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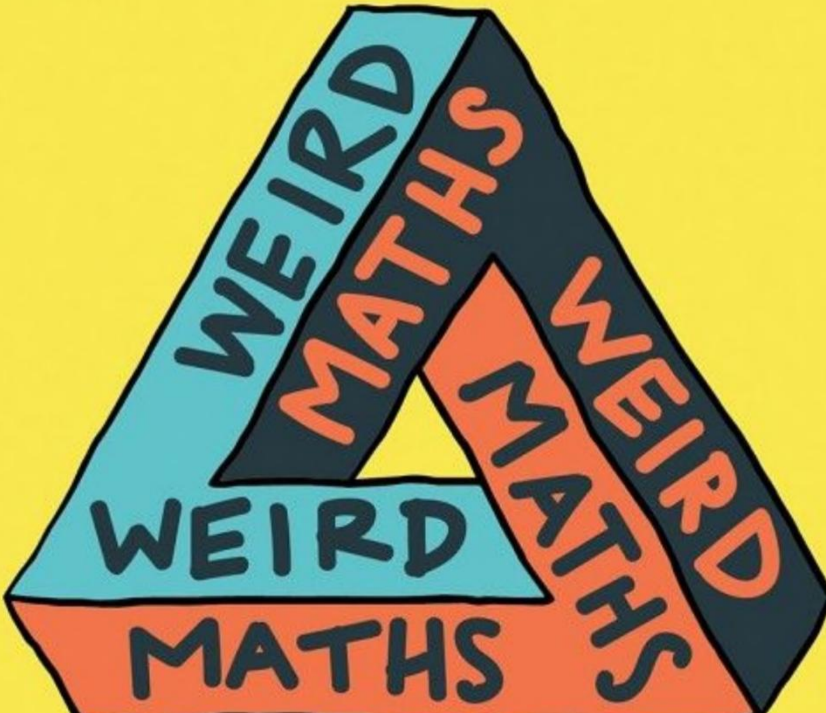

AT THE EDGE OF INFINITY AND BEYOND

Contents

Introduction

Maths *is* weird. Numbers go on forever – and there are different kinds of forever. Prime numbers help cicadas survive. A (mathematical) ball can be cut up and then put back together, without any gaps, to make a ball twice the size, or a million times the size, of the original. There are shapes that have fractional dimensions and curves that fill a plane leaving no holes. While bored by a dull presentation, physicist Stanislaw Ulam wrote out numbers, starting from 0, in a spiral form, marked in all the prime numbers, and found that many primes lie on long diagonals – a fact still not fully explained.

We forget sometimes how weird maths is because we're so used to dealing with what seem like ordinary numbers and calculations, the stuff we learn about in school or use every day. Yet the fact that our brains are so adept at thinking mathematically, and, if we choose, at doing really complex and abstract maths, is surprising. After all, our ancestors, tens or hundreds of thousands of years ago, didn't need to solve differential equations or dabble in abstract algebra in order to stay alive long enough to pass on their genes to the next generation. While searching for their next meal or a place to shelter, there was nothing to be gained from musing about geometry in higher dimensions or theories of prime numbers. Yet we're born with brains that have the potential to do these things, and to uncover, with each passing year, more and more extraordinary truths about the mathematical universe. Evolution has provided us with this skill: but how and why? Why are we, as a species, so good at doing something that has every appearance of being just an intellectual game?

Somehow maths is woven into the very fabric of reality. Dig deep enough and we find that what seemed to be tangible bits of matter or energy – electrons or photons, for instance – dissolve into immateriality, becoming mere waves of probability, and all we're left with is a ghostly calling card in the form of some intricate but beautiful set of equations. In some sense, mathematics underpins the physical world around us, forming an invisible infrastructure. Yet it also goes beyond this, into abstract realms of possibility that may forever remain purely exercises of the mind.

We've chosen in this book to highlight some of the more extraordinary and fascinating areas of maths, including those where exciting new developments are in the offing. In some cases, they have links with science and technology – particle physics, cosmology, quantum computers, and the like. In others, they represent, for now at least, maths for maths sake, and are adventures into an unfamiliar land that exists only in the mind's eye. We've chosen not to shy away from certain subjects just because they're hard. One of the challenges in describing many aspects of maths for a general audience is that they're far removed from everyday experience. But, in the end, some way can always be found to link what today's explorers and pioneers at the frontiers of mathematics are doing with the world of the familiar, even if the language we have to use isn't as precise as academics would ideally choose. It's perhaps true to say that if something, however obscure, can't be explained reasonably well to a person of normal intelligence then the explainer needs to improve their understanding!

This book came about in an unusual way. One of us (David) has been a science writer for more than 35 years and has written many books on astronomy, cosmology, physics, and philosophy, and even an encyclopaedia of recreational maths. The other (Agnijo) is a brilliant young mathematician and child genius, with an IQ of at least 162 according to Mensa, who, at the time of writing, has just finished training in Hungary in preparation for the 2017 International Mathematics Olympiad. Agnijo started coming to David for tuition in maths and science at the age of 12. Three years later, we decided to write a book together.

