one small formula be quoted as an epitome of what Euler achieved:

$$e^{i\pi} + 1 = 0.$$

Was it not Felix Klein who remarked that all analysis was centred here? Every symbol has its history—the principal whole numbers 0 and 1; the chief mathematical relations + and = ; π the discovery of Hippocrates; i the sign for the 'impossible' square root of minus one; and e the base of Napierian logarithms.

CHAPTER IX

MACLAURIN AND LAGRANGE

AMONG the contemporaries of Euler there were many excellent mathematicians in England and France, such as Cotes, Taylor, Demoivre, D'Alembert, Clairaut, Stirling, Maclaurin, and, somewhat later, Ivory, Wilson and Waring. This by no means exhaustive list contains the names of several friends of Newton—notably Cotes, Maclaurin and Demoivre. They were Newton's disciples, and each was partly responsible for making the work of the Master generally accessible. Cotes and Maclaurin were highly gifted geometers: the others of their time were interested in analysis. It was therefore a loss not only to British but to European mathematics that Cotes and Maclaurin should both have died young.

COLIN MACLAURIN (1698–1746), a Highlander from the county of Argyle, was educated at the University of Glasgow. Such was his outstanding ability that, at the age of nineteen, he was elected Professor of Mathematics in Aberdeen. Eight years later, when he acted as deputy Professor in Edinburgh, Newton wrote privately offering to pay part of the salary, as there was difficulty in raising the proper sum. Maclaurin took an active part in opposing the march of the Young Pretender in 1745 at the head of a great Highland army, which overran the country and finally seized Edinburgh. Maclaurin escaped, but the hardships of trench warfare and the subsequent flight to York proved fatal, and in 1746 he died.

Stirred by the brilliant work of Cotes, which luckily came into his hands, Maclaurin wrote a wonderful account of higher geometry. He dealt with the part which is called the *organic description of plane curves,* a subject belonging to Euclid, Pappus, Pascal and Newton. It is the mathematics of rods and bars, constrained by pivots and guiding rails—the abstract replica of valve gears and link motions familiar to the engineer—and it fascinates the geometer who 'likes to see the wheels go round'. Maclaurin carried on what Pascal had begun with, the celebrated mystic hexagram (which at that date still lay hid), and in so doing he reached a result of great generality. It provided a basis for the advances in pure geometry that were made a century later by Chasles, Salmon and Clifford.

In this kind of geometry the Cartesian method of co-ordinates fails to keep pace with the purely geometrical. In it men breathe a rarer air, akin to that in the theory of numbers.

The very success of Maclaurin partakes of the tragic. For there are huge tracts of mathematics where co-ordinates provide the natural medium —where, for any but a supreme master, analysis succeeds and pure geometry leaves one helpless. When Maclaurin wrote his essay on the equilibrium of spinning planets, which gained him the honours of the Paris Academy, he set out on a course wherein few could follow: for the problem was rendered in the purest geometry. When, in addition to this, Maclaurin produced a great geometrical work on fluxions, the scale was so heavily loaded that it diverted England from Continental habits of thought. During the remainder of the century British mathematics were relatively undistinguished, and there was no proper revival until the differential calculus began to be taught in Cambridge, according to the methods of Leibniz—a change which took place about a hundred years ago. This delay was the unhappy legacy of the Newton-Leibniz controversy, which need never have arisen.

The circumstances that prompted Maclaurin to adopt a geometrical style in his book on fluxions, extended beyond his partiality for geometry. Many philosophical influences were at work, and there were logical difficulties to face, which seemed to be insurmountable except by recourse to geometry. The difficulties were focused on the word *infinitesimal*—which Eudoxus had so carefully excluded from the vocabulary of Greek mathematics (the mere fact that it is a Latin, and not a Greek, word is not without its significance; so many of our ordinary mathematical terms have a Greek derivation). By an infinitesimal is meant something, distinguishable from zero, yet which is exceedingly small—so minute indeed that *no* multiple of it can be made into a finite size. It evades the axiom of Archimedes. Practically all analysts, from Kepler onwards, believed in the efficacy of infinitesimals, until Weierstrass taught otherwise. The differential calculus of Leibniz was founded on this belief, and its tremendous success, in the hands of the Bernoullis, Euler and Lagrange, obscured the issue. Men were disinclined to reject a doctrine which worked so brilliantly, and they turned a deaf ear to the philosophers, ancient and modern. In our own country a lively attack on infinitesimals was headed by the Irish philosopher and theologian, Bishop Berkeley. His criticism of the calculus was not lost upon Maclaurin, who was also well versed in Greek mathematics and the careful work of Eudoxus. So Maclaurin made up his mind to put Fluxions upon a sound basis and for this reason threw the work into a geometrical frame. It was his tribute to Newton, the master 'whose caution', said Maclaurin, 'was almost as distinguishing a part of his character as his invention.'

