THINKING BETTER

THE ART OF THE SHORTCUT IN MATH AND LIFE

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TO ALL MATHEMATICS TEACHERS, BUT
ESPECIALLY MR. BAILSON, WHO SHOWED ME
MY FIRST MATHEMATICAL SHORTCUT

BASIC
BOOKS

DEPARTURE

You have a choice. The obvious path is a long slog, with no beautiful vistas on the way. It is going to take you forever and sap all your energy, but it will eventually get you to your destination. There is a second path, however. You've got to be sharp to spot it veering off the main path, seemingly taking you away from your destination. But you spot a signpost that says SHORTCUT. It promises a quicker off-road route that will get you to your destination faster and with minimal energy expenditure. There might even be the chance of a stunning view on the way. It's just that you are going to have to keep your wits about you to navigate this path. It's your choice.

This book is pointing you toward that second path. It's your shortcut to the better thinking you'll need to negotiate this unorthodox route and get you to where you want to go.

It was the lure of the shortcut that made me want to become a mathematician. As a rather lazy teenager, I was always looking for the most efficient path to my destination. It was not that I wanted to cut corners; I just wanted to achieve my goal with as little effort as possible.

So when my mathematics teacher revealed to me at the age of twelve that the subject we were learning in school was really a celebration of the shortcut, my ears pricked up. It started with a simple story featuring a nine-year-old boy named Carl Friedrich Gauss. Our teacher transported us back to 1786 in the town of Brunswick, near Hanover, where the young Gauss grew up. It was a small town, and the local school only had one teacher, Herr Büttner, who had to somehow teach the town's one hundred children in just one classroom.

My own teacher, Mr. Bailson, was a rather dour Scot who kept strict discipline, but it sounded like he was a softy compared to Herr Büttner. Gauss's teacher would stride up and down the benches brandishing a cane to maintain discipline among the rowdy class. The classroom itself, which I've subsequently visited on a recent mathematical pilgrimage, was a drab room with a low ceiling, little light, and uneven floors. It felt like a medieval prison, and Büttner's regime sounded as if it matched the setting.

The story goes that during one arithmetic lesson Büttner decided to set the class a rather tedious task that would occupy them long enough so that he could take a nap. "Class, I want you to add up the numbers from 1 to 100 on your slates. As soon as you are done, bring your slates to the front of the class and place them on my desk."

Before he'd even finished the sentence, Gauss was on his feet and had placed his slate on the desk, declaring in Low German, "Ligget se"—there it is. The teacher looked at the boy, shocked at his impertinence. The hand holding the cane quivered with anticipation, but he decided to wait until all the students had submitted their slates for inspection before upbraiding the young Gauss. Eventually the class had finished and Büttner's desk was a tower of slates covered in chalk and calculations. Büttner began to work his way through the pile, starting with the last slate placed on the top. Most of the calculations were wrong, the students having invariably made some arithmetic slip on the way.

Eventually he arrived at Gauss's slate. He began preparing his rant at the young upstart, but when he turned over the slate, there was the correct answer: 5050. And there were no extraneous calculations. Büttner was shocked. How had this nine-year-old found the answer so quickly?

The story goes that the precocious young student had spotted a shortcut that helped him avoid the hard work of actually doing much arithmetic. What he had realized was that if you add up the numbers in pairs:

$$1 + 100$$
$$2 + 99$$
$$3 + 98$$
$$\dots$$

Time and again a gear change in civilization was effected by the discovery of new mathematics. The explosion of mathematics during the Renaissance and beyond, which gave us tools such as calculus, offered scientists extraordinary shortcuts to efficient engineering solutions. And today mathematics is behind all the algorithms being implemented on our computers to assist us through the modern digital jungle, offering shortcuts that help us find the best routes to our destinations, the best websites for our internet searches, and even the best partners for a journey through life.

It is interesting to note, however, that humans weren't the first to exploit the power of mathematics to access the best way to tackle a challenge. Nature has been using mathematical shortcuts to solve problems long before we arrived. Many of the laws of physics are based on Nature always finding a shortcut. Light travels along the path that gets it to its destination fastest, even if that involves bending around a large object like the sun. Soap films create the shapes that cost the lowest amount of energy—the bubble makes a sphere because this symmetrical shape is the one with the smallest surface area and therefore costs the minimum energy. Bees make hexagonal cells in their hives because the hexagon uses the least amount of wax to contain a fixed area. Our bodies have found the most energy-efficient way of walking to transport us from A to B.

Nature is lazy, like humans, and wants to find the lowest-energy solutions. As the eighteenth-century mathematician Pierre-Louis Maupertuis wrote: "Nature is thrifty in all its actions." It is extremely good at sniffing out shortcuts. Invariably it has a mathematical rationale to it. And often the discoveries of shortcuts by humans materialize out of our observations of how Nature solves a problem.

The Journey Ahead

In this book I want to share with you the arsenal of shortcuts that mathematicians such as Gauss have developed over the centuries. Each chapter will introduce a different sort of shortcut with its own particular flavor. But all of them have the aim of transforming you from someone who has to slog through the hard work of solving a problem to someone who can hand in their slate with the answer before everyone else.

I have chosen to take Gauss as a companion on our journey. His classroom success launched him on a career that marks him out for me as the prince of the shortcut. Indeed, the plethora of breakthroughs he made during his lifetime span many of the different shortcuts that I will introduce throughout the book.

By telling the stories of the shortcuts that mathematicians have amassed over the centuries, I hope this book will act as a tool kit for all those who want to save time doing one thing so that they can spend more time doing something more exciting. Very often these shortcuts are transferable to problems that don't at first glance seem mathematical in nature. Mathematics is a mindset for navigating a complex world and finding the pathway to the other side.