We sat down and brainstormed the topics we wanted to cover. David, for instance, came up with higher dimensions, the philosophy of maths, and the maths of music, while Agnijo was keen to write about large numbers (his personal passion), computation, and the mysteries of primes. Right from the start we chose to lean towards anything unusual or downright weird and to connect this weird maths, where possible, with real-world issues and everyday experience. We also made a commitment not to shy away from subjects just because they were tough, adopting as a mantra that if you can't explain something in plain language then you don't properly understand it. David generally took on the historical, philosophical, and anecdotal aspects of each chapter while Agnijo grappled with the more technical aspects. Agnijo fact-checked David's work, and David combined all the writing into finished chapters. It all worked surprising well! We hope you enjoy the result.

A Note to the Reader

In glancing through the pages of this book, you may notice that it contains some symbols, including x's, ω's (omegas), and even the odd א (aleph). You'll find an occasional equation or an unfamiliar-looking combination of characters, such as $3 \uparrow \uparrow 3 \uparrow \uparrow 3$ (especially in the chapters on large numbers and infinity). If you're a non-mathematician, don't be put off. They're just shorthand for ideas that, hopefully, we explain well enough in advance and thereby help us delve a little faster and deeper into the subject than would otherwise be possible. One of us (David) has taught maths privately to students for many years and has yet to come across one who can't be good at it once they believe in themselves. The fact is we're all natural mathematicians, whether we realise it or not. So, with that in mind, let's take the plunge…

CHAPTER 1

The Maths Behind the World

> Even stranger things have happened;
> and perhaps the strangest of all is the
> marvel that mathematics should be
> possible to a race akin to the apes.
>
> – Eric T. Bell, *The Development of Mathematics*

> Physics is mathematical not because
> we know so much about the physical
> world, but because we know so little;
> it is only its mathematical properties
> that we can discover.
>
> – Bertrand Russell

In terms of intellectual ability, *Homo sapiens* hasn't changed much, if at all, over the past 100,000 years. Put children from the time when woolly rhinos and mastodons still roamed the Earth into a present-day school and they would develop just as well as typical twenty-first century youngsters. Their brains would assimilate arithmetic, geometry, and algebra. And, if they were so inclined, there'd be nothing to stop them delving deeper into the subject and someday perhaps becoming professors of maths at Cambridge or Harvard.

Our neural apparatus evolved the potential to do advanced calculations, and understand such things as set theory and differential geometry, long before it was ever applied in this way. In fact, it seems a bit of a mystery *why* we have this innate talent for higher mathematics when it has no obvious survival value. At the same time, the reason our species emerged and endured is because it had an edge over its rivals in terms of intelligence and an ability to think logically, plan ahead, and ask 'what if?' Lacking other survival skills, such as speed and strength, our ancestors were forced to rely on their cunning and foresight. A capacity for logical thought became our one great super-power, and from that, in time, flowed our ability to communicate in a complex way, to symbolise, and to make rational sense of the world around us.

Like all animals, we effectively do a lot of difficult maths on the fly. The simple act of catching a ball (or avoiding predators or hunting a prey) involves solving multiple equations simultaneously at high speed. Try programming a robot to do the same thing and the complexity of calculations involved becomes clear. But the great strength of humans was their ability to move from the concrete to the abstract – to analyse situations, to ask if/then questions, to plan ahead.

The dawn of agriculture brought the need to track the seasons accurately, and the coming of trade and settled communities meant that transactions had to be carried out and accounts kept. For both these practical purposes, calendars and business transactions, some kind of reckoning had to be developed, and so elementary maths made its first appearance. One of the regions where it sprang up was the Middle East. Archaeologists have unearthed Sumerian clay trading tokens dating back to about 8,000 bc, which show that these people dealt with representations of number. But it seems that, at this early period, they didn't treat the concept as being separate from the thing being counted. For example, they had different shaped tokens for different items, such as sheep or jars of oil.

When a lot of tokens had to be exchanged between parties, the tokens were sealed inside containers called bullae, which had to be broken open to check the contents. Over time, markings began to appear on the bullae to indicate how many tokens there were within. The symbolic representations then evolved into a written number system, while tokens became generalised for counting any kind of object and eventually morphed into an early form of coinage. Along the way, the concept of number became abstracted from the type of object being counted, so that, for example, five was five whether it referred to five goats or five loaves of bread.

The Egyptians had a good understanding of practical mathematics and put this to effect in the construction of the Pyramid of Khafre at Giza, shown here together with the Sphinx.

The connection between maths and everyday reality seems strong at this stage. Counting and record keeping are practical tools of the farmer and the merchant, and if these methods do the job who cares about the philosophy behind it all? Simple arithmetic looks well rooted in the world 'out there': one sheep plus one sheep is two sheep, two sheep plus two sheep is four sheep. Nothing could be more straightforward. But look more closely and we see that already something a bit strange has happened. In saying 'one sheep and one sheep' there's the assumption that the sheep are identical or, at least, for the purposes of counting, that any differences don't matter. But no two sheep are alike. What we've done is to abstract a perceived quality to do with the sheep – their 'oneness', or apartness – and then operate on this quality with another abstraction, which we call addition. That's a big step. In practice, adding one sheep and one sheep might mean putting them together in the same field. But, also in practice, the sheep are different and, digging a little deeper, what we call 'sheep' – like anything else in the world – isn't really separate from the rest of the universe. On top of this, there's the slightly disturbing fact that what we take to be objects (such as sheep) 'out there' are constructions in our brains built up from signals that enter our senses. Even if we grant that a sheep has some external reality, physics tells us that it's a hugely complicated, temporary assemblage of subatomic particles that's in constant flux. Yet, somehow, in counting sheep we're able to ignore this monumental complexity or, rather, in everyday life, not even be aware of it.