One of the chief admirers of Maclaurin was Lagrange, the great French analyst, whose own work offered a complete contrast to that of the geometer. Maclaurin had dealt in lines and figures—those characters, as Galileo has finely said, in which the great book of the Universe is written. Lagrange, on the contrary, pictured the Universe as an equally rhythmical theme of numbers and equations; and was proud to say, of his masterpiece, the *Mécanique Analytique,* that it contained not a single geometrical diagram. Nevertheless he appreciated the true geometer, declaring that the work of Maclaurin surpassed that of Archimedes himself, while as for Newton, he was 'the greatest genius the world has ever seen—and the most fortunate, for only once can it be given a man to discover the system of the Universe!'

JOSEPH-LOUIS LAGRANGE (1736–1813) came of an illustrious Parisian family which had long connection with Sardinia, and some trace of noble Italian ancestry. He spent his early years in Turin, his active middle life in Berlin, and his closing years in Paris, where he attained his greatest fame. Foolish speculation on the part of his father threw Lagrange, at an early age, upon his own resources, but this change of fortunes proved to be no great calamity, 'for otherwise', he says, 'I might never have discovered my vocation.' At school his boyish interests were Homer and Virgil, and it was not until a memoir of Halley came his way, that the mathematical spark was kindled. Like Newton, but at a still earlier age, he reached to the heart of the matter in an incredibly short space of time. At the age of sixteen he was made Professor of Mathematics in the Royal School of Artillery at Turin, where the diffident lad, possessed of no tricks of oratory and very few words, held the attention of men far older than himself. His winning personality elicited their friendship and enthusiasm. Very soon he was conducting a youthful band of scientists who became the earliest members of the Turin Academy. With a pen in his hand Lagrange was transfigured; and from the first, his writings were elegance itself. He would set to mathematics all the little themes on physical inquiries which his friends brought him, much as Schubert would set to music any stray rhyme that took his fancy.

At the age of nineteen he won fame by solving the so-called isoperimetrical problem, that had puzzled the mathematical world for half a century. He communicated his proof in a letter to Euler, who was immensely interested in the solution, particularly as it agreed with a result that he himself had found. With admirable tact and kindness Euler replied to Lagrange, deliberately withholding his own work, that all the credit might fall on his young friend. Lagrange had indeed not only solved a problem, he had also invented a new method, a new *Calculus of Variations,* which was to be the central subject of his life-work. This calculus belongs to the story of Least Action, which began with the reflecting mirrors of Hero (p. 110) and continued when Descartes pondered over his curiously shaped oval lenses. Lagrange was able to show that the somewhat varied Newtonian postulates of matter and motion fitted in with a broad principle of economy in nature. The principle has led to the still more fruitful results of Hamilton and Maxwell, and it continues today in the work of Einstein and in the latest phases of Wave Mechanics.

Lagrange was ready to appreciate the fine work of others, but he was equally able to detect a weakness. In an early memoir on the mathematics of sound, he pointed out faults even in the work of his revered Newton. Other mathematicians ungrudgingly acknowledged him first as their peer, and later as the greatest living mathematician. After several years of the utmost intellectual effort he succeeded Euler in Berlin. From time to time he was seriously ill from overwork. In Germany King Frederick, who had always admired him, soon grew to like his unassuming manner, and would lecture him for his intemperance in study which threatened to unhinge his mind. The remonstrances must have had some effect, because Lagrange changed his habits and made a programme every night of what was to be read the next day, never exceeding the ration. For twenty years he continued to reside in Prussia, producing work of high distinction that culminated in his *Mécanique Analytique*. This he decided to publish in France, whither it was safely conveyed by one of his friends.

