Problem Books in Mathematics

Unsolved Problems
in Intuitive Mathematics
Volume II

Hallard T. Croft
Kenneth J. Falconer
Richard K. Guy

Unsolved Problems in Geometry

Springer Science+Business Media, LLC

Unsolved Problems in Intuitive Mathematics
Volume II

Hallard T. Croft   Kenneth J. Falconer
Richard K. Guy

Unsolved Problems in Geometry

With 66 Figures

Springer Science+
Business Media, LLC

Hallard T. Croft
Peterhouse,
Cambridge, CB2 1RD
England

Richard K. Guy
Department of Mathematics and Statistics
The University of Calgary
Calgary, Alberta T2N 1N4
Canada

Kenneth J. Falconer
School of Mathematics
Bristol University
Bristol, BS8 1TW
England

Series Editor:
Paul R. Halmos
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053
USA

AMS Classification: 52-02

Library of Congress Cataloging-in-Publication Data
Croft, Hallard T.
Unsolved problems in geometry / Hallard T. Croft, Kenneth J. Falconer, Richard K. Guy.
p. cm. — (Problem books in mathematics) (Unsolved problems in intuitive mathematics : v. 2)
Includes bibliographical references and index.
1. Geometry. I. Falconer, K. J., 1952– . II. Guy, Richard K., 1916– III. Title. IV. Series. V. Series: Guy, Richard K. Unsolved problems in intuitive mathematics : v. 2.
QA43.G88 vol. 2
[QA447]
510 s—dc20 90-25041
[516]

ISBN 978-1-4612-6962-5 ISBN 978-1-4612-0963-8 (eBook)
DOI 10.1007/978-1-4612-0963-8

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

Contents

Preface v

Other Problem Collections ix

Standard References xi

Notation and Definitions 1

Sets. *1* Geometrical transformations. *3* Length, area, and volume. *4*

A. Convexity 6

A1. The equichordal point problem. *9* **A2.** Hammer's x-ray problems. *11* **A3.** Concurrent normals. *14* **A4.** Billiard ball trajectories in convex regions. *15* **A5.** Illumination problems. *18* **A6.** The floating body problem. *19* **A7.** Division of convex bodies by lines or planes through a point. *20* **A8.** Sections through the centroid of a convex body. *21* **A9.** Sections of centro-symmetric convex bodies. *22* **A10.** What can you tell about a convex body from its shadows? *23* **A11.** What can you tell about a convex body from its sections? *24* **A12.** Overlapping convex bodies. *25* **A13.** Intersections of congruent surfaces. *26* **A14.** Rotating polyhedra. *26* **A15.** Inscribed and circumscribed centro-symmetric bodies. *27* **A16.** Inscribed affine copies of convex bodies. *28* **A17.** Isoperimetric inequalities and extremal problems. *28* **A18.** Volume against width. *29* **A19.** Extremal problems for elongated sets. *30* **A20.** Dido's problem. *30* **A21.** Blaschke's problem. *32* **A22.** Minimal bodies of constant width. *34* **A23.** Constrained isoperimetric problems. *34* **A24.** Is a body fairly round if all its sections are? *35* **A25.** How far apart can various centers be? *36* **A26.** Dividing up a piece of land by a short fence. *37* **A27.** Midpoints of diameters of sets of constant width. *38* **A28.** Largest convex hull of an arc of a given length. *38* **A29.** Roads on planets. *39* **A30.** The shortest curve cutting all the lines through a disk. *39* **A31.** Cones based on convex sets. *41* **A32.** Generalized ellipses. *42* **A33.** Conic sections through five points. *43* **A34.** The shape of worn stones. *43* **A35.** Geodesics. *44* **A36.** Convex sets with universal sections. *45* **A37.** Convex space-filling curves. *46* **A38.** *m*-convex sets. *46*

B. Polygons, Polyhedra, and Polytopes 48

B1. Fitting one triangle inside another. *50* **B2.** Inscribing polygons in curves. *51* **B3.** Maximal regular polyhedra inscribed in regular polyhedra. *52* **B4.** Prince Rupert's problem. *53* **B5.** Random polygons and polyhedra. *54* **B6.** Extremal problems for polygons. *57* **B7.** Longest chords of polygons. *58* **B8.** Isoperimetric inequalities for polyhedra. *58* **B9.** Inequalities for sums of edge lengths of polyhedra. *59* **B10.** Shadows of polyhedra. *60* **B11.** Dihedral angles of polyhedra. *61* **B12.** Monostatic polyhedra. *61* **B13.** Rigidity of polyhedra. *61* **B14.** Rigidity of frameworks. *63* **B15.** Counting polyhedra. *65* **B16.** The sizes of the faces of a polyhedron. *68* **B17.** Unimodality of *f*-vectors of polytopes. *69* **B18.** Inscribable and circumscribable polyhedra. *70* **B19.** Truncating polyhedra. *72* **B20.** Lengths of paths on polyhedra. *72* **B21.** Nets of polyhedra. *73* **B22.** Polyhedra with congruent faces. *75* **B23.** Ordering the faces of a polyhedron. *75* **B24.** The four color conjecture for toroidal polyhedra. *75* **B25.** Sequences of polygons and polyhedra. *76*

