# Think like a MATHEMATICIAN

Get to grips with the language of numbers and patterns

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# Chapter 26: How much have you drunk?

INTRODUCTION

# What is mathematics, really?

Mathematics is all around us. It is the language that lets us work with numbers, patterns, processes and the rules that govern the universe. It provides a way for us to understand our surroundings, and both model and predict phenomena. The earliest human societies began to investigate mathematics as they tried to track the movements of the Sun, Moon and planets, and to construct buildings, count flocks and develop trade. From Ancient China, Mesopotamia, Ancient Egypt, Greece and India, mathematical thought flowered as people discovered the beauty and wonder of the patterns that numbers make.

Mathematics is a global enterprise and an international language. Today, it underlies all areas of life.

Trade and commerce are built on numbers. The computers that are integral to all aspects of society run on numbers. Much of the information we are presented with on a daily basis is mathematical. Without a basic understanding of numbers and mathematics, it's impossible to tell the time, plan a schedule or even follow a recipe. But that's not all. If you don't understand mathematical information, you can be deceived and misled – or you might simply miss out.

Mathematics can be commandeered for both honourable and nefarious purposes. Numbers can be used to illuminate, explain and clarify – but also to lie, obfuscate and confuse. It's good to be able to see what's going on.

Computers have made mathematics a lot easier by making possible some calculations that could never have been achieved before. You will meet examples of this later in the book. For example, pi (symbol π, which defines the mathematical relationship between the circumference of a circle and its radius) can now be calculated to millions of places using computers. Prime numbers (which are only divisible by one and themselves) are now listed in their millions, again thanks to computers. But in some ways computers could be making mathematics less logically rigorous.

**PURE AND APPLIED MATHEMATICS**

Most of the mathematics in this book falls under the heading of 'applied mathematics' – it's mathematics that is being used to solve real-world problems, applied to practical situations in the world, such as how much interest is charged on a loan, or how to measure time or a piece of string. There is another type of mathematics which preoccupies many professional mathematicians, and that is 'pure' mathematics. It is pursued regardless of whether it will ever have a practical application, to explore where logic can take us and to understand mathematics for its own sake.

Now that it's possible to process very large amounts of data, far more reliable information can be extracted from empirical data (that is, data that can be directly observed) than ever before. This means that more of our conclusions can be – apparently safely – based on looking at stuff rather than working stuff out. For instance, we could examine lots and lots of data about weather and then make predictions based on what has happened in the past. We would not need any understanding of weather systems to do this, it would just work from what has been observed before on the assumption that – whatever forces lie behind it – the same will happen in the future with a certain degree of probability. It might well work, but that's not really science or mathematics.

## Look first or think first?

There are two fundamentally different ways of working with data and knowledge, and so of coming up with mathematical ideas. One starts from thinking and logic, and the other starts from observations.

**Think first:** Deduction is the process of reasoning through logic using specific statements to produce predictions about individual cases. An example would be starting with the statement that all children have (or once had) parents, and the fact that Sophie is a child, to deduce that Sophie must therefore have (or have once had) parents. As long as the two original statements are verified and the logic is sound, the prediction will be accurate.

**Look first:** Induction is the process of inferring general information from specific instances. If we looked at a lot of swans and found they were all white we might infer from this (as people once did) that all swans must be white. But this is not robust – it just means we haven't yet seen a swan that is not white (see Chapter 10).

## Being right and being wrong

Mathematicians are not always right, whether they begin with inductive or deductive methods. On the whole, though, deduction is more reliable and has been enshrined in pure mathematics since its origins with the Greek mathematician Euclid of Alexandria.

## How it can go wrong

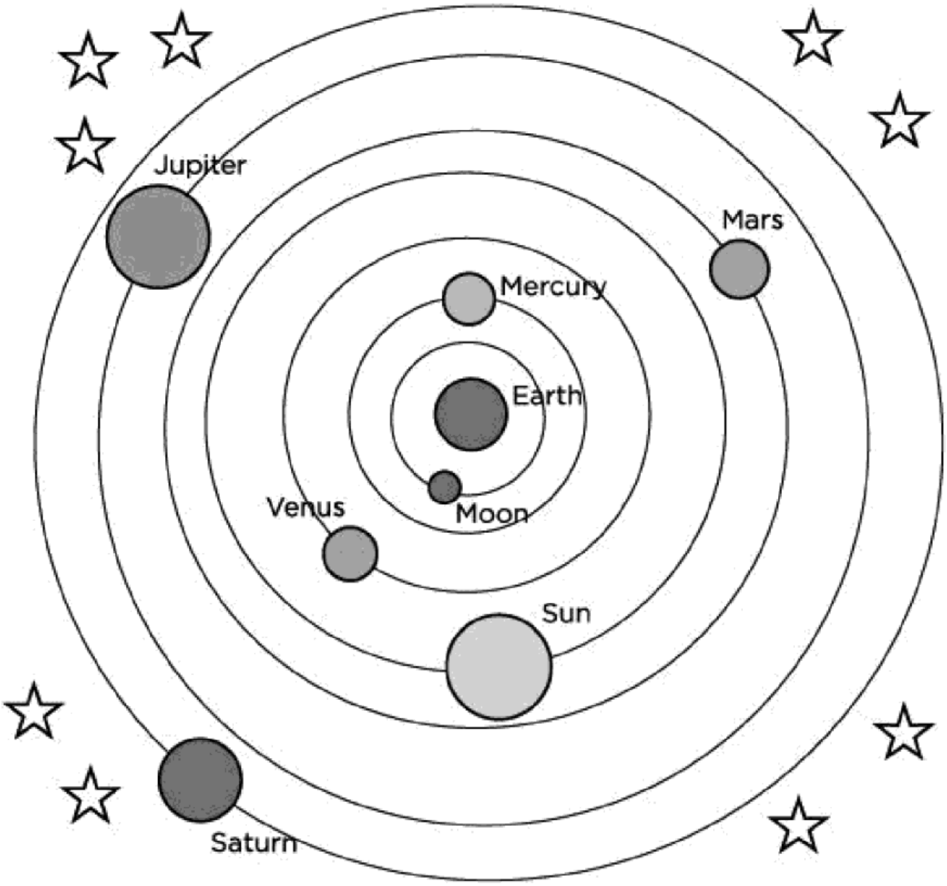
Our ancestors thought the Sun orbited the Earth, rather than the other way round. How would the movement of the Sun appear if it did go around the Earth? The answer is: exactly the same.