This is why mathematics really deserves to be a core subject in the educational curriculum. Not because it is absolutely essential that we all know how to solve a quadratic equation; frankly, when has anyone ever needed to know that? The essential skill is understanding the power that algebra and algorithms play in solving such a problem.

I begin the journey to better thinking with one of the most powerful shortcuts mathematicians have developed: patterns. A pattern is often the best sort of shortcut. Spot the pattern and you've found the shortcut to continuing the data into the future. This ability to spot an underlying rule is the basis of mathematical modeling.

Quite often the role of the shortcut is to understand the foundational principle that unites a whole slew of seemingly unrelated problems. The beauty of Gauss's shortcut is that even if the teacher tries to make it harder by asking you to add the numbers up to 1000 or 1,000,000, the shortcut still works. While adding numbers up one by one would get increasingly time-consuming, Gauss's trick is unaffected by an increase in the number of numbers you're adding up. To add the numbers up to 1,000,000, just pair them up again (1 + 1,000,000, 2 + 999,999...) to get 500,000 pairs that each add up to 1,000,001. Multiply these two together (500,000 × 1,000,001) and bingo: you've got your answer. The tunnel that provides a shortcut through the mountain is unaffected by the mountain getting taller.

The power of creating and changing language turns out to be a very effective shortcut. Algebra helps us recognize the underlying principles behind a whole range of different-looking problems. The language of coordinates turns geometry into numbers and often reveals shortcuts that were not visible in the geometric setting. Creating language is an amazing tool for understanding. I remember wrestling with an extraordinarily complex setup that needed many conditions to pin down. My doctoral supervisor's suggestion that I "give it a name" was a revelation—it truly allowed me to shortcut thought.

Whenever I mention the idea of the shortcut, invariably people think I am trying to cheat somehow. The word "cut" sounds like you could be cutting corners, so it's important right from the outset to distinguish between shortcuts and cutting corners. I'm interested in the clever path to get to the correct solution. I'm not interested in finding some shoddy approximation to the answer. I want complete understanding, but without unnecessary hard work.

That said, some shortcuts are about approximations that are good enough to solve the problem at hand. In some sense, language itself is a shortcut. The word "chair" is a shortcut to a whole host of different sorts of things we can sit on. But it is not efficient to come up with a different word for every distinct instance of an example of a chair. Language is a very clever low-dimensional representation of the world around us that allows us to efficiently communicate to others and facilitates our path through the multifaceted world we live in. Without the shortcut of single words for multiple instances, we would be overwhelmed by noise.

In mathematics too I will reveal how throwing away information is often essential to finding a shortcut. The idea of topology is geometry without measurement. If you are on the London Underground, a map showing how stations are connected is much more useful for finding your way around London than a geometrically accurate map. Diagrams are also a powerful shortcut. Again, the best diagrams discard anything that is extraneous to navigating the problem at hand. But as I shall illustrate, there's often a fine line between a good shortcut and the dangers of cutting corners.

Calculus is one of humans' greatest inventions for finding shortcuts. Many engineers depend on this bit of mathematical magic to find the optimal solution to an engineering challenge. Probability and statistics have been a shortcut to knowing a lot about a huge data set. And mathematics can often help you find the most efficient path through a complex geometry or tangled network. One of the staggering revelations I had as I fell in love with mathematics was its ability to find shortcuts to navigate even the infinite—a shortcut to get from one end of an infinite path to the other.

Each chapter begins not with an epigram but a puzzle. Often these puzzles involve a choice: the long slog or, if you can find it, the shortcut. Each puzzle has a solution that takes advantage of the shortcut that is at the heart of that puzzle's chapter. They are worth tackling before you read the chapter, as often the more time you spend battling to get to your destination the more you appreciate the shortcut when it is finally revealed.

What I have discovered on my own journey is that there are different sorts of shortcut. Because of this, I spend time highlighting the multiple approaches you might take to the journey you are about to embark on, and show that you will get to your destination faster by using the most effective shortcut. There are shortcuts that are already waiting there in the terrain for us to take advantage of them; it's just that you might need a signpost to point you in the right direction or a map to show you the way. There are shortcuts that won't exist if you don't do a lot of hard work to carve them out—like the tunnel that takes years to dig but once there allows everyone else to follow you through to the other side. There are shortcuts that require totally escaping the space you are in—the wormhole from one side of the universe to the other, or the extra dimension that shows how two things are much closer than you imagine provided you can step out of the confines of the current world. There are shortcuts that speed things up, shortcuts that cut down the distance you need to travel, and shortcuts that reduce the amount of energy you need to expend. Somewhere there is a saving that is worth the time to find the shortcut.

But I've also recognized that there are times when the shortcut misses the point. Maybe you want to take your time. Maybe the journey is the thing. Maybe you want to expend energy in an attempt to lose weight. Why go on a walk in Nature for the day if you curtail the pleasure of the walk by taking a shortcut home? Why read a novel rather than a synopsis on Wikipedia? But it's still good to know you've got the option of a shortcut even if you decide to ignore it.

The shortcut is to some extent about our relationship to time. What do you want to spend your time doing? Sometimes it is important to experience something in time and there is little value to finding a shortcut that cheats you of the feeling. Listening to a piece of music can't be shortcutted. It takes time. But on other occasions life is too brief to spend time getting to where you want to be. A film can condense a life into ninety minutes; you don't want to witness every action of the character you are following. Taking a flight to the other side of the world is a shortcut to walking there and means you can begin your vacation sooner; if you could shorten the flight even further, you probably would. But there are times when people want to experience the slow version of getting to their destination. Pilgrimage abhors the shortcut, for instance. And I never watch film trailers, because they shortcut the film too much. But it is still worth having the choice.