Of all subjects, mathematics is the most precise and immutable. Science and other fields of human endeavour are, at best, approximations to some ideal, and are always changing and evolving over time. As the German mathematician Hermann Hankel pointed out: 'In most sciences, one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure.' From the outset, this difference between maths and every other discipline is inevitable because maths starts with the mind extracting what it recognises as being most fundamental and constant among the messages it receives via the senses. This leads to the concepts of natural numbers, as a way of measuring quantity, and of addition and subtraction as basic ways of combining quantities. Oneness, twoness, threeness, and so on, are seen as common features of collections of things, whatever those things happen to be and however different individuals of the same type of thing happen to be. So, the fact that maths has this eternal, adamantine quality to it is assured from the start – and is its greatest strength.

Mathematics exists. Of that there's no doubt. Pythagoras' theorem, for instance, is somehow part of our reality. But *where* does it exist when it's not being used or being instantiated in some material form, and where *did* it exist many thousands of years ago, before anyone had thought about it? Platonists believe that mathematical objects, such as numbers, geometric shapes, and the relationships between them, exist independently of us, and our thoughts and language, and the physical universe. Quite what sort of ethereal realm they inhabit isn't specified, but it's a common assumption that they're somehow 'out there'. Most mathematicians, it's probably fair to say, subscribe to this school of thought and therefore also to the belief that maths is discovered rather than invented. Most, too, probably don't care very much for philosophising and are happy just to get on with doing maths, in the same way that the majority of physicists, working in the lab or solving theoretical problems, don't worry a lot about metaphysics. Still, the ultimate nature of things – in this case, of mathematical things – is interesting, even if we never arrive at a final answer. The Prussian mathematician and logician Leopold Kronecker thought that only whole numbers were given, or in his words: ‘God made the integers, all the rest is the work of man.’ The English astrophysicist Arthur Eddington went further and said: ‘The mathematics is not there till we put it there.’ The debate about whether mathematics is invented or discovered, or is perhaps some combination of both, arising from a synergy of mind and matter, will no doubt rumble on and, in the end, may have no simple answer.

One fact is clear: if a piece of maths has been proven to be true, it will remain true for all time. There’s no matter of opinion about it, or subjective influence. ‘I like mathematics,’ remarked Bertrand Russell, ‘because it is not human and has nothing particular to do with this planet or with the whole accidental universe.’ David Hilbert voiced something similar: ‘Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.’ This impersonal, universal quality of maths is its greatest strength, yet doesn’t, to the trained eye, detract from its aesthetic appeal. ‘Beauty is the first test: there is no permanent place in the world for ugly mathematics,’ remarked the English mathematician, G. H. Hardy. The same sentiment, but from the field of theoretical physics was expressed by Paul Dirac: ‘It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power.’

The flip side to the universality of maths, however, is that it can seem cold and sterile, devoid of passion and feeling. As a result we may find that although intelligent beings on other worlds share the same mathematics as us, it isn’t the best way to communicate with them about a lot of the things that matter to us. ‘Many people suggest using mathematics to talk to the aliens,’ commented SETI (Search for ExtraTerrestrial Intelligence) researcher Seth Shostak. In fact the Dutch mathematician Hans Freudenthal developed an entire language (Lincos) based on this idea. ‘But,’ said Shostak, ‘my personal opinion is that mathematics may be a hard way to describe ideas like love or democracy.’

The ultimate goal of scientists, certainly of physicists, is to reduce what they observe in the world to a mathematical description. Cosmologists, particle physicists, and the like are never happier than when they’ve measured and quantified things and then found a relationship between the quantities. The idea that the universe is mathematical at its core has ancient roots, stretching back at least as far as the Pythagoreans. Galileo saw the world as a ‘grand book’ written in the language of mathematics, and, much more recently, in 1960, the Hungarian-American physicist and mathematician Eugene Wigner wrote a paper called ‘The Unreasonable Effectiveness of Mathematics in the Natural Sciences’.

We don’t see numbers directly in the real world, so it isn’t immediately obvious that maths is all around us. But we do see shapes – the near-spherical shape of planets and stars, the curved path of objects when thrown or in orbit, the symmetry of snowflakes, and so on – and these can be described by relationships between numbers. Other patterns, translatable into maths, emerge from the way electricity or magnetism behaves, galaxies rotate, and electrons operate within the confines of atoms. These patterns, and the equations describing them, underpin individual events and seem to represent deep, timeless truths underlying the changing complexity in which we find ourselves. The German physicist

Heinrich Hertz, who first conclusively proved the existence of electromagnetic waves, remarked: ‘One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them.’

It’s unquestionably true that the bedrock of modern science is mathematical in nature. But that doesn’t necessarily mean that reality itself is fundamentally mathematical. Ever since the time of Galileo, science has separated the subjective from the objective, or measurable, and focused on the latter. It’s done its best to evict anything to do with the observer and pay attention only to what it assumes lies beyond the interfering influence of the brain and senses. The way modern science has developed almost guarantees that it will be mathematical in nature. But this leaves much that science has trouble dealing with – most obviously, consciousness. It may be that someday we’ll have a good, comprehensive model of how the brain works, in terms of memory, visual processing, and so forth. But why we also have an inner experience, a feeling of ‘what it is like to be’, remains – and may always remain – outside the field of conventional science and, by extension, of mathematics.

