WEIRD MATH

A TEENAGE GENIUS & HIS TEACHER REVEAL THE STRANGE CONNECTIONS BETWEEN MATH & EVERYDAY LIFE

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Go down deep enough in anything and you will find mathematics.

—DEAN SCHLICTER

The most incomprehensible thing about mathematics is that it is comprehensible.

—KIRAN MA

PREFACE

Math *is* weird. Numbers go on forever—and there are different kinds of forever. Prime numbers help cicadas survive. A (mathematical) ball can be cut up and then put back together, without any gaps, to make a ball twice the size, or a million times the size, of the original. There are shapes that have fractional dimensions and curves that fill a plane, leaving no holes. While bored by a dull presentation, physicist Stanislaw Ulam wrote out numbers, starting from 0, in a spiral form, marked in all the prime numbers, and found that many primes lie on long diagonals—a fact that is still not fully unexplained.

We forget sometimes how weird math is because we're so used to dealing with what seem like ordinary numbers and calculations, the stuff we learn about in school or use every day. Yet the fact that our brains are so adept at thinking mathematically, and, if we choose, to doing really complex and abstract math, is surprising. After all, our ancestors, tens or hundreds of thousands of years ago, didn't need to solve differential equations or dabble in abstract algebra in order to stay alive long enough to pass on their genes to the next generation. While they searched for their next meal or a place to shelter, there was nothing to be gained from musing about geometry in higher dimensions or theories of prime numbers. Yet we're born with brains that have the potential to do these things and to uncover, with each passing year, more and more extraordinary truths about the mathematical universe. Evolution has provided us with this skill, but how and why? Why are we, as a species, so good at doing something that has every appearance of being just an intellectual game?

Somehow math is woven into the very fabric of reality. Dig deep enough, and we find that what seemed to be tangible bits of matter or energy—electrons or photons, for instance—dissolve into immateriality, becoming mere waves of probability, and all we're left with is a ghostly calling card in the form of some intricate but beautiful set of equations. In some sense, mathematics underpins the physical world around us, forming an invisible infrastructure. Yet it also goes beyond this, into abstract realms of possibility that may forever remain purely exercises of the mind.

We've chosen in this book to highlight some of the more extraordinary and fascinating areas of math, including those where exciting new developments are in the offing. In some cases, they have links with science and technology—particle physics, cosmology, quantum computers, and the like. In others, they represent, for now at least, math for math's sake and are adventures into an unfamiliar land that exists only in the mind's eye. We've chosen not to shy away from certain subjects just because they're hard. One of the challenges in describing many aspects of math for a general audience is that they're far removed from everyday experience. But in the end, some way can always be found to link what today's explorers and pioneers at the frontiers of mathematics are doing with the world of the familiar, even if the language we have to use isn't as precise as academics would ideally choose. It's perhaps true to say that if something, however obscure, can't be explained reasonably well to a person of normal intelligence, then the explainer needs to improve their understanding!

This book came about in an unusual way. One of us (David) has been a science writer for more than thirty-five years and has written many books on astronomy, cosmology, physics, and philosophy, even an encyclopedia of recreational math. The other (Agnijo) is a brilliant young mathematician and child genius, with an IQ of at least 162, according to Mensa, who, at the time of writing, has just finished training in Hungary in preparation for the 2017 International Mathematics Olympiad. Agnijo started coming to David for tuition in math and science at the age of twelve. Three years later, we decided to write a book together.

We sat down and brainstormed the topics we wanted to cover. David, for instance, came up with higher dimensions, the philosophy of math, and the math of music, while Agnijo was keen to write about large numbers (his personal passion), computation, and the mysteries of primes. Right from the start, we chose to lean toward anything unusual or downright weird and to connect this weird math, where possible, with real-world issues and everyday experience. We also made a commitment not to shy away from subjects just because they were tough, adopting as a mantra that if we can't explain something in plain language, then we don't properly understand it. David generally took on the historical, philosophical, and anecdotal aspects of each chapter, while Agnijo grappled with the more technical aspects. Agnijo fact-checked David's work, and David combined all the writing into finished chapters. It all worked surprisingly well! We hope you enjoy the result.