C. Tiling and Dissection 79

C1. Conway's fried potato problem. *80* **C2.** Squaring the square. *81* **C3.** Mrs. Perkins's quilt. *83* **C4.** Decomposing a square or a cube into *n* smaller ones. *85* **C5.** Tiling with incomparable rectangles and cuboids. *85* **C6.** Cutting up squares, circles, and polygons. *87* **C7.** Dissecting a polygon into nearly equilateral triangles. *89* **C8.** Dissecting the sphere into small congruent pieces. *90* **C9.** The simplexity of the *d*-cube. *90* **C10.** Tiling the plane with squares. *91* **C11.** Tiling the plane with triangles. *92* **C12.** Rotational symmetries of tiles. *93* **C13.** Tilings with a constant number of neighbors. *94* **C14.** Which polygons tile the plane? *95* **C15.** Isoperimetric problems for tilings. *96* **C16.** Polyominoes. *96* **C17.** Reptiles. *99* **C18.** Aperiodic tilings. *101* **C19.** Decomposing a sphere into circular arcs. *103* **C20.** Problems in equidecomposability. *104*

D. Packing and Covering 107

D1. Packing circles, or spreading points, in a square. *108* **D2.** Spreading points in a circle. *110* **D3.** Covering a circle with equal disks. *111* **D4.** Packing equal squares in a square. *111* **D5.** Packing unequal rectangles and squares in a square. *112* **D6.** The Rados' problem on selecting disjoint squares. *113* **D7.** The problem of Tammes. *114* **D8.** Covering the sphere with circular caps. *116* **D9.** Variations on the

penny-packing problem. *117* **D10.** Packing balls in space. *118*
D11. Packing and covering with congruent convex sets. *119* **D12.** Kissing numbers of convex sets. *121* **D13.** Variations on Bang's plank theorem. *121* **D14.** Borsuk's conjecture. *123* **D15.** Universal covers. *125* **D16.** Universal covers for several sets. *127* **D17.** Hadwiger's covering conjecture. *128* **D18.** The worm problem. *129*

E. Combinatorial Geometry 131

E1. Helly-type problems. *131* **E2.** Variations on Krasnosel'skiĭ's theorem. *133* **E3.** Common transversals. *135* **E4.** Variations on Radon's theorem. *136* **E5.** Collections of disks with no three in a line. *137* **E6.** Moving disks around. *138* **E7.** Neighborly convex bodies. *139* **E8.** Separating objects. *141* **E9.** Lattice point problems. *143* **E10.** Sets covering constant numbers of lattice points. *143* **E11.** Sets that can be moved to cover several lattice points. *144* **E12.** Sets that always cover several lattice points. *145* **E13.** Variations on Minkowski's theorem. *146* **E14.** Positioning convex sets relative to discrete sets. *148*

F. Finite Sets of Points 149

F1. Minimum number of distinct distances. *150* **F2.** Repeated distances. *151* **F3.** Two-distance sets. *152* **F4.** Can each distance occur a different number of times? *153* **F5.** Well-spaced sets of points. *154* **F6.** Isosceles triangles determined by a set of points. *154* **F7.** Areas of triangles determined by a set of points. *155* **F8.** Convex polygons determined by a set of points. *155* **F9.** Circles through point sets. *156* **F10.** Perpendicular bisectors. *157* **F11.** Sets cut off by straight lines. *157* **F12.** Lines through sets of points. *158* **F13.** Angles determined by a set of points. *160* **F14.** Further problems in discrete geometry. *161* **F15.** The shortest path joining a set of points. *162* **F16.** Connecting points by arcs. *164* **F17.** Arranging points on a sphere. *165*

G. General Geometric Problems 168

G1. Magic numbers. *168* **G2.** Metrically homogeneous sets. *169* **G3.** Arcs with increasing chords. *169* **G4.** Maximal sets avoiding certain distance configurations. *170* **G5.** Moving furniture around. *171* **G6.** Questions related to the Kakeya problem. *173* **G7.** Measurable sets and lines. *175* **G8.** Determining curves from intersections with lines. *175* **G9.** Two sets which always intersect in a point. *176* **G10.** The chromatic number of the plane and of space. *177* **G11.** Geometric graphs. *180* **G12.** Euclidean Ramsey problems. *181* **G13.** Triangles with vertices in sets of a given area. *182* **G14.** Sets containing large triangles. *182* **G15.** Similar copies of sequences. *183* **G16.** Unions of similar copies of sets. *184*

Index of Authors Cited 185

General Index 195

Notation and Definitions

The following brief synopsis serves to introduce some of the notation and terminology that will be needed throughout the book. More specific notions are discussed in the chapter introductions, or, in some cases, in the individual sections.

Sets

We shall require some definitions and notation from set theory.