Why has the human brain evolved to be so extraordinarily good at a subject – mathematics – that it doesn't need for survival?

On the one hand, Platonists believe maths to be a land that already exists, awaiting our exploration of it. On the other, there are those who insist that we invent mathematics as we go along to suit our purposes. Both positions have weaknesses. Platonists struggle to explain where things like pi might be outside the physical universe or the intelligent mind. Non-Platonists have a hard time denying the fact that, for example, planets would continue to orbit the Sun in ellipses whether we do the maths or not. A third school of mathematical philosophy occupies a middle ground between the other two, by pointing out that, in describing the real world, maths is not as successful as it's sometimes made out to be. Yes, equations are useful in telling us how to navigate a spacecraft to the Moon or Mars, or design a new aircraft, or predict the weather several days in advance. But these equations are mere approximations to the reality of what they're intended to describe, and, moreover, they apply to just a small portion of all the things going on around us. In touting the success of mathematics, the realist would say, we downplay the vast majority of phenomena that are too complex or poorly understood to capture in mathematical form, or that, by their very nature are irreducible to this kind of analysis.

Is it possible that the universe isn't, in reality, mathematical? After all, space and the objects it contains don't directly present anything mathematical to us. We humans rationalise and make approximations in order to model aspects of the universe. In doing so we find mathematics extremely useful in enabling us to understand it. That doesn't necessarily imply that maths is anything other than a convenience of our making. But if mathematics isn't present in the universe to start with, how is it that we're able to invent it in order to put it to such use?

Mathematics is broadly divided into two areas: pure and applied. Pure maths is maths for maths' sake. Applied mathematicians put their subject to work on real-world problems. But often, developments in pure maths, with seemingly no bearing on anything tangible, have turned out later to be surprisingly useful to scientists and engineers. In 1843, the Irish mathematician William Hamilton hatched the idea of quaternions – four-dimensional generalisations of ordinary numbers of no practical interest at the time but which, more than a century later, have turned out to be an effective tool in robotics and computer graphics and games. A question first tackled by Johannes Kepler in 1611, about the most efficient way to pack spheres in three-dimensional space, has been applied to the efficient transmission of information over noisy channels. The purest mathematical discipline, number theory, much of which was thought to have little practical value, has led to important breakthroughs in the development of secure ciphers. And the new geometry pioneered by Bernhard Riemann that dealt with curved surfaces proved ideal for the formulation of Einstein's general theory of relativity – a new theory of gravity – more than 50 years later.

In July 1915, one of the greatest scientists of all time met one of the greatest mathematicians of the age, when Einstein paid a visit to David Hilbert at the University of Göttingen. The following December, both published, almost simultaneously, the equations that described the gravitational field of Einstein's general theory. But whereas the equations themselves were the goal for Einstein, they were what Hilbert hoped would be a stepping-stone towards an even grander scheme. Hilbert's passion, the driving force behind much of his work, was a search for fundamental principles, or axioms, that might underlie all of mathematics. Part of this quest, as he saw it, was to find a minimum set of axioms from which he could deduce not only the equations of Einstein's general theory but any other theory in physics as well. Kurt Gödel, with his incompleteness theorems, undermined faith in the notion that maths might have the answers to all questions. But we remain uncertain as to what extent the world in which we live is truly mathematical or just mathematical in appearance.

Whole swathes of mathematics may never be put to use, other than to help open up yet more avenues of pure research. On the other hand, for all we know, it may be that much of pure maths is enacted, in unexpected ways, in the physical universe – or, if not this universe, then in others that might exist throughout what cosmologists suspect is a multiverse of incomprehensible scale. Perhaps everything that is mathematically true and valid is represented somewhere, sometime, somehow in the reality in which we are embedded. For now there is the journey to keep us occupied: the weird and wonderful adventure of the human mind as it explores further the frontiers of number, space, and reason.

In the chapters that follow we'll take a deep dive into subjects that are both bizarre and astonishing and yet, at the same time, have very real connections with the world we know. True, some of the maths may seem esoteric, fanciful, and even pointless, like some strange and convoluted game of the imagination. But, at its core, mathematics is a practical affair, rooted in commerce, agriculture, and architecture. Although it's developed in ways that our ancestors could never have dreamed about, still those links with our everyday lives remain at its heart.

CHAPTER 2

How to See in 4D

> One of the strangest features of string theory is that it requires more than the three spatial dimensions that we see directly in the world around us. That sounds like science fiction, but it is an indisputable outcome of the mathematics of string theory.
>
> – Brian Greene

We live in a world of three dimensions – up and down, side to side, and backwards and forwards, or any other three directions that are at right angles to each other. It's easy to imagine something in one dimension, such as a straight line, or two dimensions, such as a square drawn on a sheet of paper. But how can we possibly learn to see in an extra dimension to those we're familiar with? Where is this additional direction that's perpendicular to the three we know?

These questions may seem purely academic. If our world is three-dimensional, why worry about 4D or 5D and so forth? The fact is that science may need higher dimensions to explain what is going on at a subatomic level. These extra dimensions may hold the key to understanding the grand scheme of matter and energy. Meanwhile, on a more practical level, if we could learn to see in 4D we'd have a powerful new tool to deploy in medicine and education.