A NOTE TO THE READER

In glancing through the pages of this book, you may notice that it contains some symbols, including x's, ω's (omegas), and even the odd $\aleph$ (aleph). You'll find an occasional equation or an unfamiliar-looking combination of characters, such as $3 \uparrow \uparrow 3 \uparrow \uparrow 3$ (especially in the chapters on large numbers and infinity). If you're a nonmathematician, don't be put off. They're just shorthand for ideas that, hopefully, we explain well enough in advance and thereby help us delve a little faster and deeper into the subject than would otherwise be possible. One of us (David) has taught math privately to students for many years and has yet to come across one who can't be good at it once they believe in themselves. The fact is that we're all natural mathematicians, whether we realize it or not. So, with that in mind, let's take the plunge....

1

THE MATH BEHIND THE WORLD

> Even stranger things have happened; and perhaps the strangest of all is the marvel that mathematics should be possible to a race akin to the apes.
>
> —ERIC T. BELL

> Physics is mathematical not because we know so much about the physical world, but because we know so little; it is only its mathematical properties that we can discover.
>
> —BERTRAND RUSSELL

In terms of intellectual ability, *Homo sapiens* hasn't changed much, if at all, over the past one hundred thousand years. Put children from the time when woolly rhinos and mastodons still roamed the earth into a present-day school, and they would develop just as well as typical twenty-first-century youngsters. Their brains would assimilate arithmetic, geometry, and algebra. And if they were so inclined, there would be nothing to stop them from delving deeper into the subject and someday perhaps becoming professors of math at Cambridge or Harvard.

Our neural apparatus evolved the potential to do advanced calculations, and understand such things as set theory and differential geometry, long before it was ever applied in this way. In fact, it seems a bit of a mystery *why* we have this innate talent for higher mathematics when it has no obvious survival value. At the same time, the reason our species emerged and endured is that it had an edge over its rivals in terms of intelligence and an ability to think logically, plan ahead, and ask "what if?" Lacking other survival skills, such as speed and strength, our ancestors were forced to rely on their cunning and foresight. A capacity for logical thought became our one great superpower, and from that, in time, flowed our ability to communicate in a complex way, to symbolize, and to make rational sense of the world around us.

Like all animals, we effectively do a lot of difficult math on the fly. The simple act of catching a ball (or avoiding predators or hunting a prey) involves solving multiple equations simultaneously at high speed. Try programming a robot to do the same thing, and the complexity of the calculations involved becomes clear. But the great strength of humans was their ability to move from the concrete to the abstract—to analyze situations, to ask if-then questions, to plan ahead.

The dawn of agriculture brought the need to accurately track the seasons, and the coming of trade and settled communities meant that transactions had to be carried out and accounts kept. For both of these practical purposes, calendars and business transactions, some kind of reckoning had to be developed, and so elementary math made its first appearance. One of the regions where it sprang up was the Middle East. Archaeologists have unearthed Sumerian clay trading tokens dating back to about 8000 BC that show that these people dealt with representations of number. But it seems that, at this early period, they didn't treat the concept as being separate from the thing being counted. For example, they had different-shaped tokens for different items, such as sheep or jars of oil. When a lot of tokens had to be exchanged between parties, the tokens were sealed inside containers called bullae, which had to be broken open to check the contents. Over time, markings began to appear on the bullae to indicate how many tokens were within. The symbolic representations then evolved into a written number system, while tokens became generalized for counting any kind of object and eventually morphed into an early form of coinage. Along the way, the concept of number became abstracted from the type of object being counted, so that, for example, five was five whether it referred to five goats or five loaves of bread.