Most of our problems are posed in d-**dimensional Euclidean space**, $\mathbb{R}^d$; in particular $\mathbb{R}^1 = \mathbb{R}$ is just the set of real numbers or the real line, $\mathbb{R}^2$ is the (Euclidean) plane, and $\mathbb{R}^3$ is usual (Euclidean) space. Points in $\mathbb{R}^d$ are printed in bold type $\mathbf{x}$, $\mathbf{y}$, etc, and we will sometimes use the coordinate form $\mathbf{x} = (x_1, \ldots, x_d)$. If $\mathbf{x}$ and $\mathbf{y}$ are points of $\mathbb{R}^d$, the **distance** between them is $|\mathbf{x} - \mathbf{y}| = (\sum_{i=1}^{d} |x_i - y_i|^2)^{1/2}$.

Sets, which will generally be subsets of $\mathbb{R}^d$, are denoted by capital letters (e.g., E, F, K, etc). In the usual way, $\mathbf{x} \in E$ means that the point $\mathbf{x}$ is a member of the set E, and $E \subset F$ means that E is a subset of F. We write $\{\mathbf{x}\colon \text{condition}\}$ for the set of $\mathbf{x}$ for which "condition" is true. The **empty set**, which contains no elements, is written $\varnothing$. The set of integers is denoted by $\mathbb{Z}$ and the rational numbers by $\mathbb{Q}$. We sometimes use a superscript $^+$ to denote the positive elements of a set (e.g., $\mathbb{R}^+$ is the set of positive real numbers).

The **closed ball** of center $\mathbf{x}$ and radius r is defined by $B_r(\mathbf{x}) = \{\mathbf{y}\colon |\mathbf{y} - \mathbf{x}| \leq r\}$. Similarly, the **open ball** is $\{\mathbf{y}\colon |\mathbf{y} - \mathbf{x}| < r\}$. Thus the closed ball contains its bounding sphere, but the open ball does not. Of course, in $\mathbb{R}^2$ a ball is a disk, and in $\mathbb{R}^1$ a ball is just an interval. If $a < b$, we write $[a, b]$ for the **closed interval** $\{x\colon a \leq x \leq b\}$ and (a, b) for the **open interval** $\{x\colon a < x < b\}$.

We write $E \cup F$ for the **union** of the sets E and F (i.e., the set of points belonging to either E or F). Similarly, we write $E \cap F$ for their **intersection** (i.e., the points in both E and F). More generally $\bigcup_i E_i$ denotes the **union** of an arbitrary collection of sets $\{E_i\}$ (i.e., those points in at least one E_i) and $\bigcap_i E_i$ denotes their **intersection**, consisting of the points common to all of the sets E_i. A collection of sets is **disjoint** if the intersection of any pair is the empty set. The **difference** $E \backslash F$ consists of those points in E that are not in F, and $\mathbb{R}^d \backslash E$ is called the **complement** of E.

An infinite set E is **countable** if its elements can be listed in the form $\mathbf{x}_1, \mathbf{x}_2, \ldots$ with every element of E appearing at a specific place in the list; otherwise the set is **uncountable**. The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable but $\mathbb{R}$ is uncountable.

If E is any set of real numbers, the **supremum**, sup E, is the least number m such that $x \le m$ for every x in E. Similarly, the **infimum**, inf E, is the greatest number m such that $m \le x$ for every x in E. Roughly speaking, we think of inf E and sup E as the minimum and maximum of the numbers in E, though it should be emphasized that inf E and sup E need not themselves be in E.

We use the "floor" and "ceiling" symbols "$\lfloor\ \rfloor$" and "$\lceil\ \rceil$" to mean "the greatest integer not more than" and "the least integer not less than."

The **diameter**, diam E, of a subset E of $\mathbb{R}^d$ is the greatest distance apart of pairs of points in E; thus diam $E = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in E\}$. A set A is **bounded** if it has finite diameter, or, equivalently, is contained in some (sufficiently large) ball.

We have already used the terms "open" and "closed" in connection with intervals and balls, but these notions extend to much more general sets. Intuitively, a set is closed if it contains its boundary and open if it contains none of its boundary points. More precisely, a subset E of $\mathbb{R}^d$ is **open** if, for every $\mathbf{x}$ in E, there is some ball $B_r(\mathbf{x})$ of positive radius r, centered at $\mathbf{x}$ and contained in E. A set E is **closed** if its complement is open; equivalently if for every sequence $\mathbf{x}_r$ in E that is convergent to a point $\mathbf{x}$ of $\mathbb{R}^d$, we have $\mathbf{x}$ in E. The empty set $\varnothing$ and $\mathbb{R}^d$ are regarded as both open and closed. The union of any collection of open sets is open, as is the intersection of a *finite* collection of open sets. The intersection of any collection of closed sets is closed, as is the union of a *finite* number of closed sets.

The smallest closed set containing a set E, more precisely, the intersection of all closed sets that contain E, is called the **closure** of E. Similarly, the **interior** of a set E is the largest open set contained in E, that is the union of all open subsets of E. The **boundary** of E is defined as the set of points in the closure of E but not in its interior.

For our purposes, a subset of $\mathbb{R}^d$ is **compact** if it is closed and bounded.

A set E is thought of as connected if it consists of just one "piece"; formally E is **connected** if there do not exist open sets U and V such that $U \cup V$ contains E and with $E \cap U$ and $E \cap V$ disjoint and nonempty. A subset E of $\mathbb{R}^2$ is termed **simply connected** if both E and $\mathbb{R}^2 \backslash E$ are connected.