The model of the universe constructed by the Ancient Greek astronomer Claudius Ptolemy (*c.*AD90–168) accounted for the apparent movements of the Sun, Moon and planets across the sky. This was an inductive method: Ptolemy looked at the empirical evidence (what he observed for himself) and constructed a model to fit it.

**WHERE'S A PLANET? THERE'S A PLANET!**

In 1845–6, the mathematicians Urbain Le Verrier and John Couch Adams independently predicted the existence and position of Neptune. They used mathematics, after looking at perturbations (disturbances) in the orbit of the neighbouring planet Uranus. Neptune was discovered and identified in 1846.

As it became possible to make more accurate measurements of the movements of the planets, medieval and Renaissance astronomers devised ever more complex refinements to the mathematics of Ptolemy's Earth-centred model of the universe to make it fit their observations. The whole system became a horrible tangle as bits were added incrementally to explain every new observation.




## Putting it right

It was only when the model was overthrown in 1543 by the Polish astronomer and mathematician Nicolaus Copernicus, who put the Sun at the centre of the solar system, that the mathematics started to work. But even his calculations were not totally accurate. Later, the English scientist Isaac Newton (1642–1726) improved on Copernicus's ideas to give a mathematically coherent account of the movements of the planets which doesn't need lots of fudging to make it work. His laws of planetary motion have been validated by the observation of planets not discovered when he was alive. They have accurately predicted the existence of planets even before they were observed. But the model is not yet perfect; we still can't quite account for the motion of the outer planets, using our current mathematical model. There is more to be discovered, both in space and in mathematics.

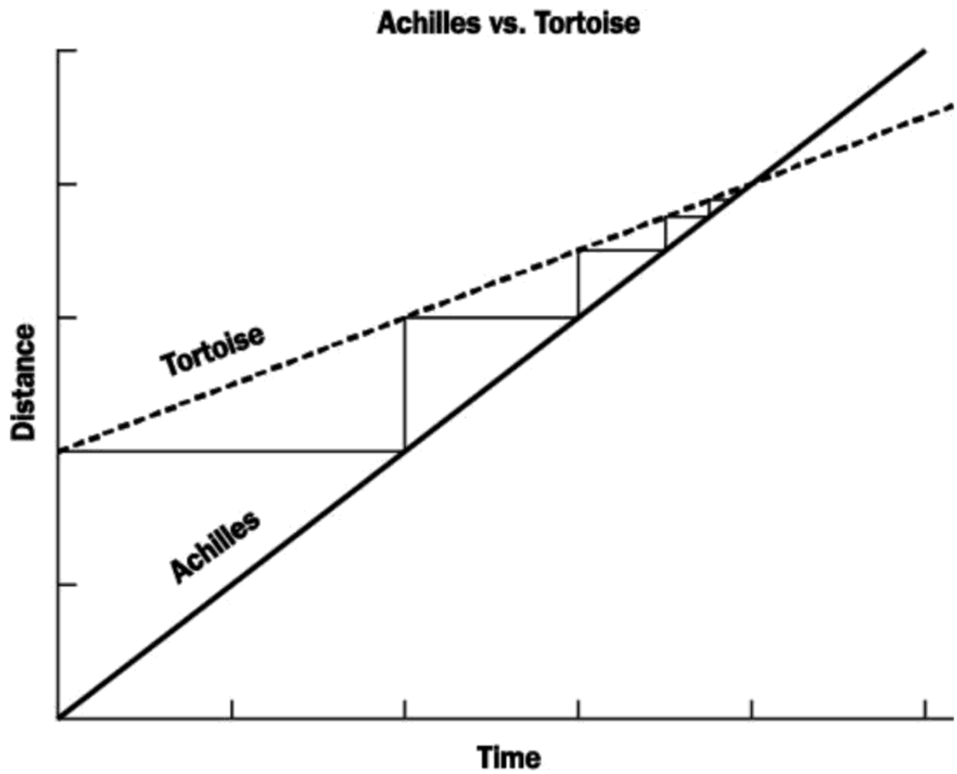
## Zeno's paradoxes

The mismatch between the world we experience and the world modelled by mathematics and logic has long been recognized.

The Greek philosopher Zeno of Elea (*c*.490–430BC) used logic to demonstrate the impossibility of motion. His 'paradox of the arrow' states that at any instant of time, an arrow is in a fixed position. We can take millions of snapshots of the arrow in all its positions between leaving the bow and reaching its target, and in any infinitely short instant of time it is motionless. So when does it move?

Another example is the paradox of Achilles and the tortoise. If the speedy Greek hero Achilles gave a tortoise a head start in a race, he would never be able to catch up with it. In the time it took Achilles to cover the distance to the tortoise's original position, the tortoise would have moved on. This would keep happening, with the tortoise covering ever-shorter distances as Achilles approached, but Achilles would never manage to overtake it.

This paradox works by treating the continuity of time and distance as a string of infinitesimal moments or positions. Logically coherent, it doesn't match reality as we experience it.



CHAPTER 1

# You couldn't make it up – or did we?

Is mathematics just 'out there', waiting to be discovered? Or have we made it up entirely?

Whether mathematics is discovered or invented has been debated since the time of the Greek philosopher Pythagoras, in the 5th century BC.

## Two positions – if you believe in 'two'

The first position states that all the laws of mathematics, all the equations we use to describe and predict phenomena, exist independently of human intellect. This means that a triangle is an independent entity and its angles actually do add up to 180°. Mathematics would exist even if humans had never come along, and will continue to exist long after we have gone. The Italian mathematician and astronomer Galileo shared this view, that mathematics is 'true'.

> *'Mathematics is the language in which God has written the universe.'*
>
> Galileo Galilei

## It's there, but we can't quite see it

The Ancient Greek philosopher and mathematician Plato proposed in the early 4th century BC that everything we experience through our senses is an imperfect copy of a theoretical ideal. This means every dog, every tree, every act of charity, is a slightly shabby or limited version of the ideal, 'essential' dog, tree or act of charity. As humans, we can't see the ideals – which Plato called 'forms' – but only the examples that we encounter in everyday 'reality'. The world around us is ever-changing and flawed, but the realm of forms is perfect and unchanging. Mathematics, according to Plato, inhabits the realm of forms.

Although we can't see the world of forms directly, we can approach it through reason. Plato likened the reality we experience to the shadows cast on the wall of a cave by figures passing in front of a fire.