The connection between math and everyday reality seems strong at this stage. Counting and record keeping are practical tools of the farmer and the merchant, and if these methods do the job, who cares about the philosophy behind it all? Simple arithmetic looks well rooted in the world "out there": one sheep plus one sheep is two sheep; two sheep plus two sheep is four sheep. Nothing could be more straightforward. But look more closely, and we see that already something a bit strange has happened. In saying "one sheep plus one sheep," there's the assumption that the sheep are identical or, at least for the purposes of counting, that any differences don't matter. But no two sheep are alike. What we've done is to abstract a perceived quality to do with the sheep—their "oneness," or apartness—and then operate on this quality with another abstraction, which we call addition. That's a big step. In practice, adding one sheep and one sheep might mean putting them together in the same field. But also in practice, the sheep are different and, digging a little deeper, what we call "sheep"—like anything else in the world—aren't really separate from the rest of the universe. On top of this, there's the slightly disturbing fact that what we take to be objects (such as sheep) "out there" are constructions in our brains built up from signals that enter our senses. Even if we grant that a sheep has some external reality, physics tells us that it's a hugely complicated, temporary assemblage of subatomic particles that's in constant flux. Yet somehow, in counting sheep we're able to ignore this monumental complexity or, rather, in everyday life not even be aware of it.

The Egyptians had a good understanding of practical mathematics and put this to effect in the construction of the Pyramid of Khafre at Giza, shown here together with the Sphinx.

Of all subjects, mathematics is the most precise and immutable. Science and other fields of human endeavor are, at best, approximations to some ideal and are always changing and evolving over time. As German mathematician Hermann Hankel pointed out: "In most sciences, one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure." From the outset, this difference between math and every other discipline is inevitable because math starts with the mind extracting what it recognizes as being most fundamental and constant among the messages it receives via the senses. This leads to the concepts of natural numbers, as a way of measuring quantity, and of addition and subtraction as basic ways of combining quantities. Oneness, twoness, threeness, and so on are seen as common features of collections of things, whatever those things happen to be and however different individuals of the same type of thing happen to be. So the fact that math has this eternal, adamantine quality to it is ensured from the start—and is its greatest strength.

Mathematics exists. Of that there's no doubt. Pythagoras's theorem, for instance, is somehow part of our reality. But where *does* it exist when it's not being used or being instantiated in some material form, and where *did* it exist many thousands of years ago, before anyone had thought about it? Platonists believe that mathematical objects, such as numbers, geometric shapes, and the relationships between them, exist independently of us, our thoughts and language, and the physical universe. Quite what sort of ethereal realm they inhabit isn't specified, but it's a common assumption that they're somehow "out there." Most mathematicians, it's probably fair to say, subscribe to this school of thought and therefore also to the belief that math is discovered rather than invented. Most, too, probably don't care very much for philosophizing and are happy just to get on with doing math, in the same way that the majority of physicists, working in the lab or solving theoretical problems, don't worry a lot about metaphysics. Still, the ultimate nature of things—in this case of mathematical things—is interesting, even if we never arrive at a final answer. Prussian mathematician and logician Leopold Kronecker thought that only whole numbers were given, or in his words: "God made the integers, all the rest is the work of man." English astrophysicist Arthur Eddington went further: "The mathematics is not there till we put it there." The debate about whether mathematics is invented or discovered, or is perhaps some combination of both, arising from a synergy of mind and matter, will no doubt rumble on and, in the end, may have no simple answer.

One fact is clear: if a piece of math has been proven to be true, it will remain true for all time. There's no matter of opinion about it or subjective influence. "I like mathematics," remarked Bertrand Russell, "because it is not human and has nothing particular to do with this planet or with the whole accidental universe." David Hilbert voiced something similar: "Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country." This impersonal, universal quality of math is its greatest strength yet doesn't, to the trained eye, detract from its aesthetic appeal. "Beauty is the first test: there is no permanent place in the world for ugly mathematics," remarked English mathematician G. H. Hardy. The same sentiment, but from the field of theoretical physics, was expressed by Paul Dirac: "It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power."