There is a further class of sets that will be mentioned occasionally, though its precise definition is indirect, and need not unduly concern the reader. The

Borel sets are, roughly speaking, the sets that can be built up from open or closed sets by repeatedly taking countable unions and intersections. More precisely, the class $\mathscr{B}$ of **Borel sets** in $\mathbb{R}^d$ is the smallest collection of sets that includes the open and closed sets, such that if $E, E_1, E_2, \ldots$ are in $\mathscr{B}$ then so are $\bigcup_{i=1}^{\infty} E_i$, $\bigcap_{i=1}^{\infty} E_i$ and $\mathbb{R}^d \setminus E$.

Occasionally we need to indicate the degree of smoothness of a curve or surface. We say that such a set is C^k ($k = 1, 2 \ldots$) if it may be defined locally, with respect to suitable coordinate axes, by a function that is k times differentiable with continuous kth derivative. A curve or surface is C^∞ if it is C^k for every positive integer k.

The notation $f(x) = o(g(x))$ means that $f(x)/g(x) \to 0$ as $x \to \infty$, and $f(x) = O(g(x))$ means that there is a constant c such that $|f(x)| \le c|g(x)|$ for all sufficiently large x. Similarly, $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.

Geometrical transformations

Let E and F be any sets. A **mapping, function**, or **transformation** f from E to F is a rule or formula that associates a point $f(\mathbf{x})$ of F with each point $\mathbf{x}$ of E. We write $f: E \to F$ to denote this situation. If $A \subset E$, we write $f(A) = \{f(\mathbf{x}): \mathbf{x} \in A\}$ for the **image** of A.

A function $f: E \to F$ is called an **injection** or **one-to-one** function if $f(\mathbf{x}) \ne f(\mathbf{y})$ whenever $\mathbf{x} \ne \mathbf{y}$ (i.e., if different elements of E are mapped to different elements of F). A function is called a **surjection** or an **onto** function if, for every $\mathbf{y} \in F$, there exists $\mathbf{x} \in E$ such that $f(\mathbf{x}) = \mathbf{y}$. A function that is both an injection and a surjection is called a **bijection** or a **one-to-one correspondence** between E and F.

Certain transformations have particular geometric significance. A transformation $S: \mathbb{R}^d \to \mathbb{R}^d$ is called a **congruence** or **isometry** if it preserves distances (i.e., if $|S(\mathbf{x}) - S(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$). Such a transformation also preserves angles and transforms sets into **congruent** ones. Special cases include **translations**, which shift points a constant distance in parallel directions, **rotations**, which have a center $\mathbf{a}$ such that $|S(\mathbf{x}) - \mathbf{a}| = |\mathbf{x} - \mathbf{a}|$ for all $\mathbf{x}$, and **reflections**, which map all points to their mirror images in a fixed $(d-1)$-dimensional plane. A congruence that may be achieved by a translation followed by a rotation is sometimes called a **rigid motion** or **direct congruence**. A transformation S is a **similarity** if there is a positive constant c such that $|S(\mathbf{x}) - S(\mathbf{y})| = c|\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, and transforms each set E into a **similar** set $S(E)$. A similarity that preserves orientation (i.e., for which the line segments $[S(\mathbf{x}), S(\mathbf{y})]$ and $[\mathbf{x}, \mathbf{y}]$ are parallel) is called a **homothety** and E and $S(E)$ are termed **homothetic**. An **affinity** or **affine transformation** transforms straight lines to straight lines and may be thought of as a shearing transformation; the contracting or expanding effect need not be the same in every direction. The effect of these transformations is shown in Figure N1.

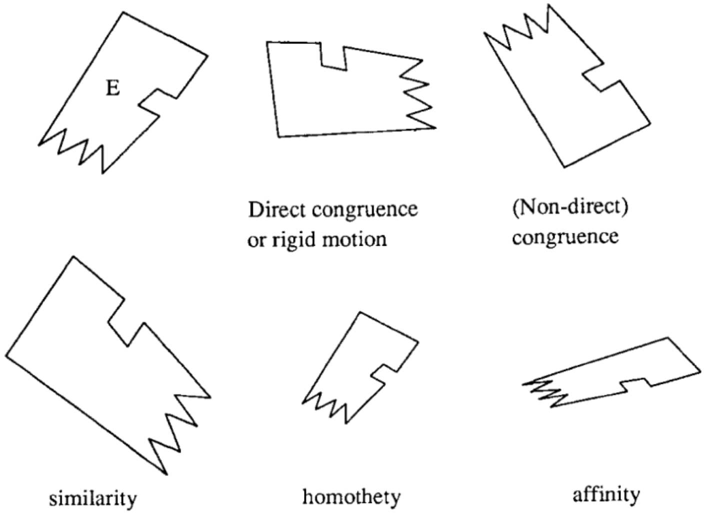

Figure N1. The effect of various transformations on a set E.

Length, area, and volume

For most of this book, an intuitive idea of length, area, and volume will be perfectly adequate. However, a few problems involve sets that may be rather irregular, and precise formulation of length, area, and volume requires a few ideas from measure theory.