If you are in the cave, facing the wall (chained up so that you can't turn around, in Plato's scenario) the shadows are all you know, so you consider them to be reality. But in fact reality is represented by the figures near the fire and the shadows are a poor substitute. Plato considered mathematics to be part of eternal truth. Mathematical rules are 'out there' and can be discovered through reason. They regulate the universe, and our understanding of the universe relies on discovering them.

## What if we made it up?

The other main position is that mathematics is the manifestation of our own attempts to understand and describe the world we see around us. In this view, the convention that the angles of a triangle add up to 180° is just that – a convention, like black shoes being considered more formal than mauve shoes. It is a convention because we defined the triangle, we defined the degree (and the idea of the degree), and we probably made up '180', too.

At least if mathematics is made up, there's less potential to be wrong. Just as we can't say that 'tree' is the wrong word for a tree, we couldn't say that made-up mathematics is wrong – though bad mathematics might not be up to the job.

> *'God created the integers. All the rest is the work of Man.'*
>
> Leopold Kronecker (1823–91)

## Alien mathematics

Are we the only intelligent beings in the universe? Let's assume not, at least for a moment (see Chapter 18).

If mathematics is discovered, any aliens of a mathematical bent will discover the same mathematics that we use, which will make communication with them feasible. They might express it differently – using a different number base, for example (see Chapter 4) – but their mathematical system will describe the same rules as ours.

If we make up mathematics, there is no reason at all why any alien intelligence should come up with the same mathematics. Indeed, it would be rather a surprise if they did – perhaps as much of a surprise as if they turned out to speak Chinese, or Akkadian, or killer whale.

For if mathematics is simply a code we use to help us describe and work with the reality we observe, it is similar to language. There is nothing that makes the word 'tree' a true signifier for the object that is a tree. Aliens will have a different word for 'tree' when they see one. If there is nothing 'true' about the elliptical orbit of a planet, or about the mathematics of rocket science, an alien intelligence will probably have seen and described phenomena in very different terms.

## How amazing!

Perhaps it is amazing that mathematics is such a good fit for the world around us – or perhaps it is inevitable. The 'it's amazing' argument doesn't really support either view. If we invented mathematics, we would create something that adequately describes the world around us. If we discovered mathematics, it would obviously be appropriate to the world around us as it would be 'right' in a way that is larger than us. Mathematics is ‘so admirably appropriate to the objects of reality’ either because it’s true or because that’s what it was designed for.

> *How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?’*
>
> Albert Einstein (1879–1955)

## Look out – it’s behind you!

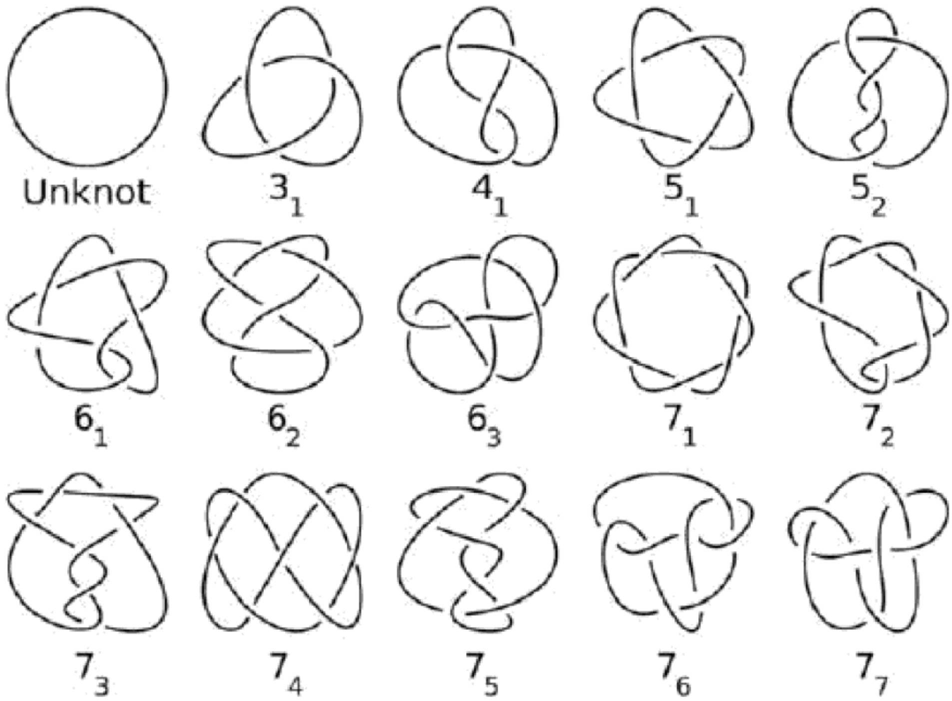
Another possibility is that mathematics seems astonishingly good at representing the real world because we only look at the bits that work. It’s rather like seeing coincidences as evidence of something supernatural going on. Yes, it’s really amazing that you went abroad on holiday to an obscure village in Indonesia and bumped into a friend – but only because you are not thinking about all the times you and other people have gone somewhere and not bumped into anyone you knew. We only remark on the remarkable; unremarkable events go unnoticed. In the same way, no one thinks to fault mathematics because it can’t describe the structure of dreams. So it would be reasonable to collate a list of areas where mathematics fails if we want to assess its level of success.

## ‘The unreasonable effectiveness of mathematics’

If mathematics is made up, how can we explain the fact that some mathematics, developed without reference to real-world applications, has been found to account for real phenomena often decades or centuries after its formulation?

As the Hungarian-American mathematician Eugene Wigner pointed out in 1960, there are many examples of mathematics developed for one purpose – or for no purpose – that have later been found to describe features of the natural world with great accuracy. One example is knot theory. Mathematical knot theory involves the study of complex knot shapes in which the two ends are connected. It was developed in the 1770s, yet is now used to explain how the strands of DNA (the material of inheritance) unzip themselves to duplicate. There are still counter-arguments. We only see what we look for. We choose the things to explain, and choose those that can be explained with the tools we have.

Perhaps evolution has primed us to think mathematically and we can’t help doing so.