Sometimes the fourth dimension is taken to be something other than an extra direction in space. After all, the word ‘dimension’, from the Latin *dimensionem*, means simply a ‘measurement’. In physics, the basic dimensions that form the building blocks of other quantities are considered to be length, mass, time, and electric charge. Very often, in a different context, physicists talk about three dimensions of space and one of time, especially since Albert Einstein showed that, in the world in which we live, space and time are always bound up together in a single entity called spacetime. Even before the theory of relativity came along, however, there had been speculation about the possibility of being able to move backwards and forwards along the dimension of time just as we can move any way we like in space. In his novel *The Time Machine*, published in 1895, H. G. Wells explains that an instantaneous cube, for instance, can’t exist. A cube that we see moment by moment is just a cross section of a four-dimensional thing having length, breadth, width, *and duration*. ‘There is no difference,’ says the Time Traveller, ‘between Time and any of the three dimensions of Space except that our consciousness moves along it.’

The Victorians were also fascinated by the idea of a fourth dimension of space, both from a mathematical point of view and in the possibilities it seemed to offer of explaining another obsession of the age – spiritualism. The late 1800s was a period when many people, including luminaries such as Arthur Conan Doyle, Elizabeth Barrett Browning, and William Crookes, were attracted to the claims of mediums and the prospect of communicating with the dead. Might the afterlife, people wondered, exist in a fourth dimension that was parallel to, or overlapped with, our own so that spirits of the deceased could pass easily into our material realm and back again?

Our failure to be able to visualise in higher dimensions makes it tempting to think that the fourth dimension is somehow mysterious or alien to anything we know. Mathematicians, though, have no trouble in working with four-dimensional objects or spaces because they don't have to imagine what they're actually like in order to describe their properties. These properties can be figured out using algebra and calculus without having to resort to any multidimensional mental gymnastics. Start with a circle, for instance. A circle is a curve made of all the points in a plane that are at the same distance (the radius) from a given point (the centre). Like a straight line, it has only length – no width or height – and so is a one-dimensional thing. Imagine yourself positioned and constrained within a line. The only freedom of movement you would have is along the line, one way or the other. It's the same with a circle. Although a circle exists in a space of at least two dimensions, if you were positioned and confined within the circle then you'd have no more or less freedom of movement than if you were positioned on a line. You could only go back and forth along the circle, effectively tied down to a single dimension of movement.

Non-mathematicians sometimes think of a circle as including its interior as well. But a 'filled-in circle' to a mathematician isn't a circle at all but a very different object called a disc. A circle is a one-dimensional object that can be 'embedded' in a two-dimensional object, a plane (a finely drawn circle on a sheet of paper is an approximation to this). The length, or circumference, of a circle is given by $2\pi r$, where r is the radius, and the area enclosed by the circle is πr^2. Moving up a dimension we come to the sphere, which consists of all the points lying at the same distance in three-dimensional space from a given point. Again, the layperson may confuse an actual sphere, which is just a two-dimensional surface, with the object that also includes all the points inside this surface. But, once more, mathematicians make a sharp distinction and call the latter a 'ball'. A sphere is a two-dimensional object that can be embedded in three-dimensional space. It has a surface area of $4\pi r^2$ and encloses a volume of $4/3\pi r^3$. Because an ordinary sphere is two-dimensional, mathematicians call it a 2-sphere, whereas a circle, using the same naming system, is a 1-sphere. Spheres in higher dimensions are said to be 'hyperspheres' and can be labelled in the same way. The simplest hypersphere, the 3-sphere, is a three-dimensional object embedded in four-dimensional space. We can't capture this in our mind's eye but we can understand it by analogy. Just as a circle is a curved line and an ordinary (2-) sphere is a curved surface, a 3-sphere is a curved volume. Using some straightforward calculus, mathematicians can show that this curved volume is given by $2\pi^2 r^3$. It's the 3-sphere equivalent of the surface area of an ordinary sphere and is also referred to as a cubic hyperarea or surface volume. The four-dimensional space enclosed by a 3-sphere has a four-dimensional volume, or quartic hypervolume, of $½\pi^2 r^4$. Proving these facts about the 3-sphere is not much more difficult than proving them about the circle or ordinary sphere and doesn't involve having to understand what a 3-sphere actually looks like.

In the same way, we may struggle to grasp the true appearance of a four-dimensional cube or 'tesseract', though, as we'll see, we can try to represent it in two or three dimensions. But it's straightforward to describe the progression from square to cube to tesseract: a square has 4 vertices (corners) and 4 edges; a cube has 8 vertices, 12 edges, and 6 faces; a tesseract has 16 vertices, 32 edges, 24 faces, and 8 'cells' – the three-dimensional equivalents of faces – consisting of cubes. This last fact is the one that defies our attempts at visualisation: a tesseract has 8 cubic cells arranged in such a way as to enclose a four-dimensional space, just as a cube has 6 square faces arranged so as to enclose a three-dimensional space.

The best we can normally do in coming to terms with the fourth dimension is to draw analogies with the third. For instance, if we were to ask, 'What would a four-dimensional hypersphere look like if it were to pass through our space?' we can get an impression by considering what happens if a sphere passes through a plane. Suppose there are two-dimensional beings who inhabit that plane. Looking along the surface of their world – which is all they can do – they see only dots or lines of different length which they can only interpret as two-dimensional figures. As our 3D sphere initially makes contact with their 2D space they see it as a dot, which then grows into a circle reaching a maximum diameter equal to the diameter of the sphere before the circle shrinks again to a dot and then disappears, as the sphere passes through. Likewise, if a 4-sphere were to intersect our space we'd see it as a dot that expanded, like a bubble, into a three-dimensional sphere of maximum size before shrinking and finally vanishing. The true nature – the extra dimensionality – of the 4-sphere would be hidden from us, although its mysterious appearance, growth, and disappearance would probably cause us to wonder what was going on!