The publication of this masterpiece aroused great interest, which was considerably augmented in 1787 by the arrival in Paris of the celebrated author himself, who had left Germany after the death of King Frederick, as he no longer found a sympathetic atmosphere in the Prussian Court. Mathematicians thronged to meet him and to show him every honour, but they were dismayed to find him distracted, melancholy, and indifferent to his surroundings. Worse still—his taste for mathematics had gone! The years of activity had told; and Lagrange was mathematically worn out. Not once for two whole years did he open his *Mécanique Analytique*: instead, he directed his thoughts elsewhere, to metaphysics, history, religion, philology, medicine, botany, and chemistry. As Serret has said, 'That thoughtful head could only change the objects of its meditations.' Whatever subject he chose to handle, his friends were impressed with the originality of his remarks. His saying that chemistry was 'easy as algebra' vastly astonished them. In those days the first principles of atomic chemistry were keenly canvassed: but it seemed odd to draw a comparison between such palpable things as chemicals, that can be handled and seen, and such abstractions as algebraic symbols.

In this philosophical and unmathematical state of mind Lagrange continued for two years, when suddenly the country was plunged into the Revolution. Many avoided the ordeal by flight abroad, but Lagrange refused to leave. He remained in Paris, wondering as he saw his friends done to death if *his* turn was coming, and surprised at his good fortune in surviving. France has reason to be glad that he was not cut down as was his friend Lavoisier, the great chemist; for in later years mathematical skill once again returned to him, and he produced many gems of algebra and analysis.

One mathematical effect of the Revolution was the adoption of the metric system, in which the subdivision of money, weights and measures is strictly based on the number ten. When someone objected to this number, naturally preferring twelve, because it has more factors, Lagrange unexpectedly remarked, what a pity it was that the number eleven had not been chosen as base, because it was prime. The M.C.C. appears to be one of the few official bodies who have followed this hint, by thinking systematically in terms of such a unit!

For music he had a liking. He said it isolated him and helped him to think, as it interrupted general conversation. 'For three bars I listen to it; thereafter I distinguish nothing, but give myself up to my thoughts. In this way I have solved many a difficult problem.' He was twice married: first when he lived in Berlin, where he lost his wife after a long illness, in which he nursed her devotedly. Then again in Paris he married Mlle. Lemonnier, daughter of a celebrated astronomer. Happy in his home life, simple and almost austere in his tastes, he spent his quiet fruitful years, till he died in 1813 at the age of seventy-six.

Lagrange is one of the great mathematicians of all time, not only for the abundance and originality of his work but for the beauty and propriety of his writings. They possess the grandeur and ease of the ancient geometers, and Hamilton has described the *Mécanique Analytique* as 'a scientific poem'. He was equally at home rivalling Fermat in the theory of numbers and Newton in analytical mechanics. Much of the contemporary and later work of Laplace, Legendre, Monge, Fourier and Cauchy, was the outcome of his inspiration. Lagrange sketched the broad design; it was left to others to fill in the finished picture. One must turn to the historians of mathematics to learn how fully and completely this was done. The breadth of the canvas attracted men of widely different interests. Nothing could afford a greater contrast to the mind of Lagrange than that of Laplace, the other great contributor to natural philosophy, whose most notable work was the *Mécanique Céleste*. To Laplace mathematics were the accidents and natural phenomena the substance—a point of view exactly opposite to that of Lagrange. To Laplace mathematics were tools, and they were handled with extraordinary skill, but any makeshift of a proof would do, provided that the problem was solved. It remained for the nineteenth century to show the faultiness of this naïve attitude. The instinct of the Greeks was yet to be justified.