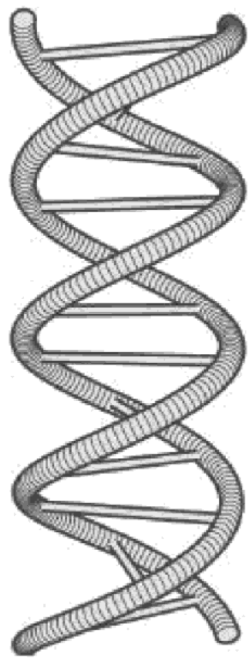

*Knot theory: the simplest possible true knot is the trefoil or overhand knot, in which the string crosses three times ($3_1$ below). There are no knots with fewer crossings. The number of knots increases rapidly thereafter.*

> *'How do we know that, if we made a theory which focuses its attention on phenomena we disregard and disregards some of the phenomena now commanding our attention, that we could not build another theory which has little in common with the present one but which, nevertheless, explains just as many phenomena as the present theory?'*
>
> Reinhard Werner (b.1954)

## Does it matter?

If you are just working out your household accounts or checking a restaurant bill, it doesn't much matter whether mathematics is discovered or invented. We operate within a consistent mathematical system – and it works. So we can, in effect, 'keep calm and carry on calculating'.

For pure mathematicians, the question is of philosophical rather than practical interest: are they dealing with the greatest mysteries that define the fabric of the universe? Or are they playing a game with a kind of language, trying to write the most elegant and eloquent poems that might describe the universe?



> *'The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.'*
>
> Eugene Wigner

Where the 'reality' of mathematics matters most is where humans are pushing against the boundaries of knowledge and of technical achievement. If mathematics is made up, we might come up against the limitations of our system and not be able to push through them to answer certain questions. We might never achieve time travel, zip to the other side of the universe, or create artificial consciousness, simply because our mathematics is not up to the task. We will deem impossible things which, with a different system of mathematics, might be perfectly easy.

On the other hand, if mathematics is discovered we can, potentially, uncover all of it and achieve right to the edges of what is possible, of what is allowed by the physical laws of the universe. It would be nice, then, if mathematics were discovered. But we can't be certain.

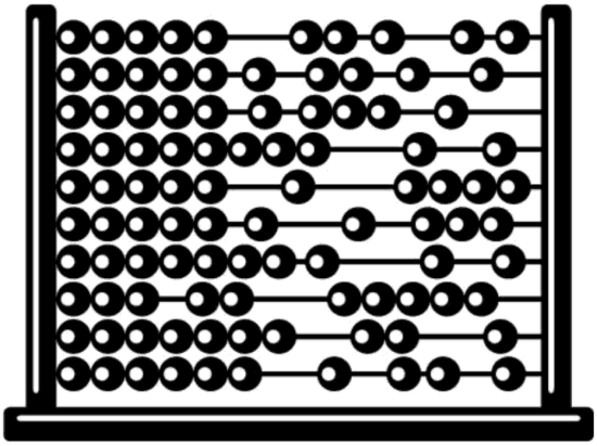
**A DREADFUL POSSIBILITY**

One possibility that doesn't usually get much consideration is that mathematics is real, but we've got it all wrong, just as Ptolemy got the model of the solar system wrong. What if the mathematics we have developed is the equivalent of the Ptolemaic Earth-centred universe? Could we throw it away and start again? It’s hard to see how that would be possible now we have invested so much in it.

## CHAPTER 2

# Why do we have numbers at all?

Getting to grips with numbers came early in the development of human society.



We are so used to numbers that we rarely give them a second thought. Children learn to count at a very early age, with numbers and colours being among the first abstract ideas they encounter.

## Tally ho!

The first human engagement with numbers that we know of was in the form of tallying. Our distant ancestors kept tallies of their flocks by marking a stick, stone or bone, with one cut for each animal, or by moving pebbles or shells from one pile to another.

Tallying doesn't need words for the numbers – it's not the same as counting. It's a simple system of correspondence, using one object or mark to represent another object or phenomenon. If you have a shell that represents each sheep, and you drop a shell into the pot as each sheep passes, it's easy to see if you have shells left over, and so sheep missing, at the end. You don't need to know whether you should have 58 sheep or 79 sheep – you just keep looking for the missing sheep and dropping a shell in the pot each time one is found until there are no shells left over.

We still use tallies to keep score in games, to keep a record of days shipwrecked, and in other circumstances in which a number is only needed at the end of a process.

Counting comes after tallying.

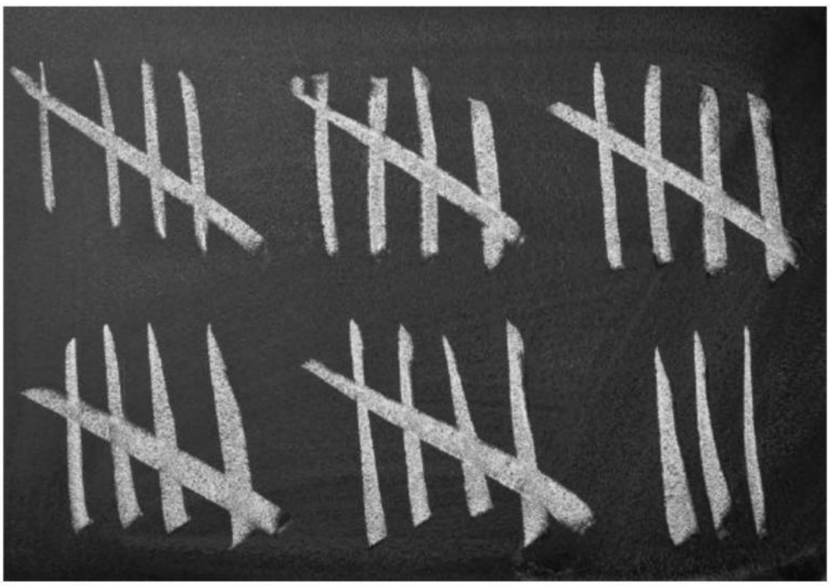
## Counting 1, tallying 0

Tallying was used by various Stone Age cultures for at least 40,000 years. Then at some point it became useful to have numbers with names.

We don't know quite when counting began, but it's easy to see that once people started keeping animals it would be more useful to be able to say 'three sheep are missing' than just 'some sheep are missing'. If you have three children and want a spear for each, it's easier to know you have to make three spears, then set out to find three strong sticks, and so on, than to make one spear, give it to the first child, realize there are still spearless children, make another spear, and so on. Once people started to trade, numbers would have been essential.

The first known written numbers emerged in the Middle East in the Zagros region of Iran around 10,000BC. Clay tokens used in counting sheep have survived. The token for a single sheep was a ball of clay with a + sign scratched into it. Clearly that's great if you have a few sheep, but needing 100 tokens for 100 sheep would be cumbersome. They developed tokens with different symbols to represent 10 sheep and 100 sheep, and could then account for any number of sheep with far fewer tokens – even 999 sheep could be represented with only 27 tokens (9 × 100-sheep tokens; 9 × 10-sheep tokens; 9 × single-sheep tokens).