Shortcuts in literature are invariably paths that lead to disaster. Little Red Riding Hood never would have met the wolf if she hadn't strayed from the path in search of a shortcut through the wood. In Bunyan's *Pilgrim's Progress*, those who take a shortcut around Difficulty Hill get lost and perish. In *The Lord of the Rings* Pippin warns that "shortcuts make long delays" (although Frodo counters that inns make even longer ones). Homer Simpson swears after his disastrous detour on the way to Itchy and Scratchy Land, "Let us never speak of the shortcut again." The dangers inherent in taking shortcuts are well summed up in the film *Road Trip*: "Of course it's difficult—it's a shortcut. If it was easy, it would just be 'the way.'" This book looks to rescue the idea of a shortcut from these literary tropes. Rather than the road to disaster, the shortcut is the road to freedom.

mode of neural processing is to cognitive ability. This mode is often suppressed when our attention is too focused on the outside world. The recent surge of interest in mindfulness suggests the value of stilling the mind as a pathway to enlightenment. Often it means you prefer to play rather than work. But play is often the place to foster creativity and new ideas. It is one of the reasons that the offices of start-ups and math departments often contain pool tables and board games as well as desks and computers.

Perhaps society's disapproval of laziness is a way of controlling and curtailing those who prefer not to conform. The real reason the lazy person is regarded with suspicion is that laziness is the mark of someone not prepared to play by the rules of the game. Gauss's teacher saw his pupil's shortcut to doing hard work as a threat to his authority.

Idleness has not always been shunned. Samuel Johnson very eloquently argued in favor of laziness: "The Idler... not only escapes labours which are often fruitless, but sometimes succeeds better than those who despise all that is within their reach." As Agatha Christie admitted in her autobiography, "Invention, in my opinion, arises directly from idleness, possibly also from laziness. To save oneself trouble." Babe Ruth, one of the best home-run hitters baseball has ever seen, apparently was motivated to hit the ball out of the stadium because he hated having to run between bases; when he hit a homer, he could take his time rounding the bases.

Choosing to Work

I do not wish to imply that all work is bad. Indeed many people get great value out of the work they do. It defines their identity. It gives them purpose. But the quality of the work is important. Generally, the work we find valuable is not a series of tedious, mindless tasks. Aristotle distinguished between two different sorts of work: *praxis*, which is action done for its own sake, and *poiesis*, or activity aimed at the production of something useful. We are happy to look for shortcuts in the second sort of work, but there seems little point in chasing the shortcut if the pleasure is in doing the work for its own sake. Most work seems to fall into the second category. But surely the ideal is to aspire to work of the first kind. That is where the shortcut aims to take you. The shortcut is not about eliminating work; it wants to lead you on a path to meaningful work.

The principle behind the new political movement Fully Automated Luxury Communism is that with advances in AI and robotics, machines can take over our menial work, leaving time for us to indulge in work we find meaningful. Work becomes a luxury. The cultivation of good shortcuts should be added to the list of technologies steering us toward a future of work that is undertaken for the joy of it rather than as a means to an end. This was Marx's aim with communism: to remove the difference between leisure and work. "In a higher phase of communist society… labour has become not only a means of life but life's prime want." The shortcuts we have created promise to take us away from what Marx called the "realm of necessity" and lead us instead into the "realm of freedom."

But there are some places where you can't get away from hard work. How can a lazy person learn a musical instrument? Write a novel? Climb Everest? Even here, though, I shall illustrate how shortcuts can help you maximize the value of the hours you put in at your desk or in training. The book is punctuated by conversations I've had with high achievers to see whether shortcuts are possible in their professions or if you just can't avoid the ten thousand hours of practice that Malcolm Gladwell says are necessary to get to the top of your profession. I've been intrigued to find out whether the shortcuts that people have found resonate with those I've learned in mathematics, or whether there might be new sorts of shortcuts that I've not been aware of but which might prompt new modes of thinking in my own work. But I'm also fascinated by those challenges where no shortcuts are possible. What is it about certain domains of human activity that preclude the power of the shortcut? Time and again, it turns out, the human body is often the limiting factor. To change or train or push the human body to do new things quite often takes time and repetition, and there are no shortcuts to speed up those physical transformations. So as I take you on the journey through the different shortcuts mathematicians have discovered, each chapter includes a pit stop to explore the shortcuts, or lack of them, in different fields of human activity.

Gauss's schoolroom success at adding the numbers from 1 to 100 using his cunning shortcut fueled his desire to pursue his mathematical talents. His teacher, Herr Büttner, wasn't up to the task of cultivating the budding young mathematician, but he had an assistant, seventeen-year-old Martin Bartels, who was equally passionate about mathematics. Although Bartels had been employed to cut quill pens for the students and assist them in their first attempts at writing, he was more than happy to share his mathematical texts with the young Gauss. Together they explored the mathematical terrain, enjoying the shortcuts that algebra and analysis provided to reach their destinations.

Bartels soon realized that Gauss needed a more challenging environment to test his skills. He managed to get Gauss an audience with the Duke of Brunswick. The Duke was so taken by the young Gauss that he agreed to become his patron, funding his education at the local college and then at the University of Göttingen. It was here that Gauss began to learn some of the great shortcuts that mathematicians had developed over the centuries and which would soon become the springboard for his own exciting contributions to mathematics.

This book is my curated guide through two thousand years of better thinking. It has taken me decades to learn how to navigate these cunning tunnels or hidden passes through the landscape, and it took mathematicians through history thousands of years to piece them together. But in this book I've tried to distill some of these clever strategies for attacking the complex problems we encounter in everyday life. This is your shortcut to the art of the shortcut.

CHAPTER 1

THE PATTERN SHORTCUT

> Puzzle: You have a flight of stairs in your house with 10 steps. You can take one or two steps at a time. For example, you could do 10 one-steps to get to the top, or 5 two-steps, or combinations of one-steps or two-steps. How many different possible combinations are there to get to the top?
>
> You could do this the long way and try to find all the combinations, running up and down the stairs. But how would our young Gauss do it?