CHAPTER X

GAUSS AND HAMILTON: THE NINETEENTH CENTURY

THE nineteenth century, which links the work of Lagrange with that of our own day, is perhaps the most brilliant era in the long history of mathematics. The subject assumed a grandeur in which all that was great in Greek mathematics was fully recovered; geometry once again came into its own, analysis further broadened its scope, and the outlets for its applications were ever enlarging. The century was marked in three noteworthy ways: there was deeper insight into the familiar properties of number; there was positive discovery of new processes of calculation, which, in the quaint words of Sylvester, ushered in 'the reign of Algebra the Second'; and there was also a philosophy of mathematics. During these years England once again rivalled mathematical France, and Germany and Italy rose to positions of scientific importance; while pre-eminent over all was the genius of one man, a mathematician worthy of a place of honour in the supreme rank with Archimedes and Newton.

CARL FRIEDRICH GAUSS was born in 1777 at Brunswick, and died in 1855, aged seventy-eight. He was the son of a bricklayer, and it was the wish of his father that he should be a bricklayer too. But at a very early age it was clear that the boy had unusual talents. Unlike Newton and Lagrange he showed the precocity of Pascal and Mozart. It is said that Mozart wrote a minuet at the age of four, while Gauss pointed out to his father an error in an account when he was three. At school his cleverness attracted attention, and eventually be came known to the Duke of Brunswick himself, who took an interest in the lad. In spite of parental protest the Duke sent him for a few years to the Collegium Carolinum and in 1795 to Göttingen. Still undecided whether to pursue mathematics or philology, Gauss now came under the influence of Kaestner—'that first of geometers among poets, and first of poets among geometers', as the pupil was proud to remark. In the course of his college career Gauss became known for his marvellous intuition in higher arithmetic. 'Mathematics, the Queen of the Sciences, and Arithmetic, the Queen of Mathematics', he would say: and mathematics became the main study of his life.

The next nine years were spent at Brunswick, varied by occasional travels, in the course of which he first met his friend Pfaff, who alone in Germany was a mathematician approximating to his calibre. After declining the offer of a Chair at the Academy in St. Petersburg, Gauss was appointed in 1807 to be first director of the new observatory at Göttingen, and there he lived a studious and simple life, happy in his surroundings, and blessed with good health, until shortly before his death. Once, in

1828, he visited Berlin, and once, in 1854, he made a pilgrimage to be present at the opening of the railway from Hanover to Göttingen. He saw his first railway engine in 1836, but except for these quiet adventures, it is said that until the last year of his life he never slept under any other roof than that of his own observatory!

His simple and direct character made a profound impression upon his pupils, who, seated round a table and not allowed to take notes, would listen with delight to the animated address of the master. Vivid accounts have been handed down of the chief figure in the group as he stood before his pupils 'with clear bright eyes, the right eyebrow raised higher than the left (for was he not an astronomer?), with a forehead high and wide, overhung with grey locks, and a countenance whose variations were expressive of the great mind within.'

Like Euler, Lagrange and Laplace, Gauss wrote voluminously, but with a difference. Euler never condensed his work; he revelled in the richness of his ideas. Lagrange had the easy style of a poet; that of Laplace was jerky and difficult to read. Gauss governed his writings with austerity, cutting away all but the essential results, after taking endless trouble to fill in the details. His pages stimulate but they demand great patience of the reader.

Gauss made an early reputation by his work in the theory of numbers. This was but one of his many mathematical activities, and, apart from all that followed, it would have placed him in the front rank. Like Fermat, he manifested that baffling genius which leaps—one knows not how—to the true conclusion, leaving the long-drawn-out deductive proof for others to formulate. A typical example is provided by the *Prime Number Theorem* which has taken a century to prove. Prime numbers were studied by Euclid, and continue to be an eternal source of interest to mathematicians. They are the numbers, such as 2, 3, 5, 7, 11, that cannot be broken up into factors. They are infinitely numerous, as Euclid himself was aware, and they occur, scattered through the orderly scale of numbers, with an irregularity that at once teases and captivates the mathematician. The question is naturally suggested: *How often, or how rarely, do prime numbers occur on the average?* Or, put in another way, What is the chance that a specified number is prime? In some form or other this problem was known to Gauss; and here is his innocent-looking answer:

$$\text{"Primzahlen unter } a \ (=\infty) \quad \frac{a}{la}.\text{"}$$

It means that when a is a very large number, the result of dividing a by its logarithm gives a good approximation to the total number of primes less than a: and the larger a is, the more precise is the result. Whether

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