If E is a subset of $\mathbb{R}$, we define the **length** or (one-dimensional) **Lebesgue measure** $L(E)$ of E as the infimum (i.e., the smallest possible value) of the sums $\sum_{i=1}^{\infty}(b_i - a_i)$ over all countable collections of intervals $\bigcup_{i=1}^{\infty}[a_i, b_i]$ that cover E. If E itself consists of a finite or countable collection of intervals, then $L(E)$ equals the sum of the interval lengths. It turns out that it is not possible to define $L(E)$ consistently on all subsets of $\mathbb{R}$, but only on a rather large class of subsets called the **Lebesgue measurable sets**. Intervals, open and closed sets, and Borel sets are all Lebesgue measurable, and finite and countable unions and complements of measurable sets are always measurable. Length is additive, indeed countably additive, on such sets, in the sense that if $E_1, E_2, \ldots$ are disjoint measurable sets, then $L(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} L(E_i)$. Any sets that can be constructed "effectively" (i.e., by specifying exactly which points are in the set and not resorting to an axiom such as the axiom of choice) are measurable, so for intuitive purposes, we may think of Lebesgue measure as length in the obvious way.

Similarly, we can make the ideas of area or volume precise by introducing 2- or 3-dimensional Lebesgue measure. Thus if $E \subset \mathbb{R}^2$, we define the **area** or **plane Lebesgue measure** of E to be $A(E)$, the infimum of the sums $\sum_{i=1}^{\infty}(b_i - a_i)(d_i - c_i)$ taken over all countable unions of rectangles

$\bigcup_{i=1}^{\infty} [a_i, b_i] \times [c_i, d_i]$ that cover E. As in the one-dimensional case, $A(E)$ is defined consistently on the familiar types of sets. Volume of sets in $\mathbb{R}^3$, and more generally, d-dimensional volume of subsets of $\mathbb{R}^d$, are defined analogously.

For a full treatment of measure theory, see, for example, Kingman & Taylor.

J. F. C. Kingman & S. J. Taylor, *Introduction to Measure and Probability*, Cambridge University Press, Cambridge, 1966; *MR* **36** #1601.

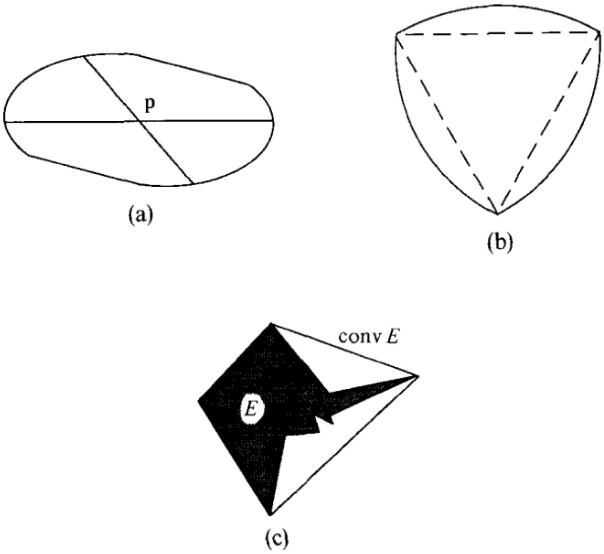

Figure A3. (a) A centro-symmetric convex set with center **p**. (b) The Reuleux triangle, with each arc centered at the opposite vertex, is a set of constant width. (c) A set E and its convex hull conv E.

triangle, consisting of three circular arcs centered at the opposite vertices [Figure A3(b)] is of constant width.

The **convex hull**, conv E, of any set E is the smallest convex set that contains E [i.e., the intersection of all convex sets containing E, which is necessarily convex, see Figure A.3(c)].

The **dimension**, dim K, of a convex set K in $\mathbb{R}^d$ is the least integer s such that K is contained in an s-dimensional flat (i.e., translate of an s-dimensional subspace). Thus a convex set in 3-dimensional space is 0-dimensional if it is a single point, 1-dimensional if it is a line segment, 2-dimensional if it is contained in a plane but not in a line, and 3-dimensional if it contains a ball of positive radius. It is sometimes convenient to define the dimension of an arbitrary set E as the dimension of its convex hull, and write dim E = dim conv E. Very often questions in convexity are only of interest for **proper** convex sets in $\mathbb{R}^d$, i.e., those with dim $E = d$, or, equivalently, those with nonempty interior.

We list below the standard references on convexity.

R. V. Benson, *Euclidean Geometry and Convexity*, McGraw-Hill, New York, 1966; *MR* **35** #844.
W. Blaschke, [Bla].
T. Bonnesen & W. Fenchel, [BF].
H. Busemann, *Convex Surfaces*, Interscience, New York, 1958; *MR* **21** #3900.
H. G. Eggleston, *Convexity*, Cambridge University Press, Cambridge, 1958; *MR* **23** #A2123.
H. G. Eggleston, *Problems in Euclidean Space—Application of Convexity*, Pergamon, New York, 1957; *MR* **23** #A3228.