WANT TO KNOW A shortcut to getting an extra 15 percent salary for doing exactly the same work? Or perhaps a shortcut to growing a small investment into a large nest egg? How about a shortcut to understanding where a stock price might be heading in the coming months? Do you feel like you are sometimes reinventing the wheel again and again, yet sense there is something that connects all these different wheels you are making? What about a shortcut to help you with your terrible memory?

I'm going to dive in and share with you one of the most potent shortcuts that humans have discovered. It is the power of spotting a pattern. The ability of the human mind to glean a pattern in the chaos around us has provided our species with the most amazing shortcut: knowing the future before it becomes the present. If you can spot a pattern in data describing the past and the present, then by extending that pattern further you have the chance to know the future. No need to wait. The power of the pattern is for me the heart of mathematics and its most effective shortcut.

Patterns allow us to see that even though the numbers might be different, the rule for how they grow can be the same. Spotting the rule underlying the pattern means that I don't have to do the same work every time I encounter a new set of data. The pattern does the work for me.

Economics is full of data with patterns that, if read properly, can guide us to a prosperous future—although, as I shall explain, some patterns can be misleading, as the world witnessed with the financial crash of 2008. Patterns in the number of those falling ill with a virus mean we can understand the trajectory of a pandemic and intervene before it kills too many people. Patterns in the cosmos allow us to understand our past and our future. Looking at the numbers that describe the way stars are moving away from us has revealed a pattern that tells us our universe began in a big bang and will end with a cold future called heat death.

It was this ability to sniff out the pattern in astronomical data that launched the aspiring young Gauss onto the world stage as the master of the shortcut.

Planetary Patterns

On New Year's Day, 1801, an eighth planet was detected orbiting around the sun somewhere between Mars and Jupiter. Christened Ceres, its discovery was regarded by everyone as a great omen for the future of science at the beginning of the nineteenth century.

But excitement turned to despair a few weeks later, when the small planet (which was in fact just a tiny asteroid) disappeared from view near the sun, lost among a plethora of stars. The astronomers had no idea where it had gone.

Then news arrived that a twenty-four-year-old from Brunswick had announced that he knew where to find this missing planet. He told the astronomers where to point their telescopes. And lo, as if by magic, there was Ceres. The young man was none other than my hero Carl Friedrich Gauss.

Since his classroom successes at age nine, Gauss had gone on to make numerous fascinating mathematical breakthroughs, Gauss had discovered the rather extraordinary fact that every number can be written as three triangular numbers added together—for example, 1796 = 10 + 561 + 1225. This kind of observation can lead to powerful shortcuts because rather than proving that something is true for all numbers, it might be enough to prove it for triangular numbers and then exploit Gauss's discovery that every number is the sum of three triangular numbers.

Here's another challenge. What's the next number in this sequence?

1, 2, 4, 8, 16...

Not too tricky: 32 is the next number. This sequence is doubling each time. Called exponential growth, this pattern controls the way a lot of things can grow, and it's important to understand how this kind of pattern evolves. For example, the sequence looks quite innocent to start with. That's certainly what the king of India thought when he agreed to pay the creator of the game of chess the price he demanded for his game. The inventor had asked for a single grain of rice to be placed on the first square of the chessboard and then to double the number of grains of rice on each subsequent square on the board. The first row looked quite innocent, with only a total of 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 = 255 grains of rice. Barely enough for a piece of sushi.

But as the king's servants added more and more rice to the board, they very quickly ran out of supplies. To get to the halfway point needs about 280,000 kg of rice. And that's the easy half of the board. How many grains of rice does the king need in total to pay the inventor? At first sight this looks like one of those problems Herr Büttner might give his poor students. There is the hard way to do this: add up 64 different numbers. Who wants to do such hard work? How might Gauss go about this sort of challenge?

There is a beautiful shortcut to making this calculation, but at first sight the shortcut looks like I'm making life harder. Often shortcuts begin by seeming to head in the opposite direction from your destination. First I'm going to give the total grains of rice a name: x. It's one of our favorite names in mathematics, and is in itself a powerful shortcut in the mathematician's arsenal, as I shall explain in Chapter 3.

I am going to kick off by doubling the amount that I am trying to work out:

$$2 \times (1 + 2 + 4 + 8 + 16 + \ldots + 2^{62} + 2^{63})$$

This looks like it's made life more difficult. But stick with me. Let's multiply this out:

$$= 2 + 4 + 8 + 16 + 32 + \ldots + 2^{63} + 2^{64}$$

Now comes the smart bit. I am going to take x away from this. At first sight that looks like I've just got us back to where we started: $2x - x = x$. So how does that help? A bit of magic happens when I replace $2x$ and x by the sums I've got:

$$2x - x = (2 + 4 + 8 + 16 + 32 + \ldots + 2^{63} + 2^{64}) \\ - (1 + 2 + 4 + 8 + 16 + \ldots + 2^{62} + 2^{63})$$

Most of these terms cancel! There is just the 2^{64} in the first part and 1 in the second part that doesn't get canceled. So all I am left with is

$$x = 2x - x = 2^{64} - 1$$

Instead of lots of calculating, all I need to do is this one calculation to discover that the number of grains of rice that the king needed in total to pay the inventor of chess is

18,446,744,073,709,551,615

That's more rice than has been produced on our planet in the last millennium. The message here is that sometimes you can play hard work off against hard work and be left with something that is much simpler to analyze.