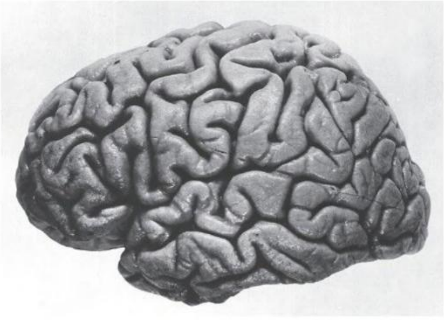
The flip side to the universality of math, however, is that it can seem cold and sterile, devoid of passion and feeling. As a result, we may find that although intelligent beings on other worlds share the same mathematics as us, it isn't the best way to communicate with them about a lot of the things that matter to us. "Many people suggest using mathematics to talk to the aliens," commented SETI (search for extraterrestrial intelligence) researcher Seth Shostak. In fact, the Dutch mathematician Hans Freudenthal developed an entire language (Lincos) based on this idea. "But," said Shostak, "my personal opinion is that mathematics may be a hard way to describe ideas like love or democracy."

The ultimate goal of scientists, certainly of physicists, is to reduce what they observe in the world to a mathematical description. Cosmologists, particle physicists, and the like are never happier than when they have measured and quantified things and then found a relationship between the quantities. The idea that the universe is mathematical at its core has ancient roots, stretching back at least as far as the Pythagoreans. Galileo saw the world as a "grand book" written in the language of mathematics, and, much more recently, in 1960, Hungarian-American physicist and mathematician Eugene Wigner wrote a paper called "The Unreasonable Effectiveness of Mathematics in the Natural Sciences."

We don't see numbers directly in the real world, so it isn't immediately obvious that math is all around us. But we do see shapes—the near-spherical shape of planets and stars, the curved path of objects when thrown or in orbit, the symmetry of snowflakes, and so on—and these can be described by relationships between numbers. Other patterns, translatable into math, emerge from the way electricity or magnetism behaves, galaxies rotate, and electrons operate within the confines of atoms. These patterns, and the equations describing them, underpin individual events and seem to represent deep, timeless truths underlying the changing complexity in which we find ourselves. German physicist

Heinrich Hertz, who first conclusively proved the existence of electromagnetic waves, remarked: “One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them.”

It’s unquestionably true that the bedrock of modern science is mathematical in nature. But that doesn’t necessarily mean that reality itself is fundamentally mathematical. Ever since the time of Galileo, science has separated the subjective from the objective, or measurable, and focused on the latter. It’s done its best to evict anything to do with the observer and to pay attention only to what it assumes lies beyond the interfering influence of the brain and senses. The way modern science has developed almost guarantees that it will be mathematical in nature. But this leaves much that science has trouble dealing with—most obviously consciousness. It may be that someday we’ll have a good, comprehensive model of how the brain works, in terms of memory, visual processing, and so forth. But why we also have an inner experience, a feeling of “what it is like to be,” remains—and may always remain—outside the field of conventional science and, by extension, of mathematics.



Why has the human brain evolved to be so extraordinarily good at a subject—mathematics—that it doesn’t need for survival?

On the one hand, Platonists believe math to be a land that already exists, awaiting our exploration of it. On the other, there are those who insist that we invent mathematics as we go along, to suit our purposes. Both positions have weaknesses. Platonists struggle to explain where things like pi might be outside the physical universe or the intelligent mind. Non-Platonists have a hard time denying the fact that, for example, planets would continue to orbit the sun in ellipses whether we do the math or not. A third school of mathematical philosophy occupies a middle ground between the two, by pointing out that, in describing the real world, math is not as successful as it's sometimes made out to be. Yes, equations are useful in telling us how to navigate a spacecraft to the moon or Mars, design a new aircraft, or predict the weather several days in advance. But these equations are mere approximations of the reality of what they're intended to describe, and, moreover, they apply to just a small portion of all the things going on around us. In touting the success of mathematics, the realist would say, we downplay the vast majority of phenomena that are too complex or poorly understood to capture in mathematical form or that, by their very nature, are irreducible to this kind of analysis.