The tokens could be strung on a cord, or were often baked into a hollow clay ball. The outside of the ball was impressed with symbols showing the number of 'sheep' inside, but it could be broken to verify the number if there was a dispute. These numbers on the outside of sheep-counting balls are the oldest surviving written number system.



## Making up numbers

Many early number systems developed directly from tallies and so used a symbol repeated for units, a different symbol for tens, and another for hundreds. Some had symbols for 5, or other intermediate numbers.

The system of Roman numerals, familiar from clockfaces and the copyright date shown at the end of a movie, began with the vertical strokes of a tallying system. The numbers 1–4 were originally represented as I, II, III, IIII. X is used for 10 and C for 100. The intermediates V (5), L (50) and D (500) make large numbers a bit shorter to write. After a while, a convention emerged of putting a I before a V or X to denote subtraction, so IV is 5 – 1, or 4. IV is shorter to write and easier to read than IIII. You can only do it within the same power of ten, so IX is 9 but you can't write IC for 99 – it has to be XCIX (or 100 – 10 and 10 – 1).

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| I | II | III | IIII later IV | V | VI | VII | VIII | VIIII later IX | X |
| **11** | **19** | **20** | **40** | **50** | **88** | **99** | **100** | **149** | **150** |
| XI | XIX | XX | XL | L | LXXXVIII | XCIX | C | CXLIX | CL |

## Limited by numbers

Using repeated symbols to stand for extra units, tens and hundreds, makes numbers cumbersome to write and makes arithmetic difficult. With a system, like the Roman one, of preceding a symbol with one to be subtracted, addition can't even be achieved by just counting up the total number of each type of symbol: XCIV + XXIX (94 + 29) would give the same answer as CXVI + XXXI (116 + 31) if we just counted Cs, Xs, Vs and Is. Although the Romans managed, the system has clear limitations: their mathematics was too inflexible. Fractions were all based on division by 12, there were no decimal fractions – and can you imagine trying to deal with complex concepts such as powers (see box on page 28) or quadratic equations using Roman numerals and with no figure for 0?

**EGYPTIAN FRACTIONS**

The Ancient Egyptian writing system used hieroglyphs (picture symbols). Like the Roman system, the Egyptians used accumulating symbols. They had a form of fraction, too.

To show a fraction, the Egyptian scribe drew the 'mouth' glyph above a number of down strokes. There was a problem, though. This method only provided unit fractions (1 over a number), and repeating a unit fraction wasn't allowed. This meant you could represent ¾ (= ½ ¼), but not fractions such as $^{7}/_{10}$

The exception was ⅔, represented by a mouth glyph over two strokes of different sizes.


$IV^{III}$ = LXIV

$XIIx^{II}$ + IVx – IX = I – I

Not surprisingly, Roman mathematics didn't develop very far.

## Place value

The Indo–Arabic numeral system we use today has only nine figures, which can be reused *ad infinitum.* It developed slowly in India from the 3rd century BC and was later refined by Arabian mathematicians before being adopted in Europe. In this system, the status of a number is indicated by its position, called *place value.* Place value increases moving towards the left. This is a much more flexible system than the Roman one.

**POWERS**

A squared number is a number multiplied by itself. For example, three squared is: 3 × 3.

We can also write it as 32.

This is read as 'three to the power two', meaning we multiply two threes together.

A cubed number is a number multiplied by itself again, so three cubed is: 3 × 3 × 3 and can also be written 33, 'three to the power three'. The superscript number (the small, raised number) is called the power or exponent.

Squared and cubed numbers have obvious applications as they relate to objects in two and three dimensions. Higher powers are used in mathematics, but unless you are a theoretical physicist you probably don't think of extra dimensions in the real world.

| Thousands | Hundreds | Tens | Units |
|---|---|---|---|
| 5 | 6 | 9 | 1 |

We can make a number such as 5,691 by combining:

**5,000 (5 × 1,000)**
**600 (6 × 100)**
**90 (9 × 10)**
**1 (1 × 1)**

Using place value, it's possible to represent even very large numbers with a small number of figures. Compare the Roman and Arabic representations:

**88 = LXXXVIII**
**797 = DCCXCVII**
**3,839 = MMMDCCCXXXIX**

## Nothing there – the start of zero

Place value is all very well as long as there is a digit in each place. If there are gaps – nothing in the 10s column (308, for instance) – how can we show this? Leaving a space, as the Chinese did, can be ambiguous unless the numbers line up carefully in columns: 9 2 could be 902 or possibly 9002, and there's a big difference between the two.

*'From place to place each is ten times the preceding.'*

The first description of place value in the Indo–Arabic method of counting, Aryabhata, Indian mathematician (AD476–550)

A space indicated an empty column in Indian numbers, too, but was later replaced by a dot or small circle. This was given the Sanskrit name *sunya,* meaning empty. When the Arabs adopted the Indian numerals, around AD800, they also took the empty place-marker, still calling it empty, which was *sifr* in Arabic and is the origin of the modern word ‘zero’.

*The earliest surviving use of a symbol for zero in decimal figures is a Cambodian inscription on stone dating from 683. The large dot stands for 0 between the figures for 6 and 5, denoting 605.*

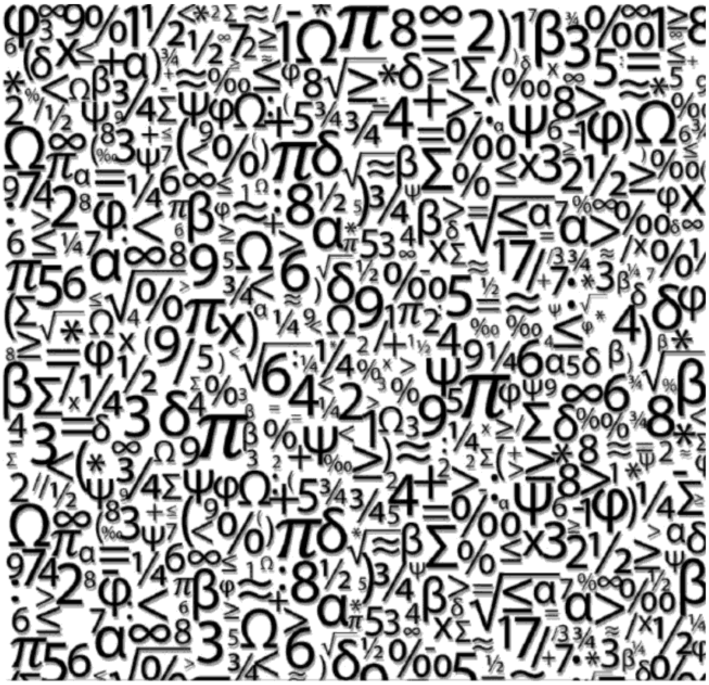
> *‘The nine Indian figures are: 9 8 7 6 5 4 3 2 1. With these nine figures, and with the sign 0… any number may be written.’*
>
> Fibonacci, *Liber Abaci* (1202)

Indo–Arabic numerals first appeared in Europe around AD1000, but it was several centuries before they were universally adopted. The Italian mathematician Leonardo Bonacci, better known today as ‘Fibonacci’, promoted their use as early as the 1200s, yet merchants continued to use Roman numerals until the 16th century.