As the king learned to his cost, doubling starts off looking innocent and then ramps up very quickly. This is the power of exponential growth. The effect is felt by those who take out loans to cover debt. At first sight the offer from a company of a $1,000 loan at 5 percent interest each month might seem like a lifesaver. After one month you only owe $1,050. But the trouble is that each month this gets multiplied by 1.05 again. After two years you already owe $3,225. By the fifth year, the debt is $18,679. Great for the person who's lending money to you, but not so great for the borrower.

The fact that people in general don't understand this pattern of exponential growth means that it can be a shortcut to penury. Payday loan companies have successfully exploited this inability to read the pattern into the future to suck vulnerable people into a contract that initially looks quite attractive. The dangers of doubling and the path it takes us down are important to know before we find ourselves lost and helpless with no way back to safety.

We all learned the frightening rate of growth of the exponential to our cost too late with the pandemic of 2020. The number of people infected doubled every three days on average. And this resulted in healthcare systems being overwhelmed.

On the other hand, the power of the exponential can also help to explain why there are (probably) no vampires on earth. Vampires need to feed on the blood of a human being at least once a month to survive. The trouble is that once you have feasted on the human, the victim too becomes a vampire. So next month there are twice as many vampires in the search for human blood to feast on. The world's population is estimated to be 6.7 billion. Each month the population of vampires doubles. Such is the devastating effect of doubling that within thirty-three months a single vampire would end up transforming the world's population into vampires.

Just in case you ever meet a vampire, here is a useful trick from the mathematician's arsenal to ward off the blood-sucking monster. In addition to the classics—garlic, mirrors, and crosses

—one rather unusual way to ward off a vampire is to scatter poppy seeds around his coffin. Vampires, it turns out, suffer from a condition called arithmomania: a compulsive desire to count things. Theoretically, before Dracula finishes trying to count how many poppy seeds are scattered around his resting place, the sun will have driven him back to his coffin.

Arithmomania is a serious medical condition. The inventor Nikola Tesla, whose studies into electricity gave us alternating current, suffered from the syndrome. He was obsessed with numbers divisible by 3: he insisted on 18 clean towels a day and counted his steps to make sure they were divisible by 3. Perhaps the most famous fictional depiction of arithmomania is the Muppets' Count von Count, a vampire who has helped generations of viewers in their first steps along the mathematical path.

Urban Patterns

Here's a slightly more challenging sequence of numbers. Can you sniff out the pattern here?

179, 430, 1033, 2478, 5949...

The trick is to divide each number by the number before it. This reveals that the multiplying factor is 2.4. Still exponential growth, but what is intriguing is what these numbers actually represent: patents issued in cities of population size 250,000, 500,000, 1 million, 2 million, 4 million, and so on. It turns out that when you double the population you don't simply get a doubling of the number of patents, as you might expect. Larger cities seem to produce more creativity. The doubling of population seems to add an extra 40 percent to creativity! And it's not just patents that seem to have this pattern of growth.

Despite the huge cultural differences between Rio de Janeiro, London, and Guangzhou, there is a mathematical pattern that connects all cities across the world from China to Brazil. We are used to describing cities by their geography and history, traits that highlight the individuality of a place such as

New York or Tokyo. But those facts are mere details, interesting anecdotes that don't explain very much. Look at the city through the eyes of a mathematician, though, and a universal character begins to emerge that transcends political and geographic boundaries. This mathematical perspective unveils the appeal of the city... and it proves that bigger is better.

The mathematics reveals that the growth of each resource in a city can be understood by a single magic number particular to that resource. Each time the population of a city doubles, the socioeconomic factors scale not simply by doubling but by doubling and a bit more. Rather remarkably, for many resources that bit more is around 15 percent. For example, if you compare a city with a population of 1 million people to a city of 2 million, then instead of the larger city having twice as many restaurants, concert halls, libraries, and schools, you find an extra 15 percent on top of what you'd expect from simply doubling the numbers.

Even salaries are affected by this scaling. Take two employees doing exactly the same job but in different-sized cities. The employee living in the city with a population of 2 million will on average have a salary 15 percent higher than the salary of the employee in the city with 1 million inhabitants. Double the city size again to 4 million, and the salary gets increased by another factor of 15 percent. The bigger the city, the more you'll get paid for doing exactly the same job.

It's spotting a pattern like this that can be the key to a business getting the most out of what it puts in. Cities come in lots of shapes and sizes. Understanding that the shape is irrelevant but the size matters means that a company can get much more for its buck by simply relocating to a city double the size.

This strange universal scaling was discovered not by an economist or a social scientist but by a theoretical physicist applying the same mathematical analysis that is usually applied in the search for the fundamental laws that underpin the universe. Geoffrey West was born in the United Kingdom, and after studying physics at Cambridge, he went to do research at Stanford exploring properties of fundamental particles. But it was his becoming president of the Santa Fe Institute that would be the catalyst for his discoveries about urban growth. The Santa down the line. They've already shaken citizen 1's hand, so they end up shaking $N - 2$ hands. As we go down the line, each citizen does one less handshake. The total number of handshakes is the sum from 1 to $N - 1$. Hello again! This is the calculation that Gauss was asked to perform. His shortcut produces a formula for this number:

$$1/2 \times (N - 1) \times N$$

What happens to this connectivity when I double N? The number of handshakes doesn't double but goes up by a factor of 2 squared—that is, 4. The number of handshakes is proportional to the square of the number of inhabitants.

This is a great example of why mathematics can spare us from having to continually reinvent the wheel. Although I was asking a completely different question about connections across a network, I found that from the analysis of the triangular numbers I already had the tools to know how this number grows. Time and again the characters might change, but the script remains the same. Understand the script and you've got a shortcut to knowing the behavior of any character inserted into the drama. In this case, the number of connections between citizens grows quadratically with the number of inhabitants.