Is it possible that the universe isn't, in reality, mathematical? After all, space and the objects it contains don't directly present anything mathematical to us. We humans rationalize and make approximations in order to model aspects of the universe. In doing so, we find mathematics extremely useful in enabling us to understand it. That doesn't necessarily imply that math is anything other than a convenience of our making. But if mathematics isn't present in the universe to start with, how is it that we're able to invent it in order to put it to such use?

Mathematics is broadly divided into two areas: pure and applied. Pure math is math for math's sake. Applied mathematicians put their subject to work on real-world problems. But often, developments in pure math, with seemingly no bearing on anything tangible, have turned out later to be surprisingly useful to scientists and engineers. In 1843 Irish mathematician William Hamilton hatched the idea of quaternions—four-dimensional generalizations of ordinary numbers of no practical interest at the time but that, more than a century later, have turned out to be an effective tool in robotics and computer graphics and games. A question first tackled by Johannes Kepler in 1611, about the most efficient way to pack spheres in three-dimensional space, has been applied to the efficient transmission of information over noisy channels. The purest mathematical discipline, number theory, much of which was thought to have little practical value, has led to important breakthroughs in the development of secure ciphers. And the new geometry pioneered by Bernhard Riemann that dealt with curved surfaces proved ideal for the formulation of Einstein's general theory of relativity—a new theory of gravity—more than fifty years later.

In July 1915, one of the greatest scientists of all time met one of the greatest mathematicians of the age when Einstein paid a visit to David Hilbert at the University of Göttingen. The following December, both published, almost simultaneously, the equations that described the gravitational field of Einstein's general theory. But whereas the equations themselves were the goal for Einstein, they were what Hilbert hoped would be a stepping-stone toward an even grander scheme. Hilbert's passion, the driving force behind much of his work, was a search for fundamental principles, or axioms, that might underlie all of mathematics. Part of this quest, as he saw it, was to find a minimum set of axioms from which he could deduce not only the equations of Einstein's general theory but any other theory in physics as well. Kurt Gödel, with his incompleteness theorems, undermined faith in the notion that math might have the answers to all questions. But we remain uncertain as to what extent the world in which we live is truly mathematical or just mathematical in appearance.

Whole swaths of mathematics may never be put to use, other than to help open up yet more avenues of pure research. On the other hand, for all we know, it may be that much of pure math is enacted, in unexpected ways, in the physical universe—or if not this universe, then in others that might exist throughout what cosmologists suspect is a multiverse of incomprehensible scale. Perhaps everything that is mathematically true and valid is represented somewhere, sometime, somehow in the reality in which we are embedded. For now there is the journey to keep us occupied: the weird and wonderful adventure of the human mind as it explores further the frontiers of number, space, and reason.

In the chapters that follow, we'll take a deep dive into subjects that are both bizarre and astonishing yet, at the same time, have very real connections with the world we know. True, some of the math may seem esoteric, fanciful, and even pointless, like some strange and convoluted game of the imagination. But at its core, mathematics is a practical affair, rooted in commerce, agriculture, and architecture. Although it has developed in ways that our ancestors could never have dreamed of, still those links with our everyday lives remain at its heart.

2

HOW TO SEE IN 4-D

> One of the strangest features of string theory is that it requires more than the three spatial dimensions that we see directly in the world around us. That sounds like science fiction, but it is an indisputable outcome of the mathematics of string theory.
>
> —BRIAN GREENE

We live in a world of three dimensions—up and down, side to side, and backward and forward, or any other three directions that are at right angles to each other. It's easy to imagine something in one dimension, such as a straight line, or two dimensions, such as a square drawn on a sheet of paper. But how can we possibly learn to see in an extra dimension than those we're familiar with? Where is this additional direction that's perpendicular to the three we know?