CHAPTER 3

# How far can you go?

Not all number systems are infinitely extendable.



Our number system is unlimited – it can go up to any number you care to imagine, just by putting down more and more digits. That has not always been the case.

## Not enough numbers?

The simplest counting systems are called 2-count. They don't provide a way of doing calculations, but allow counting of small quantities. A 2-count system has words for 1, 2 and sometimes 'many' (meaning an uncountably large number). The 2-count system used by bushmen in South Africa builds in a series of 2s and 1s. Its usefulness is limited by how many 2s people can keep track of.

1 xa

2 t'oa

3 'quo

4 t'oa-t'oa

5 t'oa-t'oa-ta

Supyire, a language spoken in Mali, has basic number-words for i, 5, 10, 20, 80 and 400. The rest of the numbers are built up from these. For example, 600 is *kdmpwdd nd kwuu shuuni nd beeshuunni,* which means 400 + (80 × 2) + (20 × 2).

The Toba in Paraguay use a system which has words for numbers up to 4, and then starts reusing words extravagantly:

| | |
|---|---|
| 1 | nathedac |
| 2 | cacayni or nivoca |
| 3 | cacaynilia |
| 4 | nalotapegat |
| 5 = 2 + 3 | nivoca cacaynilia |
| 6 = 2 × 3 | cacayni cacaynilia |
| 7 = 1 + 2 × 3 | nathedac cacayni cacaynilia |
| 8 = 2 × 4 | nivoca nalotapegat |
| 9 = 2 × 4 + 1 | nivoca nalotapegat nathedac |
| 10 = 2 + 2 × 4 | cacayni nivoca nalotapegat |

This sort of system is fine for counting your children or other things that come in relatively small quantities, but it has clear limitations.

## A small infinity

Infinity is often considered to be an uncountably large number (see Chapters 7 and 8). For the Toba and the South African bushmen using 2-count, that might well be a number below 100. In a society not concerned with abstract mathematics, there is no need to raise the bar for infinity much further than the size of a family or herd of animals.

## Less than zero

In early run-of-the-mill counting, there was no need for negative numbers. Indeed, the Ancient Greeks were highly distrustful of them, and the mathematician Diophantus, in the 3rd century AD, said that an equation such as $4x + 20 = 0$ (which is solved with a negative value for x) is absurd.

**A TAXONOMY OF NUMBERS**

Mathematicians now recognize several categories of numbers.

• *Natural* numbers are those you first learn about, the numbers we count with: 1, 2, 3, and so on.

• *Whole* numbers are the natural numbers with zero chucked in: 0, 1, 2, 3, and so on. (This might seem a bit odd, as how whole is zero? It's a lack of a number, a hole rather than a whole. Never mind, that's mathematicians for you.)

• *Integers* are whole numbers and the numbers below zero, the negative numbers: … -3, -2, -1, 0, 1, 2, 3…

• *Rational* or *fractional* numbers are numbers that can be written as fractions, such as ½, ⅓, and so on. They include the integers as they can be written as fractions: 1/1, 2/1, etc. They include all the fractions between whole numbers, as they can be written as fractions, too: 1 ½ can be written as 3/2, and so on. All rational numbers can be written as either terminating or repeating decimals. So ½ is 0.5 and ⅓ is 0.33333…

• *Irrational* numbers are those which can't be written as terminating or repeating decimals or expressed as a ratio between two whole numbers. They are decimals that go on and on in a non-repeating sequence. Examples are $\pi$, $\sqrt{2}$, and e, which can be calculated by computer to trillions of places without revealing a repeating pattern.

• *Real* numbers: all of the above.

• *Imaginary* numbers: numbers that include *i*, defined as the square root of -1. (We won't worry about that one.)

Certainly the early, tallying farmer who noticed that three sheep were missing did not need to say he or she had -3 sheep; it was good enough to say they were three short of a full flock. With commerce, though, came a need to show a debt. If you borrowed

100 coins, your account stood at -100; if you paid back 50 of them, your account stood at -50. Negative numbers were used for this purpose in India from the 7th century AD.

The first known appearance of negative numbers is even earlier. The Chinese mathematician Liu Hui established rules for arithmetic using negative numbers in the 3rd century. He used counting rods in two colours, one for gains and one for losses, which he called positive and negative. He used red counting rods for positive numbers and black for negative numbers – the opposite of the modern accounting convention.

## Counting and measuring

While many things can be counted, not all can be counted easily and some can't be counted at all. In nature, there are perhaps more things that can't easily be counted than can be.

We can count people, animals, plants and small numbers of stones or seeds. But although in theory we could count the grains of wheat in a harvest or the number of trees in a forest or ants in an anthill, it's unlikely that we would. These are things we are likely to measure instead. Humans began measuring grain by weight or volume long ago. Some things can only be measured in this way: we measure the volume of liquids, the weight (or mass) of rocks and the area of land (see Chapter 15).

Further still from counting are the arbitrary scales for measurements such as temperature. Scales provide another use for negative numbers. Unless a scale starts at some form of absolute zero, a negative number can be useful. Thermometers most certainly need negative numbers, if working in Celsius or even Fahrenheit. Negative numbers are needed with vectors (a quantity that also includes direction), as we express one direction as positive and the opposite as negative. If we turn clockwise through 45°, that is a positive rotation, but if we then turn back 30°, that's a rotation of -30°. Ions (electrically charged particles) can have a positive or negative charge, and which charge they have indicates how they will react with other substances. You might come across negative numbers on a daily basis in circumstances such as:

- Floor -1 in a lift – a floor below ground level, which is considered to be 0
- A soccer club with a negative goal difference – more goals conceded than scored
- A negative altitude, indicating that a geographical location is below sea level
- Negative inflation (deflation) showing that retail prices are dropping.