Of course, there is no way that every inhabitant will know every other citizen in the town. A more conservative measure would be that they know the citizens in their local neighborhood. This would scale linearly; the overall size doesn't really matter.

It looks like real cities are somewhere in between the extreme case and the most limited case. A citizen has all their local connections plus a certain degree of longer-range connections across the city. It seems that the additional long-range connections are the ones that are causing the growth in connections, resulting in the extra 15 percent as the population doubles. As I will explain later in the book, this sort of network arises in many different scenarios and turns out to be a very efficient setup for creating shortcuts across the network.

Misleading Patterns

Although patterns are incredibly powerful, we should still be careful with how we use them. You can set off on a path and think you know where you are heading. But sometimes that path can veer off in a weird and unexpected direction. Take the sequence that I challenged you with earlier in this chapter:

1, 2, 4, 8, 16...

What if I told you that 31, not 32, was the next number in this sequence?

If I take a circle, add points on the edge of the circle, and join up all the points with lines, what is the largest number of regions the circle gets divided into? If I have just one point on the circle, then there are no lines and I've got just 1 region. If I add a point, then I can join the two points to get 2 regions, divided by the line I've drawn. Now add a third point. Draw in all the lines connecting points, and I have a triangle figure with three sectors of the circle surrounding the triangle: 4 regions.

Figure 1.1. The first five circle division numbers

If I keep doing this, then it seems like a pattern begins to emerge. Here is the data showing the number of regions as I add another point on the circle:

1, 2, 4, 8, 16...

A good guess at this point would be that adding a point doubles the number of regions. The trouble is the pattern breaks down as soon as I add a sixth point on the edge of the circle. No matter how hard you try, the maximum number of regions cut out by the lines is 31. Not 32!

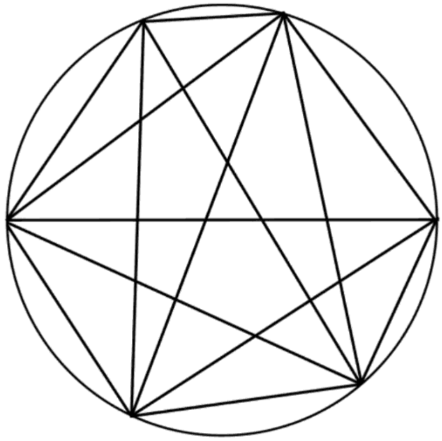

Figure 1.2. The sixth circle division number

There is a formula that will give you the number of regions, but it is a little more complicated than simple doubling. If there are *N* points on the circle, then the maximum number of regions you'll get when you join the points is

$$1/24\ (N^4 - 6N^3 + 23N^2 - 18N + 24)$$

The message here is that it is still important to know what your data is describing and not to rely simply on the numbers themselves. Data science is dangerous if it is not combined with a deep understanding of where the data comes from.

Here is another warning about this shortcut. What is the next number in this sequence?

2, 8, 16, 24, 32...

Lots of powers of 2 in there. But a slightly unusual 24? Well, if you could identify 47 as the next number in this sequence, then I recommend you buy a lottery ticket next Saturday. These were the winning lottery tickets for the UK National Lottery on September 26, 2007. We are so addicted to looking for patterns that we often see them in places where we can't expect a pattern. Lottery tickets are random. No patterns. No secret formulas. No shortcuts to becoming a millionaire. But that said, I shall explain in Chapter 8 that even random things have patterns in them that we can exploit as potential shortcuts. When it comes to randomness, the shortcut is to stand back and take the long view.

The concept of pattern can be used as a shortcut to understanding when something is truly random or not, and it relates to how memorable a sequence of numbers is.

A Shortcut to a Good Memory

Given that there is so much data being spewed out every second on the internet, companies are on the lookout for clever ways to store it. Finding patterns in the data actually offers a way to compress the information such that you don't need as much space to store it. This is the key behind compression technologies such as JPEG and MP3.

Take a picture that is just black and white pixels. The idea is that in any picture there might be a large swath of white pixels in one corner. Instead of recording each pixel as white and using as much memory to store the picture as there is data in the image, you could take a potential shortcut. Record instead the location of the boundary of the region and just add the instruction to fill in the region with white pixels. The bit of code that I can write to do this will in general be much smaller than recording that each pixel in this region is white.

Any patterns that you can discern in the pixels can be exploited to write code that will record the picture using far less memory than saving the data pixel by pixel. For example, take a chessboard. The image has a very obvious pattern, which allows us to write code that simply says repeat white-black 32 times across the board. Even if you had an enormous chessboard, the code would not grow any bigger.

I believe patterns are also key to how humans store data. I must admit that I have a very bad memory. I think it was one of the reasons I was drawn to mathematics. Mathematics has always been my weapon against my terrible memory for names and dates and random information that I can make no logical sense of. I haven't a clue on what date Queen Elizabeth I died, and if you tell me it was in 1603, I'll have forgotten it ten minutes later; in French I always had difficulty recalling all the different forms of the irregular verb *aller*; in chemistry, was it potassium or sodium that burns purple? But in mathematics I could reconstruct everything from the patterns and logic I'd identified in the subject. Spotting patterns replaced the need for a good memory.

I suspect this is one of the ways our brains store memories. Memory depends on identifying pattern and structure to help our brains store a condensed program from which to regenerate the stored memory. Here's a little challenge. Stare at the squiggles contained in the following 6 × 6 grid. Then close the book. Can you reproduce the grid from memory? The key is not to try to remember each of the 36 squares in the image individually but to find a pattern that helps you to generate the image.

have chosen the first two-step in position 1 and the second in position 2, or I could have done it in the other order. The result would be the same. So the total number now is 8 × 7 / 2 = 28. There is actually a mathematical name for this number. It is "8 choose 2," and it is denoted by

$$\binom{8}{2}$$

More generally, the way of choosing 2 numbers from N numbers is given by the formula $1/2\ (N - 1)\ N$, which is the same formula Gauss came up with for the triangular numbers. There's that wheel we invented showing up again! There is a way to translate the question of choosing 2 numbers from N numbers into the challenge of calculating the triangular numbers. I will explain in Chapter 3 how changing one problem into another can often be a great shortcut to solving a problem.