These questions may seem purely academic. If our world is three-dimensional, why worry about 4-D or 5-D and so forth? The fact is that science may need higher dimensions to explain what is going on at a subatomic level. These extra dimensions may hold the key to understanding the grand scheme of matter and energy. Meanwhile, on a more practical level, if we could learn to see in 4-D, we would have a powerful new tool to deploy in medicine and education.

Sometimes the fourth dimension is taken to be something other than an extra direction in space. After all, the word "dimension," from the Latin *dimensionem*, means simply a "measurement." In physics the basic dimensions that form the building blocks of other quantities are considered to be length, mass, time, and electric charge. Very often, in a different context, physicists talk about three dimensions of space and one of time, especially since Albert Einstein showed that, in the world in which we live, space and time are always bound up together in a single entity called spacetime. Even before the theory of relativity came along, however, there had been speculation about the possibility of being able to move backward and forward along the dimension of time just as we can move any way we like in space. In his novel *The Time Machine*, published in 1895, H. G. Wells explains that an instantaneous cube, for instance, can't exist. A cube that we see moment by moment is just a cross section of a four-dimensional thing having length, breadth, width, *and duration*. "There is no difference," says the Time Traveller, "between Time and any of the three dimensions of Space except that our consciousness moves along it."

The Victorians were also fascinated by the idea of a fourth dimension of space, both from a mathematical point of view and in the possibilities it seemed to offer of explaining another obsession of the age—spiritualism. The late 1800s was a period when many people, including luminaries such as Arthur Conan Doyle, Elizabeth Barrett Browning, and William Crookes, were attracted to the claims of mediums and the prospect of communicating with the dead. Might the afterlife, people wondered, exist in a fourth dimension that was parallel to, or overlapped with, our own so that spirits of the deceased could pass easily into our material realm and back again?

Our failure to be able to visualize in higher dimensions makes it tempting to think that the fourth dimension is somehow mysterious or alien to anything we know. Mathematicians, though, through. Likewise, if a 4-sphere were to intersect our space, we would see it as a dot that expanded, like a bubble, into a three-dimensional sphere of maximum size before shrinking and finally vanishing. The true nature—the extra dimensionality—of the 4-sphere would be hidden from us, although its mysterious appearance, growth, and disappearance would probably cause us to wonder what was going on!

Four-dimensional beings would have seemingly magical powers in our world. They could, for example, pick up a right-footed shoe, flip it over in the fourth dimension, and put it back as a left-footed one. If this seems hard to understand, think of a two-dimensional shoe, which would be like an infinitely thin sole shaped for one foot or the other. We could cut such a shape out of a piece of paper, lift it up, turn it over, and put it back down, so that we changed its footedness. A 2-D creature would find this utterly astonishing, but to us, with the benefit of the extra dimension, the trick would seem obvious.

In principle, a 4-D being could flip around a whole (3-D) person in the fourth dimension, although the absence of cases of people having suddenly had everything switched right to left or left to right suggests that this hasn't actually happened. In his short tale "The Plattner Story," H. G. Wells describes the remarkable case of Gottfried Plattner, a teacher who disappears for nine days following an explosion in a school chemistry lab. Upon his return, he is effectively a mirror image of his previous self, though his recollections of what happened during the period of absence are met with incredulity. Being flipped over for real in the fourth dimension would be bad for your health, apart from the shock of seeing yourself look different in the mirror (faces are surprising asymmetrical). Many of the crucial chemicals in our bodies, including glucose and most amino acids, have a certain handedness. Molecules of DNA, for example, which take the form of a double helix, always twist like a right-handed screw. If all these chemicals had their handedness reversed, we would quickly die of malnutrition because much of the essential nutrients in our food, from plants and animals, would now be in a form we couldn't assimilate.