## Who counts?

Although we think of mathematics as a uniquely human activity, some other animals seem to be able to count. Scientists have found that some types of salamander and fish can distinguish between different sized groups as long as the ratio of one to the other is greater than two. Honeybees can apparently distinguish numbers up to four. Lemurs and some types of monkey have limited numerical abilities, and some types of bird can count well enough to know if their eggs or chicks are missing.

This sort of system is fine for counting things that come in relatively small quantities, but it has clear limitations.

**ARE NUMBERS REAL?**

Of all the candidates for reality in mathematics, the whole numbers seem to have the best claim. Even the Polish mathematician Leopold Kronecker accepted them.

Whole numbers seem quite healthy until you look closely, as though they could be found in nature. Perhaps three wolves run through the forest. That's an event in the natural world which looks as though it works with whole numbers. But we can't actually put a rigid boundary around each wolf. There are always atoms flying off the wolf, moving in and out of it; it's picking up more electrons from getting a static charge by rubbing against another wolf; even most of its cells are not actually bits of wolf. There is an entity that is approximately one wolf, but it's ever-changing. We can go smaller and smaller, down to subatomic particles, and even then we find a 'thing' is a cloud or pulse of energy that might or might not be in a particular position at any moment. Hard to count.

Are whole numbers a snapshot of a moment? How short is the moment? How are we measuring it? The measurement of a continuity such as time is entirely arbitrary. And, as Zeno's paradoxes show (see page 13), if we break time into ever shorter moments the logical results don't match the reality we observe.

CHAPTER 4

# How many is 10?

Ten is generally considered to be one more than nine – but it doesn't have to be.

We say our number system uses base-10, which means that when we get as far as nine, we start again with 0 in the units column and 1 in the next column, which we designate 'tens'. Succeeding numbers use two digits, one showing the tens and one showing the units. When we get to 99, we've run out of digits we can put in both places and start another column, for hundreds.

It doesn't have to be this way – there is no rule that says 9 has to be the highest digit we can put in a column. We could use more or fewer digits.

## What is base 10?

The name 'base 10' tells us nothing; at whichever number we stop counting units, the first number using a new column is always going to be '10'. An alien race that counts in base 9 will also call their system base-10 and will have no digit for, say, '9' (0, 1, 2, 3, 4, 5, 6, 7, 8, 10). We really need a new name (and squiggle) for the '10' we use just to name the base.

## Fingers, toes, legs and tentacles

We have probably developed a base-10 number system because we have ten fingers and thumbs, so that makes counting in tens easy. If, instead of humans, three-toed sloths had become the dominant species, perhaps they would have developed a base-6 or base-3 number system – or even base-12 if they were happy to use the toes on their hind limbs as well as those on their forelimbs. A base-3 system would count like this:

| Base 3 – counting in sloth #1 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 10 | 11 | 12 | 20 | 21 | 22 | 100 |
| **Base 6 – counting in sloth #2** | | | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 10 | 11 | 12 | 13 |
| **Base 10 – counting in human** | | | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

If octopuses had become the dominant species, they might have counted in base-8 (octal). In fact, as they are very intelligent creatures, they might well count in base-8 for all we know.

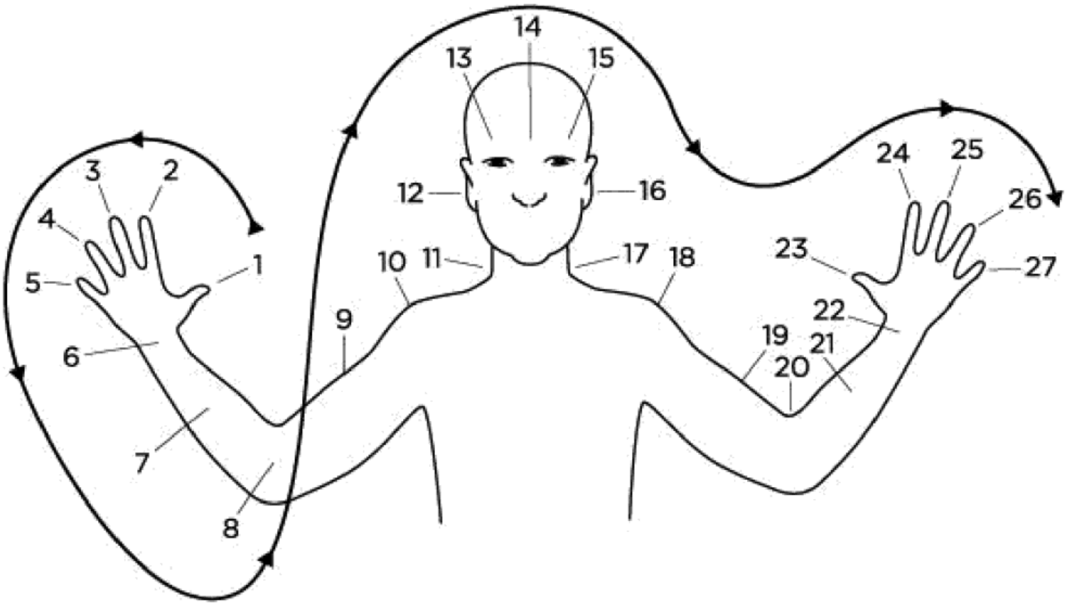
| Base 8 – counting in octopus | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 10 | 11 |
| **Base 10 – counting in human** | | | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

## 10, 20, 60…

We don’t even need to switch species to see different bases at work. The Babylonians worked in base 60 (see Chapter 6) and the Mayans used base 20.

Two-count systems use base-2 (see page 42). We have used base 12 as the basis for quite a few systems of measurement (12 inches in a foot, 12 pennies in an old shilling, 12 eggs in a dozen). Starting with the human body doesn’t mean we have to end up with base 10, either.

The Oksapmin of New Guinea use base 27, derived from counting body parts starting with the thumb of one hand and moving up the arm to the face and down the other side to the opposite hand (see image below).




## Computer counting

We don’t use base-10 for everything. Many computing tasks use base-16, called hexadecimal. As we don’t have any digits for numbers above 9, the letters at the start of the alphabet are co-opted to stand for the numbers from 10 to 15 in hexadecimal.

| Base 10 – counting in human | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| Base 16 – counting in computer #1 | | | | | | | | | | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F | 10 |

You might have noticed codes such as #a712bb labelling colours on the computer. These are triplets of hexadecimal numbers – a7, 12, bb – which give a value for each of the three principal colours – red, green and blue – from which all other colours are built on a computer. These numbers, if converted to decimal (base-10) would be 23 (a7=16+7); 18 (12=16+2); and 191(bb=(11x16)+15). Using hexadecimal means that larger numbers (up to 255=ff) can be stored using only two digits.