Four-dimensional beings would have seemingly magical powers in our world. They could, for example, pick up a right-footed shoe, flip it over in the fourth dimension, and put it back as a left-footed one. If this seems hard to understand, think of a two-dimensional shoe, which would be like an infinitely thin sole shaped for one foot or the other. We could cut such a shape out of a piece of paper, lift it up, turn it over, and put it back down, so that we changed its footedness. A 2D creature would find this utterly astonishing but to us, with the benefit of the extra dimension, the trick would seem obvious.

In principle a 4D being could flip around a whole (3D) person in the fourth dimension, although the absence of cases of people having suddenly had everything switched right-to-left or left-to-right, suggests that this hasn't actually happened. In his short tale 'The Plattner Story', H. G. Wells describes the remarkable case of Gottfried Plattner, a teacher who disappears for nine days following an explosion in a school chemistry lab. Upon his return, he is effectively a mirror image of his previous self, though his recollections of what had happened during the period of absence are met with incredulity. Being flipped over for real in the fourth dimension would be bad for your health, apart from the shock of seeing yourself look different in the mirror (faces are surprising asymmetrical). Many of the crucial chemicals in our bodies, including glucose and most amino acids, have a certain handedness. Molecules of DNA, for example, which take the form of a double helix, always twist like a right-handed screw. If all these chemicals had their handedness reversed we'd quickly die of malnutrition because many of the essential nutrients in our food, from plants and animals, would now be in a form we couldn't assimilate.

Mathematical interest in a fourth spatial dimension began in the first half of the nineteenth century with the work of the German Ferdinand Möbius. He's best remembered for his study of a shape that's now named after him – the Möbius band – and as a pioneer of the field known as topology. It was he who first realised that in a fourth dimension a three-dimensional form could be rotated into its mirror image. In the second half of the nineteenth century, three mathematicians stood out as explorers of the new realm of multidimensional geometry: the Swiss Ludwig Schläfli, the Englishman Arthur Cayley, and the German Bernhard Riemann.

Schläfli began his magnum opus, *Theorie der Vielfachen Kontinuität* (*Theory of Continuous Manifolds*), by saying: 'The treatise ... is an attempt to found and develop a new branch of analysis that would, as it were, be a geometry of n dimensions, containing the geometry of the plane and space as special cases for $n = 2, 3$.' He went on to describe multidimensional analogues of polygons and polyhedrons, which he called 'polyschemes'. These are now commonly known as polytopes, a term coined by the German mathematician Reinhold Hoppe and introduced to English researchers by Alicia Boole Stott, daughter of the English mathematician and logician George Boole, who devised Boolean algebra, and Mary Everest Boole, a self-taught mathematician and writer on the subject.

Also to Schläfli's credit is the discovery of the higher-dimensional relatives of the Platonic solids. By Platonic solid is meant a convex shape (one with all the corners pointing outwards) with regular polygon faces and the same number of faces meeting at each corner. There are five of them: the cube, tetrahedron, octahedron, (12-sided) dodecahedron, and (20-sided) icosahedron. The four-dimensional equivalents of the Platonic solids are the convex regular 4-polytopes (also called polychora), of which Schläfli found there were six, named after the number of cells they have. The simplest 4-polytope is the 5-cell, which has 5 tetrahedral cells, 10 triangular faces, 10 edges, and 5 vertices, and is analogous to the tetrahedron. Then there is the 8-cell, or tesseract, and its 'dual', the 16-cell, sect devoted to polygamy and free love, played a part in Charles' behaviour. In any event, Hinton was found guilty of bigamy at an Old Bailey trial and jailed for several days. With his (first) family he then fled to Japan, where he taught for some years, before becoming an instructor of mathematics at Princeton University. There, in 1897, he designed a species of baseball gun, which, with the help of gunpowder charges, fired out balls at speeds of 40 to 70 miles per hour. *The New York Times* in its March 12 edition of that year described it as, 'a heavy cannon, with a barrel about two and a half feet in length, with a rifle attachment in the rear'. Its cleverest trick, throwing curveballs, was accomplished with the help of 'two curved rods, which are inserted in the barrel of the cannon'. For a few seasons, the Princeton Nine used it, on and off, before abandoning it as a safety hazard. Whether the injuries it caused were a factor in Hinton's dismissal from the college is unclear, but they didn't prevent him reintroducing the machine at the University of Minnesota where, briefly, in 1900, he held a teaching post before joining the US Naval Observatory in Washington, D.C.