Mathematical interest in a fourth spatial dimension began in the first half of the nineteenth century with the work of German mathematician Ferdinand Möbius. He's best remembered for his study of a shape that's now named after him—the Möbius band—and as a pioneer of the field known as topology. It was he who first realized that in a fourth dimension, a three-dimensional form could be rotated into its mirror image. In the second half of the nineteenth century, three mathematicians stood out as explorers of the new realm of multidimensional geometry: Swiss Ludwig Schläfli, Englishman Arthur Cayley, and German Bernhard Riemann.

Schläfli began his magnum opus, *Theorie der Vielfachen Kontinuität* (Theory of Continuous Manifolds), by saying, "The treatise... is an attempt to found and develop a new branch of analysis that would, as it were, be a geometry of n dimensions, containing the geometry of the plane and space as special cases for $n = 2, 3$." He went on to describe multidimensional analogues of polygons and polyhedrons, which he called "polyschemes." These are now commonly known as polytopes, a term coined by German mathematician Reinhold Hoppe and introduced to English researchers by Alicia Boole Stott, daughter of English mathematician and logician George Boole, who devised Boolean algebra, and Mary Everest Boole, a self-taught mathematician and writer on the subject (and George's wife).

Also to Schläfli's credit is the discovery of the higher-dimensional relatives of the Platonic solids. By Platonic solid we mean a convex shape (one with all the corners pointing outward) with regular polygon faces and the same number of faces meeting at each corner. There are five of them: the cube, tetrahedron, octahedron, (twelve-sided) dodecahedron, and (twenty-sided) icosahedron. The four-dimensional equivalents of the Platonic solids are the convex regular 4-polytopes (also called polychora), of which Schläfli found there were six, named after the number of cells they have. The simplest 4-polytope is the 5-cell, which has 5 tetrahedral cells, 10 triangular faces, 10 edges, and 5 vertices and is analogous to the tetrahedron. Then there is the 8-cell, or tesseract, and its "dual," the 16-cell, obtained by replacing cells with vertices, faces with edges, and vice versa. The 16-cell has 16 tetrahedral cells, 32 triangular faces, 24 edges, and 8 vertices and is the four-dimensional analogue of the octahedron. Two other 4-polytopes are the 120-cell, an analogue of the dodecahedron, and the 600-cell, an analogue of the icosahedron. Finally, there is a 24-cell, which has 24 octahedral cells and no three-dimensional counterpart. Interestingly, Schläfli found, the number of convex regular polytopes in all higher dimensions is the same—just three.

Through the work of Cayley, Riemann, and others, mathematicians learned how to do complex algebra in 4-D and branch out into multidimensional geometries that went beyond the rules prescribed by Euclid. But what they still couldn't do was actually see in four dimensions. Could anybody? This was a problem that intrigued British mathematician, teacher, and writer of scientific romances Charles Howard Hinton. In his twenties and early thirties, Hinton taught at two private schools in England: first at Cheltenham College in Gloucestershire and then at Uppingham School in Rutland, where a fellow teacher—in fact, Uppingham's first mathematical master—was Howard Candler, a friend of Edwin Abbott. It was during this period, in 1884, that Abbott published his now classic satirical novel *Flatland: A Romance of Many Dimensions*. Four years earlier, Hinton had penned an article of his own on alternative spaces called "What Is the Fourth Dimension?" in which he put forward the idea that particles moving around in three dimensions might be thought of as successive cross-sections of lines and curves existing in four dimensions. We, ourselves, might really be four-dimensional beings "and our successive states the passing of them through the three-dimensional space to which our consciousness is confined." The extent of the relationship between Abbott and Hinton isn't clear, but they certainly knew of each other's work (and acknowledged as much in their writings), and some social contact would have taken place, if only via their mutual friend and colleague. Candler would surely have discussed with Abbott the young teacher at Uppingham who wrote and spoke so openly about other dimensions.