These tools for calculating choices, called binomial coefficients, were actually some of the formulas that Gauss and his classroom assistant pored over in their algebra books at school together.

But to solve this puzzle, next I will have to calculate how to choose 3 locations out of a choice of 7 to place our 3 two-steps up the staircase. Although this seems like a good systematic way to build the possibilities up, it is going to require us to generalize these choice functions. It's beginning to look like a hard slog heading along this path. It doesn't really feel like a shortcut.

So here's a better shortcut exploiting what I have shown you in this chapter. With puzzles like this, I find a very powerful strategy is to consider a smaller number of steps and see if there is a pattern in the way the numbers are falling out.

Here are the possibilities for staircases with 1, 2, 3, 4, and 5 steps, which can be worked out quickly by hand:

1 step: 1
2 steps: 11 or 2
3 steps: 111 or 12 or 21
4 steps: 1111 or 112 or 121 or 211 or 22

5 steps: 11111 or 1112 or 1121 or 1211 or 2111 or 122 or 212 or 221

So the number of possibilities is going 1, 2, 3, 5, 8... Now, you might already have spotted a pattern. You get the next number by adding the two previous numbers together. You might even know a name for these numbers—the Fibonacci numbers, named after the twelfth-century mathematician who discovered that they are the key to the way things grow in the natural world. Petals on flowers, pine cones, shells, populations of rabbits—the numbers all seemed to follow the same pattern.

Fibonacci discovered that Nature was using a simple algorithm in order to grow things. The rule of adding the two previous numbers together was Nature's shortcut for building complex structures like a shell or pine cone or flower. Each organism just uses the two last things it built as ingredients for the next move.

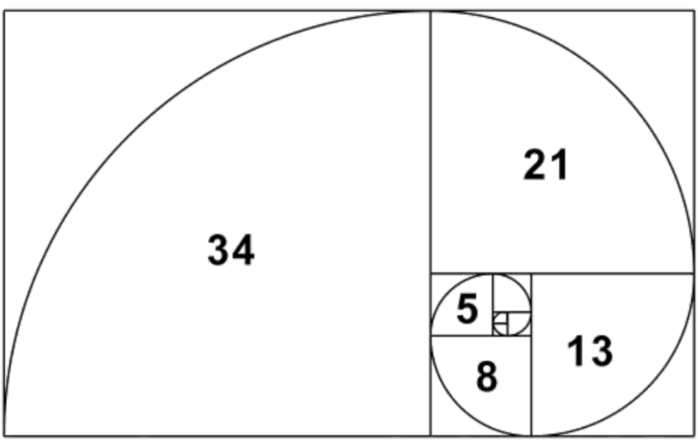

Figure 1.4. How to use the Fibonacci numbers to grow a spiral

Using a pattern to evolve structures is a key shortcut for Nature. Take, for example, the way Nature builds a virus. Viruses come in very symmetrical structures. This is because symmetry requires a simple algorithm to implement to make the structure. If a virus is in the shape of symmetrical dice, the DNA that replicates the molecule just has to make several copies of the same protein that will make up the faces and then the same rule is used across the virus to build its structure. A pattern makes building the virus fast and efficient—and that's part of what can make it so deadly.

But are we really sure just from this small amount of data that the Fibonacci rule is the secret to climbing the stairs?

Actually, the rule explains exactly how to work out the next possibilities with 6 steps on the staircase. Take all the possible steps up to the fourth step and then add a two-step on the end. Or all the possible routes to the fifth step and add a one-step onto these. This gives all the ways up to the sixth step. It is a combination of the two previous numbers in the sequence.

The answer to the puzzle is to calculate the 10th number in the sequence:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89

There are 89 different routes. The pattern is the shortcut to knowing how many ways to get to the top of the staircase. And the pattern will help crack this conundrum even if there are 100 or 1000 steps instead of just 10.

Although these numbers are named after Fibonacci, he wasn't the first to discover them. In fact, they were first discovered by Indian musicians. Tabla players are interested in showing off the different rhythms they can make on their drum. As they explored the different sorts of rhythms they could make from long and short beats, it led them to the Fibonacci numbers. If the long beat is twice the length of the short beat, then the number of rhythms the tabla player can cook up is the same answer as climbing the staircase. Each one-step corresponds to a short beat, each two-step to a long beat. The number of rhythms the drummer can make of a given duration is the same answer as the number of ways to climb the staircase. So the number of rhythms is given by the Fibonacci rule. And the rule even gives the tabla player an algorithm to construct them out of the previous shorter rhythms.

There is something exciting here about seeing the same pattern explaining so many different things. For Fibonacci, it was the way Nature grows things. For the Indian tabla player, the pattern generates rhythms. The pattern explains the number of ways to climb the staircase in ones and twos. There are some in the financial sector who even believe that these numbers can be used to predict when a stock that is falling will eventually bottom out and start rising again. It is this power of revealing the underlying structure behind different facades that can be so powerful as a shortcut. One pattern solves a multitude of very different-looking challenges. When you are faced with a new problem, it is often worth checking whether it might be an old problem in a new disguise that you already have found a way to solve.

Connecting Shortcuts

I can't resist adding a little coda to this story, because it makes use of the earlier hard work. My first strategy for calculating the number of routes to the top of the staircase started leading me into the question of how to choose 3 things from a group of 7 objects. Mathematicians actually found a clever way to shortcut calculating all those choices I was trying to make. It's something called Pascal's triangle (although, like with the Fibonacci numbers, it turns out that Pascal was beaten to the discovery by the ancient Chinese).