Hinton's fascination with the fourth dimension, stretching back to his early days as a teacher in England, began at a time when others were writing about the subject and often speculating about its possible links with spiritualism. In 1878, Friedrich Zöllner, professor of astronomy at the University of Leipzig, published a paper called 'On space of four dimensions' in *The Quarterly Journal of Science* (edited by the chemist and prominent spiritualist William Crookes). Zöllner started on solid mathematical ground by referencing Bernhard Riemann's seminal paper 'On the hypotheses which underlie geometry', published in 1868, two years after Riemann's death and 14 years after its contents were first delivered as a lecture by

Riemann while still a student at the University of Göttingen. Riemann developed the concept, first hinted at by his supervisor at Göttingen, the great Carl Gauss, that three-dimensional space could be curved (just as a two-dimensional surface, such as a sphere, can be) and extended this idea of the curvature of space into an arbitrary number of dimensions. The result, known as elliptic or Riemannian geometry, later formed a cornerstone of Albert Einstein's general theory of relativity. Zöllner also borrowed the notion, described in an 1874 paper by the young projective geometer Felix Klein, that knots could be undone and rings unlinked simply by lifting them into a fourth dimension and turning them over. In this way Zöllner set the scene for his explanation of how spirits, existing, as he saw it, on a higher plane, could perform the various phenomena – especially the knot-untying tricks – that he'd witnessed at séance experiments with the famous (and, as it turned out, utterly fraudulent) medium Henry Slade. Hinton, like Zöllner, was inclined to think that mere habit of perception limited us to a three-dimensional viewpoint and that a fourth dimension might be all around us and become visible to us if only we could train ourselves to see it.

Although something that's four-dimensional is hard to imagine, it's easy to do a 2D sketch of one. This is especially true in the case of the four-dimensional equivalent of a cube for which Hinton coined the name 'tesseract'. Start by drawing two squares, slightly offset, and connecting their corners by straight lines. This can be visualised as a perspective drawing of a cube, the squares being separated, in our mind's eye, in the third dimension. Next draw two cubes joined at their corners. With 4D vision we'd be able to see this as two cubes separated in the fourth dimension – in fact, a perspective of a tesseract. Unfortunately, flat representations of 4D objects aren't much help to us in being able to see them for what they really are. Hinton realised that a more fruitful approach to training our minds to see in four dimensions might be through three-dimensional models that could be rotated to show different aspects of a 4D shape: at least that way we'd only be dealing with a perspective of the real thing rather than a perspective of a perspective. To this end, he developed an intricate visual aid in the form of a set of one-inch wooden cubes in different colours. A complete set of Hinton cubes consisted of 81 cubes painted in 16 different colours, 27 'slabs' used to represent, by analogy, how a 3D object can be built up in two dimensions, and 12 multicoloured 'catalogue cubes'. By elaborate manipulations, described in detail in his book *The Fourth Dimension*, first published in 1904, he was able to represent the various cross sections of a tesseract and then, by memorising the cubes and their many possible orientations, gain a window on this higher-dimensional world.

Rotation of a tesseract. *(Top)* The traditional ‘cube within a cube’ view of a tesseract. *(Centre)* The tesseract has rotated slightly. The central cube has started to move and is in the process of becoming the right cube. *(Bottom)* The tesseract has rotated further and the central cube is now much closer to where the right cube was originally. Finally, the tesseract rotates fully back to its starting position. What is important is that the tesseract has not in any way deformed. Instead, the changes are due to a shift in perspective.

Did Hinton actually learn to create four-dimensional images in his brain? In addition to the familiar up and down, forward and back, and side to side, could he see ‘kata’ and ‘ana’ – his names for the two opposite directions along the fourth dimension? Without getting inside his head we can’t know. Certainly he wasn’t alone in building 3D representations of 4D shapes. He introduced his cubes to his sister-in-law, Alicia Boole Stott, who became an intuitive geometer of the fourth dimension herself and adept at making card models of 3D cross sections of 4D polytopes. The question remains whether, by such means, a person can develop true four-dimensional vision or just the ability to understand and appreciate the geometry of higher-dimensional objects.

In a way, being able to see an extra dimension is like being able to see a new colour – one outside all our previous experience. The French impressionist painter Claude Monet underwent surgery in 1923, at the age of 82, to remove the lens from his left eye, which had become hopelessly clouded by cataracts. Subsequently, the colours he chose to use in his art changed from mostly reds, browns, and other earthy tones to blues and violets. He even repainted some of his earlier works so that, for example, what had been white water lilies took on a bluish hue – an indication, it's been claimed, that he could now see into the ultraviolet region of the spectrum. This idea is supported by the fact that the lens of the eye blocks out wavelengths shorter than about 390 nanometres (billionths of a metre), at the far end of the violet range, even though the retina has the potential to detect wavelengths down to about 290 nanometres, which is in the ultraviolet. There's also plenty of evidence, in more recent times, of young children, and of older people who have missing lenses, being able to see beyond the violet end of the spectrum. One of the best-documented cases is that of a retired air force officer and engineer from Colorado, Alek Komarnitsky, who had a cataract-affected natural lens replaced by an artificial one that can transmit some UV light. In 2011, Komarnitsky underwent tests using a monochromator at a Hewlett-Packard lab where he reported being able to see wavelengths down to 350 nanometres, as a deep purple hue, and some variation in brightness even further into the UV, down to 340 nanometres.

the plane … Painting leaving the plane and entering space … [And] sculpture stepping out of closed, immobile forms.' Next, Sirató said, there would be 'the artistic conquest of four-dimensional space, which to date has been completely art-free'. Salvador Dalí's *Crucifixion (Corpus Hypercubus)*, completed in 1954, unites a classical portrayal of Christ with an unfolded tesseract. In a 2012 lecture given at the Dalí Museum, geometer Thomas Banchoff, who advised Dalí on mathematical issues connected with his paintings, explained how the artist was trying to use 'something from a three-dimensional world and take it beyond … The exercise of the whole thing was to do two perspectives at once – two superimposed crosses.' Dalí, like the nineteenth-century scientists who tried to rationalise spiritualism in terms of existence in some higher space, used the idea of the fourth dimension to connect the religious with the physical.