Hinton was nothing if not unconventional. At the time he was teaching in England, he married Mary Ellen Boole, daughter of the above-mentioned Mary Everest Boole (herself the niece of George Everest, after whom the tallest mountain is named) and George

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published by *Astounding Science Fiction* in February 1941, tells the tale of an ingenious architect who designs a house with eight cubical rooms laid out like a net of a tesseract in 3-D. Unfortunately, an earthquake shakes the building, shortly after its completion, and causes it to fold into an actual hypercube, with bewildering results for those who first venture through its door. In "A Subway Named Moebius" (1950), Boston's underground train network becomes so convoluted that part of it flips into another dimension along with a train full of passengers, although all arrive safely at their intended stations in the end. Written by A. J. Deutsch, an astronomer at Harvard (one of the stops on the system), it plays on the themes of the Möbius band and Klein bottle, the latter being a one-sided shape that can exist only in four dimensions.

Artists too have tried to capture the essence of 4-D in their work. In his 1936 *Dimensionist Manifesto*, Hungarian poet and art theorist Charles Tamkó Sirató claimed that artistic evolution had led to "Literature leaving the line and entering the plane... Painting leaving the plane and entering space... [And] sculpture stepping out of closed, immobile forms." Next, Sirató said, there would be "the artistic conquest of four-dimensional space, which to date has been completely art-free." Salvador Dalí's *Crucifixion (Corpus Hypercubus)*, completed in 1954, unites a classical portrayal of Christ with an unfolded tesseract. In a 2012 lecture given at the Dalí Museum, geometer Thomas Banchoff, who advised Dalí on mathematical issues connected with his paintings, explained how the artist was trying to use "something from a three-dimensional world and take it beyond.... The exercise of the whole thing was to do two perspectives at once—two superimposed crosses." Dalí, like the nineteenth-century scientists who tried to rationalize spiritualism in terms of existence in some higher space, used the idea of the fourth dimension to connect the religious with the physical.

Twenty-first-century physicists have a new reason to be interested in higher dimensions: string theories. Here, subatomic particles, such as electrons and quarks, are treated as being not point-like but one-dimensional vibrating "strings." One of the strangest aspects of string theories is that, in order to be mathematically consistent, they require that the space and time in which we live have extra dimensions. A version called superstring theory calls for a total of ten dimensions, an extension of this known as M-theory involves eleven, while another scheme by the name of bosonic string theory demands twenty-six. All of these additional dimensions are said to be "compactified," meaning that they're significant only on a fantastically small scale. Maybe someday we'll learn how to amplify or uncurl these dimensions or observe them as they actually are. But for now and the foreseeable future, we're stuck with our familiar three macroscopic dimensions of space. So, the question remains: Is there any way we can visualize, in our minds, what a four-dimensional object is really like?

Our visual experience of the world comes about from light entering our eyes, striking our retinas, and creating two flat images. The light-sensitive cells in the retina generate electrical signals, which travel to the visual cortex in the brain where a 3-D reconstruction takes place based on essentially 2-D information. Having two eyes means that we can see objects from two slightly different angles, and the brain learns, when we're young, to interpret these as differences in perspective and, from them, build a three-dimensional view. But even with one eye closed, we don't suddenly switch to interpreting things as if they were in 2-D. Enough clues from perspective, illumination, and shading still arrive via monocular vision to enable us to add depth in our mind's eye. In addition, we can move around or rotate our head to change the angle of sight and add to this other sensory data, such as hearing and touch, to flesh out the 3-D impression. We're so adept at adding a dimension in this way that when we watch a movie on a TV screen, we automatically inject depth, even without the aid of 3-D technology.

The question is, if we have the ability to build 3-D pictures from 2-D input, could we use 3-D visual input to create an impression in our minds of the fourth dimension? Our natural retinas are flat, but electronic technology doesn't have such a limitation. By using enough cameras or other image-gathering devices, stationed in different places, we can collect information from as many directions and perspectives as we like. This alone,