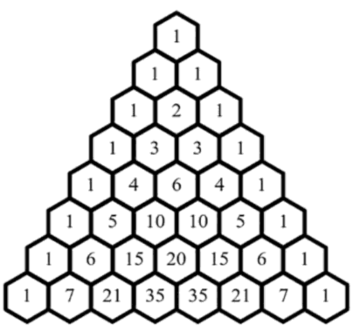

Figure 1.5. Pascal's triangle

The triangle has a rule similar to the Fibonacci rule, but you build the numbers in the layers below by adding the two numbers that sit above that number. The table is easy to build using this rule. But the great fact is that it contains all the choice numbers I was after. Suppose I run a pizza restaurant and I want to boast about the number of different pizzas I offer. If I want to know the number of ways of choosing 3 toppings from a choice of 7 different toppings, then I go to the (3 + 1)th number in the (7 + 1)th row: 35. That is my shortcut to knowing that there are 35 different pizzas I can make. In general, to choose *m* things from *n* things you go to the (*m* + 1)th number in the (*n* + 1)th row. But because these choice numbers were one way to solve our staircase problem, this means that the Fibonacci numbers are actually hiding inside Pascal's triangle. Add up numbers in diagonal lines through the triangle and the Fibonacci numbers appear.

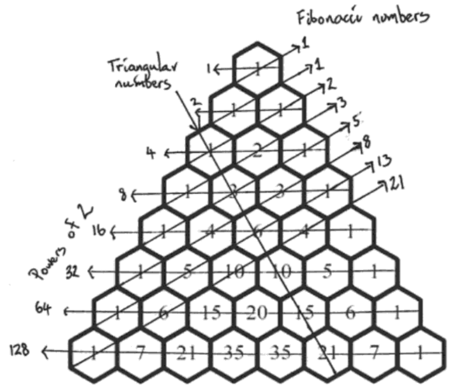

Figure 1.6. The Fibonacci numbers, triangular numbers, and powers of 2 inside Pascal's triangle

This kind of connection is one of the things that I love about mathematics. Who would have thought that the Fibonacci numbers were hiding inside Pascal's triangle? Yet by looking at the puzzle in two different ways I've found a secret tunnel, a shortcut, that links these two seemingly different corners of the mathematical world! And look how the triangular numbers and also the powers of 2 are all hidden inside this triangle. The triangle numbers are sitting along one of the diagonals through the triangle, while you get the powers of 2 by adding up all the numbers in each of the rows. Mathematics is full of these strange tunnels providing shortcuts that we can exploit to change one thing into another.

PIT STOP 1

MUSIC

A FEW YEARS AGO, I DECIDED to learn to play the cello. But it is taking me longer than I had hoped, so I am eager to sniff out any cunning shortcuts that might help. If mathematics is the science of patterns then music is the art of patterns. Could exploiting these patterns be the key?

The cello isn't the first instrument I learned to play. That same year that Mr. Bailson shared the story of the young Gauss with my math class, the music teacher at my school asked the class whether anyone wanted to learn a musical instrument. Three of us put up our hands. At the end of the lesson the teacher led us into the instrument storeroom. It was pretty bare except for three trumpets stacked on top of each other. So the three of us ended up playing the trumpet.

I don't regret the choice. The trumpet is a wonderfully flexible instrument. I cut my teeth playing for the local town band, participated in the local county orchestra, even tried my hand at a bit of jazz. But as I sat counting bars of rest in the orchestra I would stare across at the cellists in front of me, who seemed to be playing all the time. I must admit I was a little envious.

As an adult, I decided that I would buy a cello with a bit of money that my godmother left me in her will. I would use what was left over to take some lessons. But I was slightly concerned whether I'd be up to learning a new instrument. As a child, the time it took to learn an instrument didn't bother me. I was at school and we had years of learning ahead of us. But as adults we have fewer years ahead of us and so become much more impatient. I wanted to be able to play the cello now, not in seven years' time. Was there any shortcut to learning an instrument?

Malcolm Gladwell's book *Outliers* popularized the theory that to become an expert in anything requires putting in a minimum of 10,000 hours of practice. It controversially proposed that this might be enough to become internationally recognized in your field, although the team that produced the original research said that this was a misinterpretation of their work. But was there really no way I could shortcut the 10,000 hours of practice before I could play a Bach cello suite on the stage? An hour a day would mean more than twenty-seven years of practice!

I decided to seek the advice of Naomi Clein, one of my favorite cellists of all time. Clein first came to international attention as one of the youngest winners of the prestigious BBC Young Musician of the Year competition in 1994, when she performed the Elgar cello concerto. What had been her trajectory to international fame?

Clein started playing the cello at the age of six but didn't get serious about it until a few years later. "By fourteen or fifteen," she told me, "I was trying to do four to five hours a day. There are some who do much more. There are kids out there doing about eight hours a day practice when they are sixteen. There are colleagues from places like Russia or the Far East where they're put into this disciplined mode of hard work much earlier than we are in the West."

This level of discipline, Clein explained, was necessary to achieve the motor memory and control that mastering an instrument requires: "There's certainly a minimum number of hours you need to put in when you're learning an instrument, three or four hours a day in your teenage years, which you have to cover, because you physically just don't get the motor mechanics otherwise."

Take Jascha Heifetz, for example. Heifetz was one of the greatest violinists of all time. Famously, he practiced scales every morning for most of his life, thousands of hours in total, just on scales.

In this way, cellists are similar to athletes. You can't run a marathon or win a 100-meter sprint without putting in the hours to physically train your body. The physical aspect of tuning the body and mind so that it can play passages at speed requires