Twenty-first-century physicists have a new reason to be interested in higher dimensions: string theories. In these, subatomic particles, such as electrons and quarks, are treated as being not point-like, but one-dimensional vibrating 'strings'. One of the strangest aspects of string theories is that, in order to be mathematically consistent, they require that the space and time in which we live have extra dimensions. A version called superstring theory requires a total of 10 dimensions, and an extension of this, known as M-theory, involves 11, while another scheme, by the name of bosonic string theory, demands 26. All of these additional dimensions are said to be 'compactified', meaning that they're significant only on a fantastically small scale. Maybe someday we'll learn how to amplify or uncurl these dimensions or observe them as they actually are. But, for now and the foreseeable future, we're stuck with our familiar three macroscopic dimensions of space. So, the question remains: is there any way we can visualise, in our minds, what a four-dimensional object is really like?

Our visual experience of the world comes about from light entering our eyes, striking our retinas, and creating two flat images. The light-sensitive cells in the retina generate electrical signals, which travel to the visual cortex in the brain where a 3D reconstruction takes place based on essentially 2D information. Having two eyes means that we can see objects from two slightly different angles and the brain learns, when we're young, to interpret these as differences in perspective and, from them, build a three-dimensional view. But even with one eye closed, we don't suddenly switch to interpreting things as if they were in 2D. Enough clues from perspective, illumination, and shading still arrive via monocular vision to enable us to add depth in our mind's eye. In addition, we can move around or rotate our head to change the angle of sight and add to this other sensory data, such as hearing and touch, to flesh out the 3D impression. We're so adept at adding a dimension in this way that when we watch a movie on a TV screen we automatically inject depth, even without the aid of 3D technology.

The question is, if we have the ability to build 3D pictures from 2D input, could we use 3D visual input to create an impression in our minds of the fourth dimension? Our natural retinas are flat, but electronic technology doesn't have such a limitation. By using enough cameras or other image gathering devices, stationed in different places, we can collect information from as many directions and perspectives as we like. This alone, however, wouldn't be enough to form the basis of a 4D view. A genuine four-dimensional observer looking at something in our world would be able to see everything inside a thing simultaneously, in addition to its three-dimensional surface. So, for example, if you had some valuable items locked up in a safe, a 4D being would see not only all sides of the safe at a single glance but everything inside it as well (and would be able to reach in and take those things if it so chose!). This isn't because the being would have something like X-ray vision that allowed it to see through the walls of the safe, but simply because it had access to an extra dimension. We would similarly have a privileged view of an enclosed space in a 2D world. Draw a square on a piece of paper, to represent a two-dimensional safe, with some items of jewellery inside it. A Flatlander, embedded in the 2D surface, could see only a view of the outside of his safe – a mere line. We, looking from above the sheet of paper that was his world, would be able to see the lines that formed the walls of the safe and all its contents at a single glance and could reach in and lift out the 2D pieces of jewellery. It would mystify the Flatlander how the inside of the safe could be observed, or its contents removed, with no gaps in its walls. But, in the same way, an observer from the vantage point of a fourth dimension would be able to see all parts, inside and out, of something in 3D, whether it was a house, a machine, or a human body.

A way to create the illusion of 4D vision, then, if not 4D vision itself, would be to have a 3D retina, consisting of many layers, each layer of which could hold the image of a unique cross section of a 3D object. The information from this artificial retina would then be fed directly to a person's brain in such a way that they would have simultaneous access to all of the cross sections, exactly as a true four-dimensional observer would have. The result would not be an actual 4D image but something like the view we would have of a 3D thing if we could look 'down' on it from a fourth dimension, which could have some very valuable applications. The first part of the technology required – the 3D retina – is effectively already available in the form of medical scanners that build up a solid picture of part of the human body from 2D slices. The second part is at present beyond us, because we don't yet have sufficiently advanced brain-computer interfaces or the neurological knowledge needed to feed into the visual cortex so that the brain can construct an all-perspective, all-at-once image of the thing being observed. However, the dawn of 'Human 2.0' may be only a decade or two away. Futurist Ray Kurzweil believes that by the 2030s we'll be enhancing our brains with nanobots, tiny robotic implants that connect to cloud-based computer networks. In 2017, technology entrepreneur Elon Musk launched Neuralink, a venture to merge the human brain with AI through cortical implants.

As well as putting the technology in place and making the right connections to the brain, to see using a 3D retina a person would presumably have to go through a lengthy process of learning how to create mental pictures in this radically new way. However, such an ability could prove invaluable to those involved in areas such as medical diagnosis, surgery, scientific research, and education.

The more difficult step of enabling a person to experience seeing a thing in four dimensions could only be done with simulations, since 4D objects don't physically exist in our world. Perhaps a computer simulation of a tesseract – the object used by Hinton – would be the simplest place to start. When we look at a 3D model of a tesseract we see only one aspect, or projection, of the true four-dimensional shape. To grasp the thing in all its 4D glory would involve combining multiple projections, seamlessly and simultaneously, in the visual processing parts of our brain. Again, even with all the

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