Ultimately, all operations on a computer are reduced to binary, or base-2. This uses only two digits – 0 and 1 – as counting starts again with a new place every time we reach 2.

| Base 2 – counting in computer #2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 |
| Base 10 – counting in human | | | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Binary allows all numbers to be represented by one of two states, on/off or positive/negative. It means that anything can be coded on a magnetic disk or tape by the presence or absence of a charge.

## Alien alert

If there are intelligent beings anywhere else in the universe, which seems quite possible (see Chapter 18), how would they count? They might have 17 tentacles and count in base-17. It is highly likely, though, that at some point they will have discovered and used binary (assuming numbers are not just a human construct). It could be that binary is the way we will be able to communicate with them.

The plaques fixed to the outside of the *Pioneer* spacecraft (see image on page 44) launched in 1972 and 1973 showed the binary states of hydrogen, with electron spin up and down. The difference between the two is used as a measure of time and distance and, being the same everywhere in the universe, should be recognized by a civilization capable of space travel.

All logarithmic graphs, no matter what the base of the logarithms, cross the x-axis at 1 as any number raised to a power of zero is 1:

$$10^0 = 1$$

$$2^0 = 1$$

$$15.67^0 = 1$$

Clearly, the numbers go below 0, too. Negative powers yield values less than one as the minus sign tells us to put 1 over the number (the reciprocal of the number), making a fraction:

$$2^{-1} = ½$$

$$2^{-2} = ½^2 = ¼$$

And just in case you thought logarithms have to be in base 10 – they don't. For example, the logarithm in base 2 of 16 is 4:

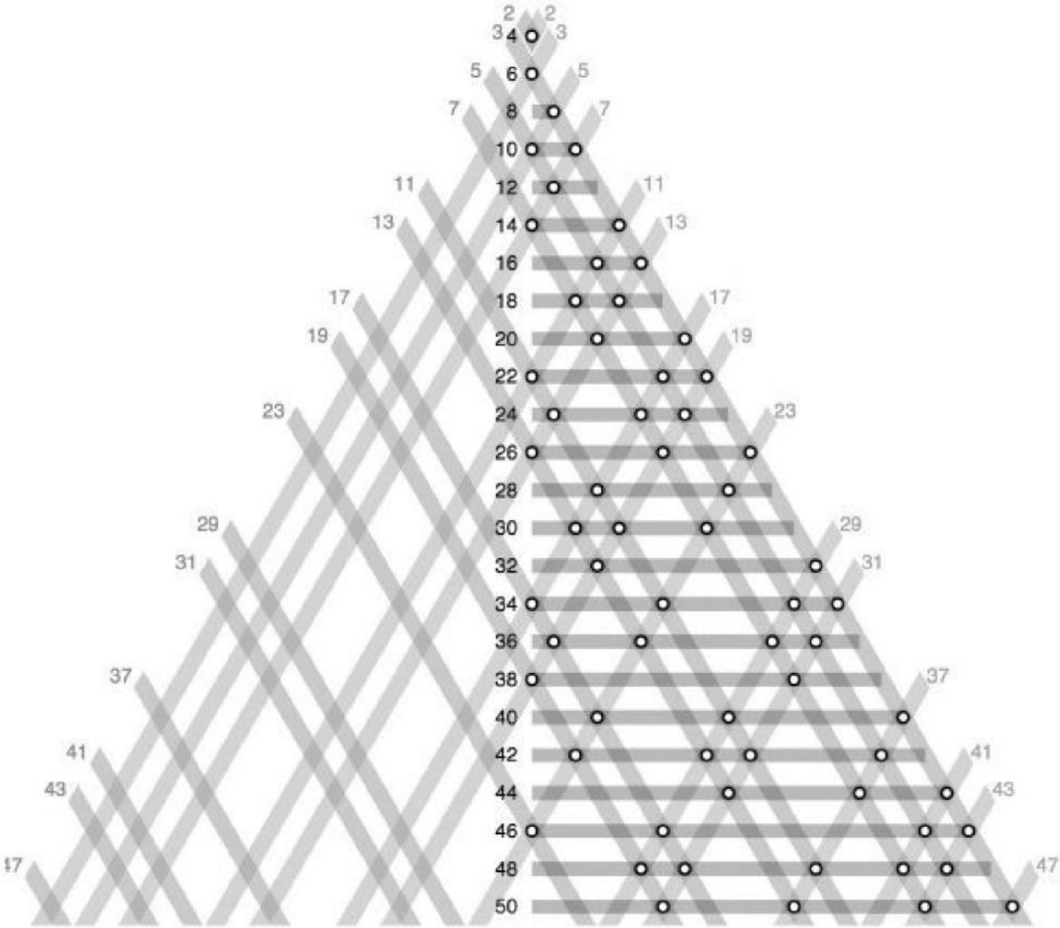
$$16 = 2^4, \text{ so } 4 = \log_2(16)$$

A lot of science, engineering and even financial applications use so-called 'natural logarithms'. These are logarithms to the base e, which is an irrational number (a number with an unending decimal fraction) that starts 2.718281828459…

## All about e

The number called 'e', or Euler's number, is defined by mathematicians with this scary-looking expression:

and onwards…




## First and prime?

Although ‘first’ and ‘prime’ are synonyms in some contexts, the number 1 is not actually considered a prime number. The definition of prime numbers excludes it: ‘any number greater than 1 that has no factors besides itself and 1.’ There are other reasons, that are increasingly complex, but let’s just take it as read that 1 is not a prime because it’s too special.

In fact, Goldbach *did* consider 1 to be a prime. He had a second idea, now called the weak Goldbach conjecture, which stated that every odd number greater than 2 could be expressed as the sum of three primes. That has had to be rephrased to say every odd whole number greater than 5, so that we don’t have to co-opt 1 into a role it’s no longer allowed to occupy. (The weak conjecture was proven by the Peruvian mathematician Harald Helfgott in 2013.)

Euler, unwisely, was rather dismissive of Goldbach’s idea. As it turned out, although Goldbach could try it out with a lot of numbers and it held up, he could not prove it. In mathematics, it’s really not good enough for something to work with every number you try it with – there has to be a proof.

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