L. Fejes Tóth, [Fej].
L. Fejes Tóth, [Fej′].
H. Guggenheimer, *Applicable Geometry—Global and Local Convexity*, Krieger, New York, 1977; *MR* **56** #1198.
H. Hadwiger, [Had].
H. Hadwiger, H. Debrunner & V. Klee, [HDK].
H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957; *MR* **21** #1561.
P. J. Kelly & M. L. Weiss, *Geometry and Convexity*, Wiley, New York, 1979; *MR* **80h**:52001.
S. R. Lay, *Convex Sets and their Applications*, Wiley, New York, 1982; *MR* **83e**:52001.
K. Leichtweiss, *Konvexe Mengen*, Springer, Berlin, 1980; *MR* **81j**:52001.
L. A. Lyusternik, *Convex Figures and Polyhedra*, Dover, New York, 1963, Heath, Boston, 1966; *MR* **19**, 57; **28** #4427; **36** #4435.
F. A. Valentine, *Convex Sets*, McGraw-Hill, New York, 1964; *MR* **30** #503.
I. M. Yaglom & V. G. Boltyanskii, *Convex Figures*, Moscow, 1951; English transl. Holt, Rinehart, and Winston, New York, 1961; *MR* **23** #A1283.

The following general articles also provide an introduction to aspects of convexity:

V. Klee, What is a convex set? *Amer. Math. Monthly* **78** (1971) 616–631; *MR* **44** #3202.
J. Dubois, Sur la convexité et ses applications, *Ann. Sci. Math. Québec* **1** (1977) 7–31; *MR* **58** #24002.
P. M. Gruber, Seven small pearls from convexity, *Math. Intelligencer* **5** (1983) No. 1, 16–19; *MR* **85h**:52001.
P. M. Gruber, Aspects of convexity and its applications, *Exposition. Math.* **2** (1984) 47–83; *MR* **86f**:52001.

The following volumes of conference proceedings, which are detailed in full under "Standard References" on pages xi–xii, will be invaluable to any serious student of convexity: [BF], [DGS], [Fen], [GLMP], [GW], [Kay], [Kle], [KB], [RZ], [TW].

A1. The equichordal point problem. Perhaps the most notorious of all problems in plane convexity was posed by Fujiwara and by Blaschke, Rothe & Weitzenböck in 1917. Is there a plane convex set having two distinct equichordal points? An **equichordal point** has the property that every chord through it has the same length, which we may take to equal 1 (see Figure A4). A number of incorrect "proofs" of the conjecture have been published, but a complete solution still seems far way. To quote Rogers: "If you are interested in studying the problem, my first advice is 'Don't'. My second is 'If you must, do study the work of Wirsing and Butler,' and the third is 'You may well have to develop a sophisticated technique for obtaining uniform and extremely accurate asymptotic expansions for the solutions of a certain recurrence relation giving sequences of points on the boundary of such sets.'"

Quite a bit is known about sets with two equichordal points *if* they exist. Wirsing showed that they must be symmetric about the line L through the two points and also about their perpendicular bisector. He also showed that

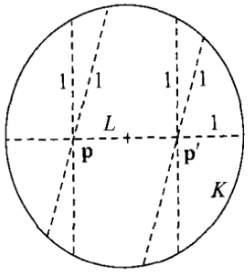

Figure A4. **p** and **p**′ are equichordal points of K, with all chords shown of length one.

the boundary must be a real-analytic curve, and gave a recurrence relationship for the coefficients of its power-series expansion near its intersections with L. It follows that there are at most countably many such sets to within congruence. Calling a the distance between the equichordal points, Ehrhart has shown that $a < 0.5$ and Michelacci has shown that a must be one of a discrete set of numbers with $a < 0.33$. Extensive computation is required in this work.

Wirsing has suggested the following generalized problem: Let C be a closed convex curve, symmetrical about the origin $\mathbf{o}$. Let C_+, C_- be C shifted a distance to the right and left, respectively. Let $f_+(\theta)$ and $f_-(\theta)$ be the defining functions for C_+ and C_- in polar coordinates with respect to $\mathbf{o}$. Characterize the curves that we can obtain as $r(\theta) = \frac{1}{2}\{f_+(\theta) + f_-(\theta)\}$. If we can get a circle, then C has two equichordal points.

The corresponding problem on the surface of a sphere turns out to be easier. Spaltenstein specified a family of convex sets on the sphere each with two equichordal points. (Here "convexity" and "chord" are defined in terms of segments of great circle arcs). Similarly, Petty & Crotty showed that there are (real) normed spaces in which there are convex sets with two equichordal points.

Klee asked the related question whether there exist nonelliptical convex curves C with two equireciprocal points. (A point $\mathbf{p}$ is **equireciprocal** if every chord $[\mathbf{x}, \mathbf{y}]$ of C through $\mathbf{p}$ satisfies $|\mathbf{x} - \mathbf{p}|^{-1} + |\mathbf{y} - \mathbf{p}|^{-1} = c$ for some constant c.) Falconer, see also Hallstrom, showed that, except for certain unlikely possibilities, any curve with two equireciprocal points must have the same value of c at each point. Further, any twice differentiable convex curve with two equireciprocal points must be an ellipse, but on the other hand there are nonelliptical convex curves with two equireciprocal points. One can generalize the problem to seek curves with pairs of points satisfying $|\mathbf{x} - \mathbf{p}|^\alpha + |\mathbf{y} - \mathbf{p}|^\alpha = c$ for any α. Do such curves exist for α other than -1 and 0?

W. Blaschke, W. Rothe & R. Weitzenböck, Aufgabe 552, *Arch. Math. Phys.* **27** (1917) 82.

G. J. Butler, On The "equichordal curve" problem and a problem of packing and covering, PhD thesis, London, 1969.

G. A. Dirac, Ovals with equichordal points, *J. London Math. Soc.* **27** (1952) 429–437; **28** (1953) 256; *MR* **14**, 309.

L. Dulmage, Tangents to ovals with two equichordal points, *Trans. Roy. Soc. Canada Sect. III* (3) **48** (1954) 7–10; *MR* **16**, 740.
E. Ehrhart, Un ovale à deux points isocordes? *Enseign. Math.* **13** (1967) 119–124; *MR* **37** #823.
K. J. Falconer, On the equireciprocal point problem, *Geom. Dedicata* **14** (1983) 113–126; *MR* **84i**:52004.
M. Fujiwara, Über die Mittelkurve zweier geschlossenen konvexen Kurven in Bezug auf einen Punkt, *Tôhoku Math. J.* **10** (1916) 99–103.
R. J. Gardner, Chord functions of convex bodies, *J. London Math. Soc.* (2) **36** (1987) 314–326; *MR* **88h**:52006.
H. Hadwiger, Ungelöste Problem 3, *Elem. Math.* **10** (1955) 10–19.
A. P. Hallstrom, Equichordal and equireciprocal points, *Bogasici Univ. J. Sci.* **2** (1974) 83–88.
V. Klee, Can a plane convex body have two equireciprocal points? *Amer. Math. Monthly* **76** (1969) 54–55, correction **78** (1971) 1114.
D. G. Larman & N. K. Tamvakis, A characterization of centrally symmetric convex bodies in E^n, *Geom. Dedicata* **10** (1981) 161–176; *MR* **82j**:52008.
G. Michelacci, A negative answer to the equichordal problem for not too small excentricities, *Bol. Un. Mat. Ital. A*(7) **2** (1988) 203–211; *MR* **89f**:52003.
C. M. Petty & J. M. Crotty, Characterization of spherical neighbourhoods, *Canad. J. Math.* **22** (1970) 431–435; *MR* **41** #2538.
C. A. Rogers, Some problems in the geometry of convex bodies, in [DGS], 279–284; *MR* **84b**:52001.
C. A. Rogers, An equichordal problem, *Geom. Dedicata* **10** (1981) 73–78; *MR* **82j**:52009.
N. Spaltenstein, A family of curves with two equichordal points on a sphere, *Amer. Math. Monthly* **91** (1984) 423.
W. Suss, Einbereiche mit ausgezeichneten Punkten, Inhalts, und Umfangspunkt, *Tôhoku Math. J.* (1) **25** (1925) 86–98.
E. Wirsing, Zur Analytisität von Doppelspeichkurven, *Arch. Math.* **9** (1958) 300–307; *MR* **21** #2205.

A2. Hammer's x-ray problems. Suppose a homogeneous solid contains a convex hole K and x-ray photographs are taken so that the "darkness" at each point on a photograph determines the length of the chord of K along an x-ray line [see Figure A5(a)]. How many pictures must be taken to permit exact reconstruction of K if

(a) the x-rays issue from a point source, or
(b) the x-rays are assumed parallel?

In the plane case the problems may be expressed mathematically as follows:

(a′) We are given points $\mathbf{x}_1, \dots, \mathbf{x}_k$ and functions $f_1, \dots, f_k$: $[0, \pi) \to \mathbb{R}$, and seek a compact convex set K such that K intersects the line through $\mathbf{x}_i$ making an angle θ with some fixed axis in a chord of length $f_i(\theta)$ [Figure A5(b)]. (We say that K has **chord function** f_i at $\mathbf{x}_i$.) For the sake of generality we allow the $\mathbf{x}_i$ to be either interior or exterior to K.

(b′) We are given angles $\theta_1, \dots, \theta_k$ and functions $F_1, \dots, F_k$: $\mathbb{R} \to \mathbb{R}$, and seek a compact convex set K such that K intersects the line in the direction θ_i and at perpendicular distance t from the origin in a chord of length $F_i(t)$ [Figure A5(c)].

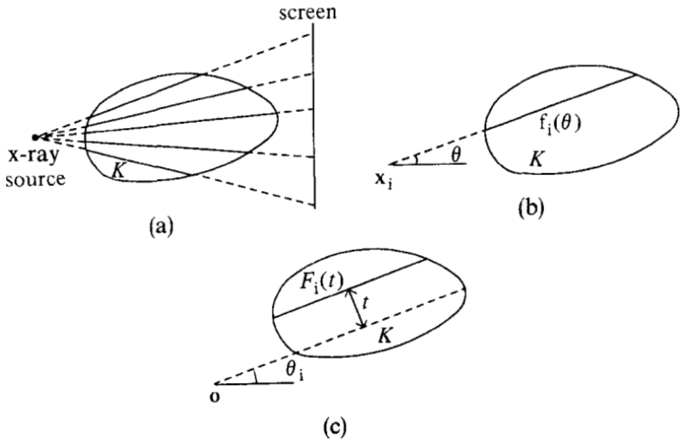

Figure A5. Hammer's x-ray problem. (a) The intensity of the shadow on the screen determines the lengths of chords of K. (b) The point source problem. (c) The parallel chord problem.

There are a number of interesting questions which we state for case (a'); the analogs for (b') should be clear.

(i) Uniqueness. When does a set of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ have the property that at most one convex set K corresponds to any set of chord functions $f_1, \ldots, f_k$?

(ii) Reconstruction. Given that a set of points and chord functions correspond to a unique convex K, reconstruct K from this information.

(iii) Relative uniqueness. Given a convex set K, find a "small" set of points such that the chord functions at these points distinguish K from all other convex sets.

(iv) Existence. Given a set of points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ find necessary or sufficient conditions on functions $f_1, \ldots, f_k$ for there to exist a convex set with chord function f_i at $\mathbf{x}_i$ $(1 \leq i \leq k)$.

(v) Descriptive. Deduce qualitative properties of a convex set K from properties of its chord functions at a set of points. For instance, if $f_1, \ldots, f_k$ are all r times differentiable, then is the boundary of K an r times differentiable curve?

A good deal of progress has been made on these problems in the last few years, but there are still many open questions. For the point source problem, Falconer has shown, using methods from dynamical systems, that, given points $\mathbf{x}_1$ and $\mathbf{x}_2$ and functions f_1 and f_2, there is at most one convex set with $\mathbf{x}_1$ and $\mathbf{x}_2$ as interior points and chord functions f_1 and f_2 at these points. Moreover, in principle at least, the method is constructive. Similarly, there are at most two such convex sets K with $\mathbf{x}_1$ and $\mathbf{x}_2$ as exterior points and the line through $\mathbf{x}_1$ and $\mathbf{x}_2$ intersecting the interior of K. One possibility has $\mathbf{x}_1$ and $\mathbf{x}_2$ on the same side of K, the other on opposite sides. Surely it should always be possible logy. If K is centro-symmetric, there are d **double normals** (that is chords that are normal at both ends) through the center, so the conjecture holds in this case (see Lyusternik & Schnirelmann). If K has a twice differentiable boundary, and either the insphere or the circumsphere of K touches at exactly $d + 1$ points, it is obvious that $d + 1$ normals pass through the center of the sphere. However in this situation we do even better, since in fact there must be $2d + 2$ normals through the center.

Zamfirescu has shown, rather surprisingly, that, in the sense of Baire category based on the Hausdorff distance between convex sets, "most" interior points of most convex bodies lie on infinitely many normals. (By most we mean for all but a countable union of nowhere dense sets.) However, we should remember that in the category sense most convex bodies are "exceptional" in that they have highly irregular boundaries that are not even twice differentiable.

For bodies of constant width all normals are double normals. In this case we would expect to find an interior point that lies on normals from $4d - 2$ distinct boundary points; again this has been proved for $d = 2$ and 3.

In the plane, Guggenheimer asks the intriguing question of whether every point in a certain (curvilinear) triangular region must lie on normals from four boundary points.

N. Deo & M. S. Klamkin, Existence of four concurrent normals to a smooth closed curve, *Amer. Math. Monthly* **77** (1970) 1083–1084; *MR* **42** #6718.
H. W. Guggenheimer, Does there exist a "four normals triangle"? *Amer. Math. Monthly* **77** (1970) 177–179.
E. Heil, Concurrent normals and critical points under weak smoothness assumptions, [GLMP] 170–178; *MR* **87d**:52006.
E. Heil, Existenz eines 6-Normalenpunktes in einem konvexen Körper, *Arch. Math. (Basel)* **32** (1979) 412–416, correction **33** (1979) 496; *MR* **80j**:52002.
L. A. Lyusternik & L. G. Schnirelmann, *Méthodes Topologiques dans les Problèmes Variationels*, Hermann, Paris, 1934.
T. Zamfirescu, Intersecting diameters in convex bodies, [RZ], 311–316; *MR* **87a**:52013.
T. Zamfirescu, Points on infinitely many normals to convex surfaces, *J. Reine Angew. Math.* **350** (1984) 183–187; *MR* **85d**:52003.

A4. Billiard ball trajectories in convex regions. Let K be a plane convex region with boundary curve C. An idealized point "billiard ball" travels across K in a straight line at constant speed and rebounds with equal angles of incidence and reflection on hitting C. Study of billiard ball trajectories involves complex ideas from ergodic theory and dynamical systems; we mention here a few of the more intuitive problems.

It should be noted that some care is required in setting up these problems. Halpern points out that even when C is a three times differentiable curve it is actually possible for the billiard to pass from inside to outside K if the angle of reflexion law is strictly adhered to! This paradox can occur when there are infinitely many bounces in a finite time, but is avoided if the third derivative of C is continuous. The two cases of particular interest are when C is smooth, say infinitely differentiable, and when C is a polygon. In the latter case some